Proc. Indian Acad. Sci. (Math. Sci.), Vol. 98, Nos 2 and 3, December 1988, pp. 109–116.


# Generalized epimorphism theorem

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MS received 12 January 1988; revised 24 March 1988

**Abstract.** Let $R[X, Y]$ be a polynomial ring in two variables over a commutative ring $R$ and let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ (a polynomial ring in one variable). In this set-up we prove that $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$ if either $R$ contains a field of characteristic zero or $R$ is a seminormal domain of characteristic zero.

**Keywords.** Epimorphism theorem; polynomial ring; seminormal domain; characteristic zero.

## 1. Introduction

Let $k$ be a field of characteristic zero. Let $k[X, Y]$ be a polynomial ring in two variables over $k$ and $F \in k[X, Y]$ such that $k[X, Y]/(F) = k[Z]$ (a polynomial ring in one variable). In this set-up the famous epimorphism theorem of Abhyankar and Moh ([2], Theorem 1.2) says that $k[X, Y] = k[F, G]$ for some $G \in k[X, Y]$. Russell and Sathaye had obtained the following analog of the epimorphism theorem ([6], Theorem 2.6.2): If $R$ is a locally factorial Krull domain of characteristic zero and $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$, then $R[X, Y] = R[F, G]$. Therefore one asks the following natural question:

*Is the foregoing result valid for an arbitrary commutative domain R of characteristic zero?*

In this paper we answer this question affirmatively under the assumption that $R$ is seminormal. We prove:

**Theorem A.** *Let $R$ be a seminormal commutative domain of characteristic zero. Let $I$ be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$. Then $I$ is a principal ideal say generated by $F$ and $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.*

Moreover we give an example (Example 3.8) to show that $I$ need not be principal if $R$ is not seminormal. When $R$ contains a field of characteristic zero we prove the following (weaker) epimorphism theorem:

**Theorem B.** *Let $R$ be a commutative ring containing a field of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$. Then $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.*

## 2. Preliminaries

Throughout this paper all rings will be commutative.

In this section we set up notations and state some results for later use.

$R$ will denote a commutative ring.

$R^{[n]}$: polynomial ring in $n$ variables over $R$.

$R^n$: free $R$-module of rank $n$.

For a finitely generated $R$-algebra $A$,

$\Omega_{A/R}$: universal module of $R$-differentials of $A$.

For a prime ideal $\mathfrak{P}$ of $R$,

$k(\mathfrak{P})$: $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$.

DEFINITION

A reduced ring $R$ is said to be *seminormal* if it satisfies the condition: for $b$, $c\in R$ with $b^3=c^2$, there is an $a\in R$ with $a^2=b$, $a^3=c$.

*Lemma* (2.1). *Let $R$ be a noetherian ring and let $s\in R$ be a non-zero divisor. Let $M$ be a finitely generated $R$-module. If $M_s$ is a projective $R_s$-module of rank $d$ and $M/sM$ is $R/sR$-projective of rank $d$ then $M$ is $R$-projective of rank $d$.*

*Proof.* Without loss of generality we can assume that $R$ is local.

Since $M/sM$ is $R/s$-projective and $R$ is local there exists a surjective $R$-linear map $\beta$: $R^d\twoheadrightarrow M$ ($d=$ rank $M/sM$). Let $N=\ker\beta$. Since $M_s$ is $R_s$-projective of rank $d$ and $\beta$ is surjective we get $N_s=0$. But $s$ is a non-zero-divisor of $R$ and $N\subset R^d$, therefore $N_s=0\Rightarrow N=0$ and $\beta$ is an isomorphism.

*Lemma* (2.2). *Let $R$ be a noetherian ring and $I$ be an ideal of $R^{[n]}$ such that $R^{[n]}/I\approx R^{[n-1]}$ as $R$-algebras. Then for an ideal $\mathfrak{G}$ of $R$, $I\cap\mathfrak{G}R^{[n]}=\mathfrak{G}I$. Moreover if $I$ is a principal ideal of $R^{[n]}$ say generated by $F$, then*

(i) *$F$ is a non-zero-divisor of $R^{[n]}$.*

(ii) *$F$ is a algebraically independent over $R$*, i.e. $R[F]\approx R^{[1]}$.

(iii) $R[F]\cap\mathfrak{G}R^{[n]}=\mathfrak{G}R[F]$ *for any ideal* $\mathfrak{G}$ of $R$.

*Proof.* Since for any non-negative integer $l$, $R^{[l]}$ is a free $R$-module, the exact sequence $0\to I\to R^{[n]}\xrightarrow{\alpha}R^{[n-1]}\to 0$; $\alpha$: $R$-algebra homomorphism of $R$-modules gives rise to the exact sequence

$$0\to I\bigotimes_R R/\mathfrak{G}\to R^{[n]}\bigotimes_R R/\mathfrak{G}\xrightarrow{\alpha\otimes_R 1_{R/\mathfrak{G}}}R^{[n-1]}\bigotimes_R R/\mathfrak{G}\to 0$$

proving that the canonical map $I/\mathfrak{G}I\to I+\mathfrak{G}R^{[n]}/\mathfrak{G}R^{[n]}$ of $R/\mathfrak{G}^{[n]}$-modules is an isomorphism. Hence $I\cap\mathfrak{G}R^{[n]}=\mathfrak{G}I$.

Now we assume that $I=(F)$.

(i) It is easy to see that $F\notin\mathfrak{m}R^{[n]}$ for any maximal ideal $\mathfrak{m}$ of $R$. This shows that $F$ is a non-zero-divisor of $R^{[n]}$.

(ii) Suppose $c_0+c_1F+\cdots+c_rF^r=0$ where $c_i\in R\,\forall i$, $0\leqslant i\leqslant r$. Then $0=\alpha(c_0+c_1F+\cdots+c_rF^r)=c_0$ i.e. $F(c_1+c_2F+\cdots+c_rF^{r-1})=0$. Therefore, as by (i) $F$ is a non-zero-divisor, $c_1 + c_2F + \cdots c_rF^{r-1} = 0$ showing that $c_1 = 0$. Repeating this argument we see that $c_i = 0 \forall i, 0 \leqslant i \leqslant r$.

(iii) Let $\bar{F}$ be the image of $F$ in $R/\mathfrak{G}^{[n]}$ $(= R^{[n]}/\mathfrak{G}R^{[n]})$. Then obviously $R/\mathfrak{G}^{[n]}/(\bar{F}) \approx R/\mathfrak{G}^{[n-1]}$. Therefore by (ii) $\bar{F}$ is algebraically independent over $R/\mathfrak{G}$ and hence $R[F] \cap \mathfrak{G}R^{[n]} = \mathfrak{G}R[F]$.

*Lemma* (2.3). *Let $R$ be a noetherian ring and let $\mathfrak{G}$ be the nilradical of $R$. Let $I$ be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as $R$-algebras. If $I/\mathfrak{G}I$ is a projective $R/\mathfrak{G}^{[n]}$-module of (constant) rank 1 then $I$ is a projective $R^{[n]}$-module of (constant) rank 1.*

*Proof.* Since $\mathfrak{G}$ is nilpotent, the canonical map $\mathrm{Pic}(R^{[n]}) \to \mathrm{Pic}(R/\mathfrak{G}^{[n]})$ is an isomorphism. Therefore there exists a projective $R^{[n]}$-module $L$ of constant rank 1 such that $L/\mathfrak{G}L \approx I/\mathfrak{G}I$. Hence there exists a $R^{[n]}$-linear map $\psi: L \to I$ such that the induced map $\bar{\psi}: L/\mathfrak{G}L \to I/\mathfrak{G}I$ is an isomorphism.

We claim that $\psi$ is an isomorphism.

*Surjectivity of* $\psi$: Since $\bar{\psi}$ is an isomorphism, we have $I = \psi(L) + \mathfrak{G}I$. But $\mathfrak{G}$ is nilpotent and hence $I = \psi(L)$.

*Injectivity of* $\psi$: Let $M =$ ' ɜr $\psi$. Then we get the following exact sequence of $R^{[n]}$-modules:

$$0 \to M \to L \xrightarrow{\psi} I \to 0.$$

As in Lemma 2.2 we see that $I$ is a projective $R$-module. Therefore the above exact sequence gives rise to the following exact se^uence:

$$0 \to M/\mathfrak{G}M \to L/\mathfrak{G}L \xrightarrow{\bar{\psi}} I/\mathfrak{G}I \to 0.$$

But $\bar{\psi}$ is an isomorphism. Therefore $M/\mathfrak{G}M = 0$ i.e. $M = \mathfrak{G}M$. The nilpotency of $\mathfrak{G}$ shows that $M = 0$.

Thus $\psi$ is an isomorphism.

## 3. Main theorems

In this section we prove Theorem A and Theorem B which are quoted in the introduction. For the proof of these theorems we need some lemmas and a proposition. Lemma 3.1 is well known but for the lack of a proper reference we give a proof.

*Lemma 3.1. Let $R$ be a noetherian ring and $S$ be a noetherian $R$-algebra. Let $\pi \in R$ be such that $S_\pi$ is a flat $R_\pi$-algebra and $S/\pi$ $S$ is a flat $R/\pi$ $R$-algebra. Moreover assume that* $\mathrm{Tor}_1^R(S, R/\pi R) = 0$. *Then $S$ is a flat $R$-algebra.*

*Proof.* Let $M$ and $N$ be finitely generated $R$-modules and let $f: M \to N$ be a $R$-linear injective map. Then we want to show that the map $f \otimes 1_S: M \otimes_R S \to N \otimes_R S$ is injective. Let $K = \ker(f \otimes 1_S)$.

Since $S_\pi$ is $R_\pi$-flat we have $K_\pi = 0$. Let $T = 1 + \pi R$ and $T' = 1 + \pi S$. Then since $\mathrm{Tor}_1^{R_T}(S_{T'}, R_T/\pi R_T) = \mathrm{Tor}_1^R(S, R/\pi R) \otimes_S S_{T'} = 0$ and $R_T/\pi R_T = R/\pi R$, $S_{T'}/\pi S_{T'} = S/\pi S$, by ([1], Theorem 3.2, p. 91) $S_{T'}$ is flat over $R_T$ and hence $K_{T'} = 0$.

Thus $K_{T'} = 0$, $K_\pi = 0$. Therefore $K = 0$ showing that $S$ is a flat $R$-algebra.

*Lemma 3.2. Let $R$ be a noetherian ring of finite Krull dimension (denoted by* $\dim R$*). Let $F$ be an element of $R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Then $R[X, Y]$ is a flat $R[F]$-algebra.*

*Proof.* Let $\mathfrak{G}$ be the nilradical of $R$. Since $R[F]$ ($F$ being algebraically independent over $R$) and $R[X, Y]$ are flat over $R$, for every module $M$ over $R$ we have $\mathrm{Tor}_1^{R[F]}(R[X, Y], M \otimes_R R[F]) = 0$. In particular for every ideal $J$ of $R$ we have $\mathrm{Tor}_1^{R[F]}(R[X, Y], R[F]/JR[F]) = 0$. Therefore, since $\mathfrak{G}$ is nilpotent, by ([1], Theorem 3.2, p. 91) $R[X, Y]$ is flat over $R[F]$ if $R[X, Y]/\mathfrak{G}R[X, Y]$ ($= R/\mathfrak{G}[X, Y]$) is flat over $R[F]/\mathfrak{G}R[F]$. So it is enough to prove the result when $R$ is a reduced ring.

We prove the result by induction on $\dim R$. Without loss of generality we can assume that $R$ is local.

If $\dim R = 0$ then $R$ is a field, $R[F]$ is a principal ideal domain and $R[X, Y]$ is a domain. Therefore $R[X, Y]$ is $R[F]$-flat.

Now we assume that $\dim R > 0$. Let $\pi \in R$ be a nonunit non-zero-divisor of $R$. Let $\bar{F}$ denote the image of $F$ in $R/(\pi)[X, Y]$. Then $R/(\pi)[\bar{F}] = R[F]/\pi R[F]$. Since $\dim R/(\pi) < \dim R$ and $\dim R_\pi < \dim R$, by the induction hypothesis $R_\pi[X, Y]$ is flat over $R_\pi[F]$ and $R[X, Y]/\pi R[X, Y]$ is flat over $R[F]/\pi R[F]$. Moreover $\mathrm{Tor}_1^{R[F]}(R[X, Y], R[F]/\pi R[F]) = 0$. Therefore by Lemma 3.1 $R[X, Y]$ is a flat $R[F]$-algebra.

Thus the proof of Lemma 3.2 is complete.

We state a definition before stating the next lemma.

DEFINITION

An element $F$ of $R[X, Y]$ is called a *residual variable* if for every prime ideal $\mathfrak{Y}$ of $R$, $k(\mathfrak{Y})[X, Y] = k(\mathfrak{Y})[\bar{F}]^{[1]}$ where $\bar{F}$ denotes the image of $F$ in $k(\mathfrak{Y})[X, Y]$.

*Lemma 3.3. Let $R$ be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Assume that $F$ is a residual variable. Then for every prime ideal $\mathfrak{Y}'$ of $R[F]$, $k(\mathfrak{Y}') \otimes_{R[F]} R[X, Y] = k(\mathfrak{Y}')^{[1]}$.*

*Proof.* Let $\mathfrak{Y}' \cap R = \mathfrak{Y}$. Then $\mathfrak{Y}R[F] \subset \mathfrak{Y}'$ and by (2.2) $\mathfrak{Y}R[F] = \mathfrak{Y}R[X, Y] \cap R[F]$. Since $F$ is a residual variable, we have $k(\mathfrak{Y})[X, Y] = k(\mathfrak{Y})[\bar{F}]^{[1]}$ where $\bar{F}$ denotes the image of $F$ in $k(\mathfrak{Y})[X, Y]$. Moreover $k(\mathfrak{Y})[\bar{F}] = k(\mathfrak{Y}) \otimes_R R[F]$ and there exists a $R[F]$-algebra homomorphism $k(\mathfrak{Y})[\bar{F}] \to k(\mathfrak{Y}')$. Therefore

$$\begin{aligned} k(\mathfrak{Y}') \underset{R[F]}{\otimes} R[X, Y] &= k(\mathfrak{Y}') \underset{k(\mathfrak{Y})[\bar{F}]}{\otimes} k(\mathfrak{Y})[\bar{F}] \underset{R[F]}{\otimes} R[X, Y] \\ &= k(\mathfrak{Y}')^{[1]}. \end{aligned}$$

*Lemma 3.4. Let $R$ be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Assume that $F$ is a residual variable. Then $\Omega_{R[X,Y]/R[F]}$ is a free $R[X, Y]$-module of rank one.*

*Proof*. We have the following right exact sequence of $R[X, Y]$-modules

$$\Omega_{R[F]/R} \underset{R[F]}{\otimes} R[X, Y] \xrightarrow{\theta} \Omega_{R[X,Y]/R} \rightarrow \Omega_{R[X,Y]/R[F]} \rightarrow 0.$$

Since $\Omega_{R[X,Y]/R}$ is a free $R[X, Y]$-module of rank two with a basis $dX, dY$ and $\mathrm{Im}(\theta) = N$ is the cyclic submodule generated by $F_X dX + F_Y dY$ where $F_X = \partial F/\partial X$ and $F_Y = \partial F/\partial Y$, it is enough to show that the ideal $(F_X, F_Y) = R[X, Y]$.

Suppose $\mathfrak{N}$ is a maximal ideal of $R[X, Y]$ such that $(F_X, F_Y) \subset \mathfrak{N}$. Let $\mathfrak{N} \cap R = \mathfrak{Y}$. Then replacing $R$ by $R_{\mathfrak{Y}}$ and $\mathfrak{N}$ by $\mathfrak{N}_{\mathfrak{Y}}$ we can assume that $R$ is a local ring with the maximal ideal $\mathfrak{m}$, $\mathfrak{N}$ is a maximal ideal of $R[X, Y]$ with $\mathfrak{N} \cap R = \mathfrak{m}$ and $(F_X, F_Y) \subset \mathfrak{N}$. But then, since $F$ is a residual variable, we have $R[X, Y] = (F_X, F_Y) + \mathfrak{m}R[X, Y] \subset \mathfrak{N}$ which is absurd. Hence $(F_X, F_Y) = R[X, Y]$.

*Lemma 3.5. Let R be a noetherian ring such that no prime integer is a zero-divisor in R. Let $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then F is a residual variable.*

*Proof.* Let $\mathfrak{Y}$ be a prime ideal of $R$ and let $\bar{F}$ denote the image of $F$ in $k(\mathfrak{Y})[X, Y]$. Then $k(\mathfrak{Y})[X, Y]/(\bar{F}) = k(\mathfrak{Y})[Z]$.

If $ht\ \mathfrak{Y} = 0$, then since no prime integer is a zero-divisor in $R$, $k(\mathfrak{Y})$ is a field of characteristic zero. Therefore by the Abhyankar-Moh epimorphism theorem ([2], Theorem 1.2) $k(\mathfrak{Y})[X, Y] = k(\mathfrak{Y})[\bar{F}]^{[1]}$.

If $ht\ \mathfrak{Y} > 0$ then there exists a discrete valuation ring $V$ of characteristic zero with the uniformizing parameter $\pi$ and a ring homomorphism $\alpha: R \rightarrow V$ such that $\alpha^{-1}((\pi)) = \mathfrak{Y}$ and the field extension $k(\mathfrak{Y}) \rightarrow V/(\pi)$ (induced by $\alpha$) is algebraic.

Let $\tilde{F}$ denote the image (through $\alpha$) of $F$ in $V[X, Y]$. Then $V[X, Y]/(\tilde{F}) = V[Z]$. Therefore by ([6], Theorem 2.6.2) $V[X, Y] = V[\tilde{F}]^{[1]}$ and hence $V/(\pi)[X, Y] = V/(\pi)[\tilde{\tilde{F}}]^{[1]}$ where $\tilde{\tilde{F}}$ is the image of $\tilde{F}$ in $V/(\pi)[X, Y]$.

Since we have the following commutative diagram of rings

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & V \\ \downarrow & & \downarrow \\ k(\mathfrak{Y}) & \longrightarrow & V/(\pi) \end{array}$$

and $V/(\pi)$ is algebraic over $k(\mathfrak{Y})$, by ([4], Proposition 1.16) $k(\mathfrak{Y})[X, Y] = k(\mathfrak{Y})[\bar{F}]^{[1]}$.

Thus we prove that $F$ is a residual variable.

PROPOSITION 3.6.

*Let R be a ring and I be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras. Then I is a projective $R^{[n]}$-module of (constant) rank 1. Moreover if there exists a projective R-module L of rank 1 such that $L \otimes_R R^{[n]} \approx I$ as $R^{[n]}$-modules then I is a free $R^{[n]}$-module of rank 1 i.e. I is a principal ideal (necessarily generated by a non-zero-divisor of $R^{[n]}$).*

*Proof.* It is easy to see that under the hypothesis of the proposition there exists a subring $R'$ of $R$ which is finitely generated over the ring of integers and an ideal $I'$ of $R'^{[n]}$ such that $R'^{[n]}/I' \approx R'^{[n-1]}$ and $I = I'R^{[n]} \approx I' \otimes_{R'} R = I' \otimes_{R'^{[n]}} R^{[n]}$. Therefore for proving the first part of the proposition we can assume without loss of generality that $R$ is noetherian of finite Krull dimension.

We prove the result by induction on dim $R$.

Let dim $R = 0$. By Lemma 2.3 we can assume that $R$ is reduced. But then $R$ is a finite product of fields and hence, since $R^{[n]}/I \approx R^{[n-1]}$, $I$ is a principal ideal (of height 1) generated by a non-zero-divisor. Therefore $I$ is a free $R^{[n]}$-module of rank 1.

Now we assume that dim $R > 0$. Again by Lemma 2.3 we can assume that $R$ is reduced. Let $S$ be the set of non-zero-divisors of $R$. Then $R_S$ is a finite product of fields and as before we conclude that $I_S$ is a free $R_S^{[n]}$-module of rank 1. Therefore $\exists\, s \in S$ such that $I_s$ is a free $R_s^{[n]}$-module of rank 1. We may assume that $s$ is a nonunit of $R$.

Since $I \cap sR^{[n]} = sI$, $I/sI \approx I + sR^{[n]}/sR^{[n]}$ as $R/(s)^{[n]}$-modules. Therefore, since $R^{[n]}/(I + sR^{[n]}) \approx R/(s)^{[n-1]}$ and $\dim R/(s) < \dim R$, by the induction hypothesis $I/sI$ is a projective $R/(s)^{[n]}$-module. Since $s$ is a non-zero-divisor of $R$, $I \subset R^{[n]}$ and $I_s$ (resp. $I/sI$) is a projective $R_s^{[n]}$-module (resp. $R/(s)^{[n]}$-module) of rank 1, by Lemma 2.1 $I$ is a projective $R^{[n]}$-module of (constant) rank 1.

Now assume that there exists a projective $R$-module $L$ of rank 1 such that $L \otimes_R R^{[n]} \approx I$ as $R^{[n]}$-modules.

Since $R^{[n]}/I \approx R^{[n-1]}$ as $R$-algebras, we get the following right exact sequence of $R^{[n-1]}$-modules:

$$I/I^2 \to \Omega_{R^{[n]}/R}/I\Omega_{R^{[n]}/R} \to \Omega_{R^{[n-1]}/R} \to 0.$$

Since, for non-negative integer $l$, $\Omega_{R^{[l]}/R}$ is a free $R^{[l]}$-module of rank $l$ and $I/I^2$ is a projective $R^{[n-1]}$-module (as $I$ is projective over $R^{[n]}$ of rank 1) of rank 1 we see that the above sequence is also left exact and

$$\Omega_{R^{[n]}/R}/I\Omega_{R^{[n]}/R} \approx \Omega_{R^{[n-1]}/R} \oplus I/I^2.$$

Thus $I/I^2$ is a stably free $R^{[n-1]}$-module of rank 1 and therefore $I/I^2$ is free over $R^{[n-1]}$ of rank 1.

Let $\theta$: $R^{[n-1]} \to R$ be a surjective $R$-algebra homomorphism. Then composite map

$$R \to R^{[n]} \twoheadrightarrow R^{[n]}/I \approx R^{[n-1]} \overset{\theta}{\twoheadrightarrow} R$$

is the identity automorphism of $R$.

Since $L \otimes_R R^{[n]} \approx I$, we get

$$\begin{aligned} L &= L \bigotimes_R R^{[n]} \bigotimes_{R^{[n]}} R^{[n-1]} \bigotimes_{R^{[n-1]}} R \approx I \bigotimes_{R^{[n]}} R^{[n-1]} \bigotimes_{R^{[n-1]}} R \\ &= I/I^2 \bigotimes_{R^{[n-1]}} R. \end{aligned}$$

But $I/I^2$ is a free $R^{[n-1]}$-module of rank 1. Hence $L$ is a free $R$-module of rank 1 and therefore $I$ is a free $R^{[n]}$-module of rank 1 i.e. $I$ is a principal ideal.

Thus the proof of Proposition 3.6 is complete.

Now we prove Theorem A.

**Theorem 3.7.** *Let $R$ be a ring such that $R_{\text{red}}$ is seminormal and no prime integer is a zero-divisor in $R_{\text{red}}$. Let $I$ be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$ (as $R$-algebras). Then $I$ is a principal ideal say generated by $F$ and $R[X, Y] = R[F]^{[1]}$.*

*Proof.* Since $R_{red}$ is seminormal by ([7], Theorem 6.1) $\mathrm{Pic}(R) = \mathrm{Pic}(R^{[n]})$ for every $n$. Therefore by Proposition 3.6 $I$ is a principal ideal say generated by $F$.

Let $\mathfrak{G}$ be the nilradical of $R$ and let $\bar{F}$ be the image of $F$ in $R/\mathfrak{G}[X, Y]$. If $R/\mathfrak{G}[X, Y] = R/\mathfrak{G}[\bar{F}]^{[1]}$ then it is easy to see that $R[X, Y] = R[F]^{[1]}$. Therefore we can assume that $R$ is reduced. It is also easy to see that there exists a subring $S$ of $R$ which is finitely generated over the ring of integers such that $F \in S[X, Y]$ and $S[X, Y]/(F) = S[Z]$ as $S$-algebras. Note that $S$ is a noetherian ring of finite Krull dimension.

Since $S \hookrightarrow R$ and $R$ is reduced, by the hypothesis of the theorem, no prime integer is a zero-divisor in $S$. Therefore $F$ is a residual variable in $S[X, Y]$ by Lemma 3.5. Hence $\Omega_{S[X,Y]/S[F]}$ is a free $S[X, Y]$-module of rank one by Lemma 3.4. Moreover by Lemma 3.3, for every prime ideal $\mathfrak{Y}'$ of $S[F]$, $k(\mathfrak{Y}') \otimes_{S[F]} S[X, Y] = k(\mathfrak{Y}')^{[1]}$. $S[X, Y]$ is a (finitely generated) flat $S[F]$-algebra by Lemma 3.2. Therefore by ([3], Lemma 3.3) there exists a positive integer $m$ such that $S[X, Y]^{[m]} = S[F]^{[m+1]}$.

Now $S[X, Y]^{[m]} = S[F]^{[m+1]}$ implies that $R[X, Y]^{[m]} = R[F]^{[m+1]}$. Since $R$ is seminormal (we have assumed $R$ to be reduced) by ([5], Theorem 2.6) $R[X, Y] = R[F]^{[1]}$.

Thus the proof of Theorem 3.7 is complete.

The following example shows that if $R_{red}$ is not seminormal then $R[X, Y]/I = R[Z]$ need not imply that $I$ is principal.

*Example 3.8.* Let $k$ be a field of characteristic zero and let $\tilde{R} = k[[t]]$: a power series in one variable over $k$. Let $R = k[[t^2, t^3]]$, considered as a subring of $\tilde{R}$. It is obvious that $\tilde{R}$ is the normalization of $R$ and $R$ is not seminormal.

Let $\alpha$: $R[X, Y] \to R[Z]$ be the $R$-algebra homomorphism defined as: $\alpha(X) = Z + t^3Z^2$ and $\alpha(Y) = t^2Z$. Let $I = \ker \alpha$. Then

(1) $\alpha$ is surjective
(2) $I$ is not a principal ideal of $R[X, Y]$.

*Proof.* Since $\alpha(X - t^3X^2 + t^2XY^2 + Y^3) = Z$, $\alpha$ is surjective.

Let $\tilde{\alpha}$: $\tilde{R}[X, Y] \to \tilde{R}[Z]$ be the $\tilde{R}$-algebra homomorphism such that $\tilde{\alpha}(X) = \alpha(X) = Z + t^3Z^2$ and $\tilde{\alpha}(Y) = \alpha(Y) = t^2Z$. Let $\tilde{I} = \ker \tilde{\alpha}$. Then $\tilde{I}$ is a principal prime ideal of $\tilde{R}[X, Y]$ generated by $F(X, Y) = t^2X - Y - tY^2$. Moreover $\tilde{I} = I\tilde{R}[X, Y] (= I \otimes_R \tilde{R})$.

If $I$ is a principal ideal of $R[X, Y]$ say generated by $H$ then $H = uF$ $(= u(t^2X - Y - tY^2))$ where $u$ is a unit in $R$ and $ut \in R$ i.e. $t \in R$ which is a contradiction.

Thus we prove that $I$ cannot be principal.

We conclude this section with the proof of Theorem B.

**Theorem 3.9.** *Let $R$ be a ring containing a field $k$ of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Then $R[X, Y] = R[F]^{[1]}$.*

*Proof.* As in Theorem 3.7, we can assume that $R$ is reduced and $R$ contains a noetherian subring $S$ of finite Krull dimension such that $F \in S[X, Y]$ and $S[X, Y]/(F) = S[Z]$ as $S$-algebras. Moreover we can assume that $S$ contains $k$. Repeating the same arguments we see that there exists a positive integer $m$ such that $S[X, Y]^{[m]} = S[F]^{m+1}$. Now since $S$ contains $k$ (a field of characteristic zero) by ([5], Theorem 2.8), $S[X, Y] = S[F]^{[1]}$. Hence $R[X, Y] = R[F]^{[1]}$.