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A NOTE ON GEOMETRIC FACTORIALITY

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ABSTRACT. Let k be a perfect field such that $\bar{k}$ is solvable over k. We show that a smooth, affine, factorial surface birationally dominated by affine 2-space $\mathbb{A}^2_k$ is geometrically factorial and hence isomorphic to $\mathbb{A}^2_k$. The result is useful in the study of subalgebras of polynomial algebras. The condition of solvability would be unnecessary if a question we pose on integral representations of finite groups has a positive answer.

1. **Introduction.** Let k be a field and A a regular factorial, affine k-algebra. Suppose $A \subset k[Z,T]$, the polynomial algebra in two variables over k. If k is algebraically closed and $k(Z,T)$ is a separable extension of the quotient field K of A, then by a famous result of Fujita and Miyanishi-Sugie, A is itself a polynomial algebra over k ([F] and [M-S], see also [R-1] for the case when $\operatorname{char} k > 0$). This result fails when k is not algebraically closed (see [B-D], Example 4.4 and 4.1 below). On the other hand, in counterexamples known to us, $[k(Z,T) : K] > 1$ and moreover, for perfect k, Russell ([R-2], Theorem 1.3) has shown that when $k[Z,T]$ is a simple (as ring) birational extension of A, then again A is a polynomial algebra over k. We therefore raise

QUESTION 1. Let k be a perfect field and A a regular, affine factorial, birational subalgebra of $k[Z,T]$. Is A a polynomial algebra over k?

We were motivated to study this question by considering regular, factorial affine k-algebras B such that

$$k[X] \subset B \subset k[X,Z,T].$$

It is then natural to ask whether B is a polynomial algebra and, if yes, whether X is a variable in B. This obviously is true if $\dim B = 1$, and has been shown to hold if $\dim B = 2$ by Russell and Sathaye ([R-S]). If $\dim B = 3$, it is not difficult to give counterexamples to the first part of the question (see [B-D], Example 4.4 and 4.2 below), even if k is algebraically closed. A first step in studying this situation will be to consider the ring extensions

$$k(X) \subset B \bigotimes_{k[X]} k(X) \subset k(X)[Z,T].$$

In case the extension $k[X,Z,T]/B$ is birational, an affirmative answer to Question 1 would imply that B is "generically" polynomial over $k[X]$ if $\operatorname{char} k = 0$, a result of interest even if we assume to begin with that B is polynomial over k.

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The key to answering Question 1 is to ascertain that factoriality of A is preserved when the base field k is extended to L, where L/k is a finite Galois extension. We show this (see Proposition 3.4) in case L/k is solvable with the help of a result on integral representations (Proposition 2.2). If the condition of solvability could be removed there, Question 1 would be answered positively in general.

2. **A result from the representation theory.** Let G be a finite group and M a finite $\mathbf{Z}[G]$-module. For any subgroup $H \subset G$, we put $\mathrm{Inv}_H(M) = \{m \in M \mid hm = m \, \forall h \in H\}$. M is said to be a *permutation module* for G if M is free over $\mathbf{Z}$ with a basis S permuted by G. We then call S a permutable basis for M. M is said to be *transitive* if G is transitive on S. It is clear that any permutation module for G is a direct sum of transitive ones, corresponding to the decomposition of S into G-orbits.

LEMMA 2.1. *Let G be a finite group and let M be a transitive permutation left* $\mathbf{Z}[G]$*-module. Let H be a normal subgroup of G. Then* Inv_H *is a transitive permutation* $\mathbf{Z}[G/H]$*-module.*

PROOF. Let S be a transitively permutable basis of M and let $S_1, \ldots, S_t$ be the all distinct H-orbits of S. Then since H is normal in G and S is a transitively permutable basis (for G) it follows that any two distinct H-orbits have the same number of elements and given two orbits S_i, S_j there exists $g \in G$ such that $g \cdot S_i = S_j$.

Let $\omega_i = \sum_{v \in S_i} v \in M$, $1 \leq i \leq t$. Then $\mathrm{Inv}_H(M) = \bigoplus_{i=1}^{t} \mathbf{Z}\omega_i$ and given ω_i, ω_j there exists $g \in G$ such that $g \cdot \omega_i = \omega_j$.

Thus $\mathrm{Inv}_H(M)$ is a transitive permutation $\mathbf{Z}[G/H]$-module.

PROPOSITION 2.2. *Let G be a finite solvable group. Let F be a permutation* $\mathbf{Z}[G]$*-module and let M and N be* $\mathbf{Z}[G]$*-submodules of F such that* $F = M \oplus N$*. Furthermore, assume M is also a permutation* $\mathbf{Z}[G]$*-module. Then* $\mathrm{Inv}_G(N) = 0 \Rightarrow N = 0$.

PROOF. Let H be a normal subgroup of G. Since every permutation $\mathbf{Z}[G]$-module is a direct sum of transitive permutation modules, it follows from Lemma 2.1 that $\mathrm{Inv}_H(F)$ and $\mathrm{Inv}_H(M)$ are permutation $\mathbf{Z}[G/H]$-modules. Moreover, $\mathrm{Inv}_H(F) = \mathrm{Inv}_H(M) \oplus \mathrm{Inv}_H(N)$ and $\mathrm{Inv}_{G/H}\big(\mathrm{Inv}_H(N)\big) = \mathrm{Inv}_G(N)$. Therefore, as F and M are obviously permutation $\mathbf{Z}[H]$-modules, it is enough to prove the result when G is simple. But as G is solvable, this means that it is enough to prove the result when G is a cyclic group of prime order.

So we assume $|G| = p, p$ a prime integer. Let g be a generator of G and let I be the ideal of $\mathbf{Z}[G]$ (note that $\mathbf{Z}[G]$ is commutative) generated by the element $g - 1$.

Let $F = \bigoplus_{i=1}^{n} F_i$ be a direct sum decomposition of F into transitive permutation $\mathbf{Z}[G]$-submodules of F. Since G is cyclic of order p, up to isomorphism $\mathbf{Z}[G]$ has only two transitive permutation modules viz. $\mathbf{Z}[G]$ (as a module) and $\mathbf{Z}$ (with the trivial G-module structure). Therefore it follows that $\mathrm{Inv}_G(F_i) \approx F_i/IF_i = \mathbf{Z}$ and hence $\mathrm{Inv}_G(F) \approx F/IF$. Similarly $\mathrm{Inv}_G(M) \approx M/IM$.

Now $F = M \oplus N$ and $\mathrm{Inv}_G(N) = 0$. So we see that $N/IN = 0$, *i.e.* $IN = N$. Since I is the principal ideal of $\mathbf{Z}[G]$ generated $g-1$, we get $(g-1)N = N$ and hence $(g-1)^pN = N$. But g is an element of G of order p. Therefore $(g-1)^pN = N$ implies that $pN = N$.

As N is a submodule of F and F is a free abelian group (since F is a permutation $\mathbf{Z}[G]$-module), $pN = N$ implies $N = 0$.

REMARK 2.3. Let $F = \oplus_{i=1}^{s} F_i$, where $F_1, \dots, F_s$ are transitive permutation modules of rank $r_1, \dots, r_s$. Then $s = \mathrm{rank}\big(\mathrm{Inv}_G(F)\big)$ and $H^0(G,F) \simeq \oplus_{i=1}^{s} \mathbf{Z}/\tilde{r}_i\mathbf{Z}$, where $\tilde{r}_i = |G|/r_i$. (Here $H^0(G,M) = \mathrm{Inv}_G(M)/\,\mathrm{Trace}(M)$; see [L]). Moreover, $H^0(G,N) = 0$ in the situation of Proposition 2.2. So Proposition 2.2 holds for arbitrary finite G in case F, or M, is transitive. It is therefore reasonable to ask

QUESTION 2. Does Proposition 2.2 remain true without the assumption that G is solvable?

3. **Factorial surfaces dominated by $\mathbb{A}^2$.**

LEMMA 3.1. *Let k be a field and let L/k be a finite separable extension. Let X be a smooth, quasi-projective scheme over k. Let $x \in X$ be a closed point of X and let $\pi: \tilde{X} \longrightarrow X$ be the blowing up of X with the center x (this will be referred to as monoidal transformation). Then the canonical map: $\pi_L: \tilde{X}_L \longrightarrow X_L$ (obtained by base change) is the blowing up of X_L with centre $p^{-1}(x)$ where $p: X_L \longrightarrow X$ is the canonical morphism.*

PROOF. Without loss of generality, we can assume that X is affine, say $X = \mathrm{Spec}(A)$. Let m be the maximal ideal of A corresponding to the closed point x. Let $B = A \otimes_k L$ and let $I = mB$. Then, since L is separable over k, I is the defining ideal of the closed subset $p^{-1}(x)$ of $\mathrm{Spec}(B)$. Now the result follows from the definition of blowing up and the following isomorphisms of L-algebras:

$$B \oplus I \oplus I^2 \cdots \approx (A \oplus m \oplus m^2 \cdots) \bigotimes_A B = (A \oplus m \oplus m^2 \cdots) \bigotimes_k L.$$

LEMMA 3.2. *Let k be a field and let L/k be a finite Galois extension with Galois group G. Let X be a smooth, geometrically integral, quasi-projective scheme over k. Then X_L is smooth and integral. The group G acts on the class group $\mathrm{Cl}(X_L)$ inducing a (left) $\mathbf{Z}[G]$-module structure. Moreover* $\mathrm{rank}\big(\mathrm{Cl}(X)\big) = \mathrm{rank}\,\mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$.

PROOF. It is obvious that X_L is smooth, integral and G acts (in a canonical manner) on $\mathrm{Cl}(X_L)$.

Let $p: X_L \longrightarrow X$ be the canonical morphism. Let C be an irreducible closed subset of X of codimension one and let $C_1', \dots, C_n'$ be the irreducible components of $p^{-1}(C)$. Then the codimension of C_i' in X_L is 1 for $1 \le i \le n$ and $p^*(C) = \sum_{i=1}^{n} C_i'$ (as L/k is separable), where $p^*: \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X_L)$ is the group homomorphism induced by p. It is easy to see that $p^*\big(\mathrm{Cl}(X)\big) \subset \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$.

Since p is a finite morphism and X, X_L are smooth, there exists a group homomorphism $p_*: \mathrm{Cl}(X_L) \longrightarrow \mathrm{Cl}(X)$ such that $p_*p^* =$ multiplication by the integer $|G|$. This gives the equality

$$\mathrm{rank}\,\mathrm{Cl}(X) = \mathrm{rank}\big(p^*\,\mathrm{Cl}(X)\big).$$

Let $\mathrm{Tr}\colon \mathrm{Cl}(X_L) \longrightarrow \mathrm{Cl}(X_L)$ be the trace homomorphism defined by $\mathrm{Tr}(c) = \sum_{g\in G} g\cdot c$. Then it is easy to see that $\mathrm{Im}(\mathrm{Tr}) \subset \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$ and for $v \in \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$, $\mathrm{Tr}(v) = |G|v$. Therefore we get the equality

$$\mathrm{rank}\big(\mathrm{Im}(\mathrm{Tr})\big) = \mathrm{rank}\big(\mathrm{Inv}_G\,\mathrm{Cl}(X_L)\big).$$

Since $p^*\,\mathrm{Cl}(X) \subset \mathrm{Inv}_G\,\mathrm{Cl}(X_L)$, to prove the result it is enough to show the inclusion $\mathrm{Im}(\mathrm{Tr}) \subset p^*\,\mathrm{Cl}(X)$.

Let C' be an irreducible closed subset of X_L of codimension 1. Let $H = \{g \mid g \in G, g(C') = C'\}$ be the stabilizer of C' and let $p(C') = C$. Then we have $\mathrm{Tr}(C') = |H|p^*(C)$. Thus we have $\mathrm{Im}(\mathrm{Tr}) \subset p^*\,\mathrm{Cl}(X) \subset \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$. Therefore, by both of the equalities above, we have

$$\mathrm{rank}\big(\mathrm{Cl}(X)\big) = \mathrm{rank}\,\mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big).$$

LEMMA 3.3. *Let k be a field and let X be a smooth, integral, quasi-projective scheme over k. Let V be an affine open subscheme of X such that* $\mathrm{Cl}(V) = 0$ *and k^* = the group of units in $\Gamma(V)$, the ring of regular functions on V. Let $C_1, \ldots, C_n$ be the irreducible components of the closed set $X - V$. Then the codimension of C_i in X is 1 for $1 \le i \le n$ and* $\mathrm{Cl}(X)$ *is a free abelian group with basis $\{C_1, C_2, \ldots, C_n\}$.*

PROOF. Since X is quasi-projective, integral and V is affine, it is clear that the codimension of C_i in X is 1 for $1 \le i \le n$.

Since $\mathrm{Cl}(V) = 0$, $\mathrm{Cl}(X)$ is generated by $C_1, \ldots, C_n$. So it is enough to show that they are linearly independent.

Suppose $0 = \sum_{i=1}^n n_i C_i$ in $\mathrm{Cl}(X)$, where the n_i are integers. This means that there exists a non zero element f of $k(X)$ (the function field of X) such that $(f) = \sum_{i=1}^n n_i C_i$, where f is the principal divisor defined by f on X. Since $C_i \cap V = 0$ for $1 \le i \le n$, f and $1/f$ are regular on V and therefore $f \in k^*$ by assumption. But then $(f) = 0$. Therefore $n_i = 0$ for $1 \le i \le n$ and we are through.

PROPOSITION 3.4. *Let k be a perfect field and A a regular, factorial, birational subalgebra of $k[Z, T]$. Let L/k be a finite Galois extension. If the Galois group $G = G(L/k)$ is solvable, then $A \otimes_k L$ is factorial.*

PROOF. Let $X = \mathrm{Spec}(A)$ and $\mathbb{A}^2_k = \mathrm{Spec}\,k[Z, T]$. Since A is a birational subring of $k[Z, T]$, we obtain a birational morphism $f\colon \mathbb{A}^2_k \longrightarrow X$. Then by Lemma 3.1 (and well known results on "Resolution of Singularities of Surfaces") it is clear that there exists a sequence of monoidal transformations

$$X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X$$

and a morphism $g\colon \mathbb{A}^2_k \longrightarrow X_n$ such that g is an open immersion and $\pi_1 \circ \pi_2 \circ \cdots \pi_n \circ g = f$.

Put $Y = X_n$ and $\pi = \pi_1 \circ \pi_2 \circ \cdots \pi_n$. Then $\pi \circ g = f$ and hence we get a commutative triangle

$$\begin{array}{ccc} \mathbb{A}^2_L & \xrightarrow{g_L} & Y_L \\ & {\scriptstyle f_L}\searrow \quad \swarrow{\scriptstyle \pi_L} & \\ & X_L & \end{array}$$

with the following properties:

(1) g_L is an open immersion and $g_L(\mathbb{A}^2_L) = V_L$ where $g(\mathbb{A}^2_k) = V$.

Let $p: Y_L \longrightarrow Y$ denote the canonical map.

(2) Let C' be an irreducible closed subset of Y_L of codimension 1. Then C' is an irreducible component of $Y_L - V_L$ if and only if $p(C')$ is an irreducible component of $Y - V$.

(3) Let E' be an irreducible closed subset of Y_L of codimension 1. Then $\pi_L(E') =$ is a (closed) point if and only if $(\pi \circ p)(E') =$ is a (closed) point.

It is easy to establish properties (1), (2) and (3) (with the help of Lemma 3.1) and these will not be proved.

Let S be the set of all irreducible components of $Y_L - V_L$. Then since $V_L \simeq \mathbb{A}^2_L$, by Lemma 3.3, $\mathrm{Cl}(Y_L)$ is a free abelian group with S as a basis. Moreover by property (2) it follows that $\mathrm{Cl}(Y_L)$ is a permutation $\mathbf{Z}[G]$-module with S as a permutable basis.

Let T be the set of all irreducible closed subsets E' of Y_L such that $\pi_L(E')$ is a point. Then by property (3) it follows that G permutes the elements of T. Moreover, as Y is obtained from X by a sequence of monoidal transformations, it follows by Lemma 3.1 that the subgroup M of $\mathrm{Cl}(Y_L)$ generated by the elements of T is a free abelian group with basis T. Thus M is a permutation $\mathbf{Z}[G]$-module. Furthermore $\mathrm{Cl}(Y_L) = \mathrm{Cl}(X_L) \oplus M$ as $\mathbf{Z}[G]$-modules.

Since A is factorial, $\mathrm{Cl}(X) = 0$. Hence by Lemma 3.2, as $\mathrm{Cl}(X_L)$ is a free abelian group (being a direct summand of the permutation module $\mathrm{Cl}(Y_L)$), we have $\mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big) = 0$. Therefore, as G is solvable, by Proposition 2.2 we have $\mathrm{Cl}(X_L) = 0$, showing that $A \otimes_k L$ is factorial.

Let A be as in Proposition 3.4. Then there exists a finite Galois extension L/k such that, in the notation of the proof of Proposition 3.4, all fundamental points of π_L are rational over L (equivalently, all exceptional curves in Y_L are absolutely irreducible) and all irreducible components of $Y_L - \mathbb{A}^2_L$ are absolutely irreducible. Then $\mathrm{Aut}(\bar{k}/L)$ acts trivially on $\mathrm{Cl}(Y_{\bar{k}})$. If $G = G(L/k)$ is solvable, it therefore follows from Proposition 3.4 that $A \otimes_k \bar{k}$ is factorial. We will say that $f: \mathbb{A}^2 \longrightarrow X$ is "split" by L/k.

THEOREM 3.5. *Let k be a perfect field and $f: \mathbb{A}^2_k \longrightarrow X$ a birational morphism, where X is a smooth, factorial, affine surface. If f is "split" by a solvable Galois extension L/k, in particular if* $\mathrm{Gal}(\bar{k}/k)$ *is solvable, then X is isomorphic to $\mathbb{A}^2$ over k.*

PROOF. $X_{\bar{k}}$ is smooth and, by Proposition 3.4 above, factorial. By [F] and [M-S], $X_{\bar{k}} = \mathbb{A}^2_{\bar{k}}$. By the triviality of separable forms of $\mathbb{A}^2_k$ ([K], Theorem 3), $X \simeq \mathbb{A}^2_k$.

4. **Some examples.**

4.1. Let $k = \mathbb{R}$ and $A = \mathbb{R}[x,y,v]/xy - v^2 - 1$. Then A is factorial and $A \subset \mathbb{R}[Z,T]$ with $x = Z^2 + 1, y = 1 + 2ZT + (Z^2+1)T^2, v = Z + (Z^2+1)T$ (see [B-D] Example 4.4 for a more elaborate version). This extension is not birational and one of the starting points of our investigation was the question whether A can be birationally embedded in $\mathbb{R}[Z,T]$. By Theorem 3.5, this is not possible. (Note that $A \otimes_{\mathbb{R}} \mathbb{C}$ is not factorial).

4.2 . Let k be a field of characteristic 0, algebraically closed to fix the ideas. We are interested in affine, regular factorial k-algebras B such that

$$k[X] \subset B \subset k[X, Z, T]$$

and the extension $k[X, Z, T]/B$ is birational. As an example consider $B = k[x, v, t, s]$ with $st - xv = 1$. Then B is as above with $X = x$, $Z = \frac{s-1}{x}$, $T = \frac{t-1}{x}$. B is not polynomial over k, but $B \otimes_{k[x]} k(x)$ is over $k(x)$. Should Proposition 2.2 be true even for non-solvable G, we would know that this holds in general for B as above. Under the assumption that B is itself polynomial over k, we would have proved that X is "generically" a variable in B. It is of course much conjectured, but not yet proved, that then X is in fact a variable in B.

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