ON THE GLOBAL DIMENSION OF SOME FILTERED ALGEBRAS

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Introduction

Let k be a commutative field of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra over k. Let f be a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}$. We set $\mathfrak{g}(f) = T(\mathfrak{g})/U_f(\mathfrak{g})$, where $T(\mathfrak{g})$ denotes the tensor algebra of the vector space $\mathfrak{g}$ and $U_f(\mathfrak{g})$ the two-sided ideal of $T(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y] - f(x, y)$ for $x, y \in \mathfrak{g}$. It is known **[15]** that $\mathfrak{g}(f)$ is a filtered k-algebra whose associated graded algebra is isomorphic to a polynomial algebra over k and that every filtered k-algebra with this property is isomorphic to one such.

In this paper we determine (§2, Theorem 2.6) the global dimension of $\mathfrak{g}(f)$, where $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over k, and deduce some interesting corollaries. In §1 we prove some results which are used in the proof of the main theorem.

1. We begin with a proposition.

PROPOSITION 1.1. *Let A be a ring which is left and right noetherian. Let $\text{l.gl.dim}\, A < \infty$. Then there exists a simple left A-module M such that $\text{hd}_A M = \text{l.gl.dim}\, A$.*

Proof. Let $\text{l.gl.dim}\, A = 0$. Then every left A-module is projective. Therefore if M is a simple left A-module then $\text{hd}\, M = 0 = \text{l.gl.dim}\, A$. If $\text{l.gl.dim}\, A \neq 0$, then, from **[1**; Theorem 1] it follows that

$$\text{l.gl.dim}\, A = 1 + \sup_I \text{hd}\, I,$$

where I ranges over all left ideals of A. Since $\text{l.gl.dim}\, A < \infty$, there exists a left ideal I such that $\text{l.gl.dim}\, A = 1 + \text{hd}\, I$. Since A is left noetherian, there exists a left ideal I_0 such that $\text{l.gl.dim}\, A = 1 + \text{hd}\, I_0$ and I_0 is a maximal left ideal with respect to this property. If I_0 is a maximal left ideal of A, then by taking $M_0 = A/I_0$ we get a simple module M_0 of A such that $\text{hd}\, M_0 = 1 + \text{hd}\, I_0 = \text{l.gl.dim}\, A$.

If I_0 is not a maximal left ideal of A, then the family $(I_j)_{j \in J}$ of all left ideals of A which contain I_0 properly is not empty. Let $I' = \bigcap_{j \in J} I_j$. We assert that $I_0 \neq I'$. For otherwise we get a short exact sequence

$$0 \to M_0 \xrightarrow{\phi} \prod_{j \in J} M_j \xrightarrow{\psi} M' \to 0$$

of left A-modules, where $M_j = A/I_j$ for every $j \in J$ and $M' = \text{coker}\, \phi$.

Since I_j contains I_0 properly, by choice of I_0 we have

$$\text{hd}\, M_j = 1 + \text{hd}\, I_j < 1 + \text{hd}\, I_0 = \text{hd}\, M_0 = \text{l.gl.dim}\, A.$$

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For a left A-module N, let $\mathrm{w.dim}\, N$ denote the weak dimension of N. Since A is left and right noetherian, if N is finitely generated then $\mathrm{hd}\, N = \mathrm{w.dim}\, N$ [**4**; Chapter VI]. Therefore we display [**3**; p. 69]

$$\mathrm{w.dim} \prod_{j \in J} M_j = \sup_{j \in J} \mathrm{w.dim}\, M_j = \sup_{j \in J} \mathrm{hd}\, M_j < \mathrm{hd}\, M_0 = \mathrm{w.dim}\, M_0.$$

Therefore $\mathrm{w.dim}\, M' = 1 + \mathrm{w.dim}\, M_0 = 1 + \mathrm{hd}\, M_0 = 1 + \mathrm{l.gl.dim}\, A > \mathrm{l.gl.dim}\, A$, which is impossible. Therefore $I_0 \neq I'$ (i.e. I' contains I_0 properly). From the construction of I' it follows that $M = I'/I_0$ is a simple left A-module. Consider the short exact sequence

$$0 \to I'/I_0 \to A/I_0 \to A/I' \to 0.$$

Since

$$\mathrm{hd}\, A/I' = 1 + \mathrm{hd}\, I' < 1 + \mathrm{hd}\, I_0 = \mathrm{hd}\, A/I_0,$$

we get

$$\mathrm{hd}\, I'/I_0 = \mathrm{hd}\, A/I_0 = \mathrm{l.gl.dim}\, A.$$

Hence the result follows.

This completes the proof of Proposition 1.1.

Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field k of characteristic 0. Let f be a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}$ [**15**; p. 532]. Let θ be an element of $\mathrm{Hom}_k(\mathfrak{g}, k)$.

Definition. A subalgebra h of $\mathfrak{g}$ is said to be *f-subordinate to θ* if for every $h_1, h_2 \in h$ we have $\theta[h_1, h_2] + f(h_1, h_2) = 0$.

Remark. From the definition it follows that if h is a subalgebra of $\mathfrak{g}$ then the restriction of f to $h \times h$ is a coboundary if and only if there exists $\theta \in \mathrm{Hom}_k(\mathfrak{g}, k)$ such that h is f-subordinate to θ. Therefore if a subalgebra h is f-subordinate to θ then $h(f)$ is isomorphic to $h(0)$ [**15**; Theorem 3.1]. But $h(0)$ is nothing but the usual enveloping algebra of the Lie algebra h. Therefore $\mathrm{l.gl.dim}\, h(0) = \dim_k h$ [**4**; p. 283, Theorem 8.2]. Moreover the map $\theta : h \to k$ defines an $h(f)$-module structure denoted by $k(\theta, h)$ on k such that $\mathrm{hd}\, k(\theta, h) = \dim_k h = \mathrm{l.gl.dim}\, h(f)$. Since $\mathfrak{g}(f)$ is $h(f)$-free as a right as well as a left module and contains $h(f)$ as a direct summand, from [**10**; Lemma 1] it follows that

$$\mathrm{l.gl.dim}\, \mathfrak{g}(f) \geqslant \mathrm{hd}_{\mathfrak{g}(f)} \mathfrak{g}(f) \underset{h(f)}{\otimes} k(\theta, h) = \mathrm{hd}_{h(f)} k(\theta, h) = \dim_k h.$$

On the other hand from [**14**; Theorem 1] we get $\dim_k \mathfrak{g} \geqslant \mathrm{l.gl.dim}\, \mathfrak{g}(f)$. Therefore we always have inequality $\dim_k \mathfrak{g} \geqslant \mathrm{l.gl.dim}\, \mathfrak{g}(f) \geqslant \dim_k h$ for a subalgebra h of $\mathfrak{g}$ for which the restriction of f to $h \times h$ is a coboundary.

Definition. A two-sided ideal I of $\mathfrak{g}(f)$ is said to be *rational* if the centre of $\mathfrak{g}(f)/I$ is k.

The following proposition has been proved by J. Dixmier in [**6**; Theorem 3] for the usual enveloping algebra. Our proof in the general case is an easy adaptation of his proof.

PROPOSITION 1.2. *Let k be a field of characteristic* 0. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over k. Let f be a k-valued* 2-*cocycle on the "standard complex" for $\mathfrak{g}$. Let I be a rational ideal of $\mathfrak{g}(f)$. Then there exists an element $\theta \in \mathrm{Hom}_k(\mathfrak{g}, k)$*

and a subalgebra h of $\mathfrak{g}$ such that (I) *h is f-subordinate to θ,* (II) $M = \mathfrak{g}(f) \otimes_{h(f)} k(\theta, h)$ *is a simple $\mathfrak{g}(f)$-module,* (III) $\operatorname{ann} M = I$, *where* $\operatorname{ann} M$ *is the set of those elements of $\mathfrak{g}(f)$ which annihilate M.*

Proof. Let $\mathfrak{g}' = \mathfrak{g} \oplus k \cdot z$, where $k \cdot z$ is a one-dimensional Lie algebra over k. Define a Lie algebra structure on $\mathfrak{g}'$ as follows. For $x+\lambda z,\ y+\beta z \in \mathfrak{g}'$,

$$[x+\lambda z,\ y+\beta z] = [x, y]+f(x, y)z.$$

Then it is easy to see that $\mathfrak{g}'$ is a nilpotent Lie algebra over k such that z belongs to the centre of $\mathfrak{g}'$ and $\mathfrak{g}'(0)/(z-1) = \mathfrak{g}(f)$. Extend $\theta : \mathfrak{g} \to k$ to $\theta' : \mathfrak{g}' \to k$ by defining $\theta'(x+\lambda z) = \theta(x)+\lambda$. Then obviously θ' is an element of $\operatorname{Hom}_k(\mathfrak{g}', k)$. Let $h' = h \oplus k \cdot z$. Then since h is f-subordinate to θ, h' is 0-subordinate to θ'. Thus for every subalgebra h of $\mathfrak{g}$ which is f-subordinate to θ there exists a subalgebra h' of $\mathfrak{g}'$ which contains z and which is 0-subordinate to θ'. Conversely one can prove that for every subalgebra h' of $\mathfrak{g}'$ which contains z and which is 0-subordinate to θ' there exists a subalgebra h of $\mathfrak{g}$ such that $h' = h \oplus k \cdot z$ and h is f-subordinate to θ.

Let I' be a rational ideal of $\mathfrak{g}'(0)$ such that $I'/(z-1) = I$. It is proved in [6; Theorem 3] that there exists a subalgebra h' of $\mathfrak{g}'$ and an element θ' of $\operatorname{Hom}_k(\mathfrak{g}', k)$ such that (1) $\theta'(z) = 1$, (2) z' belongs to h', (3) h' is 0-subordinate to θ', (4) $M' = \mathfrak{g}'(0) \otimes_{h'(0)} k(\theta', h')$ is a simple $\mathfrak{g}'(0)$-module, (5) $\operatorname{ann} M' = I'$.

Since $z \in h'$, there exists a subalgebra h of $\mathfrak{g}$ such that $h' = h \oplus k \cdot z$. Let $i : \mathfrak{g} \to \mathfrak{g}'$ be the inclusion map and let $\theta = \theta' \circ i$. Then clearly $\theta \in \operatorname{Hom}_k(\mathfrak{g}, k)$ and h is f-subordinate to θ.

Let $\mathfrak{A}(h, \theta)$ (resp. $\mathfrak{A}(h', \theta')$) be the left ideal of $\mathfrak{g}(f)$ (resp. $\mathfrak{g}'(0)$) generated by all elements of the type $x-\theta(x)$, $x \in h$ (resp. $x-\theta'(x)$, $x \in h'$). Then it is easy to that $\mathfrak{g}(f) \otimes_{h(f)} k(\theta, h)$ (resp. $\mathfrak{g}'(0) \otimes_{h'(0)} k(\theta', h')$) is the cyclic module $\mathfrak{g}(f)/\mathfrak{A}(h, \theta)$ (resp. $\mathfrak{g}'(0)/\mathfrak{A}(h', \theta')$) and $\mathfrak{A}(h, \theta) = \mathfrak{A}(h', \theta')/(z-1)$. Therefore $M = \mathfrak{g}(f) \otimes_{h(f)} k(\theta, h)$ is a simple $\mathfrak{g}(f)$-module and $\operatorname{ann} M = \operatorname{ann} M'/(z-1) = I'/(z-1) = I$.

Thus the proof of Proposition 1.2 is complete.

2. We begin this section with the following theorem.

THEOREM 2.1. *Let k be an algebraically closed field of characteristic* 0. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over k. Let f be a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}$. Then there exists an element θ of* $\operatorname{Hom}_k(\mathfrak{g}, k)$ *and a subalgebra h of $\mathfrak{g}$ such that*

(I) *h is f-subordinate to θ,*

(II) $\operatorname{hd} \mathfrak{g}(f) \underset{h(f)}{\otimes} k(\theta, h) = \operatorname{l.gl.dim} \mathfrak{g}(f)$.

For the proof of this theorem, we require some lemmas. In each of these lemmas k, $\mathfrak{g}$ and f are as in the statement of Theorem 2.1. We define the *f-centre* of $\mathfrak{g}$ denoted by $Z(f)$ to be the subset of $\mathfrak{g}$ consisting of elements x with the property $[x, y] = 0 = f(x, y)$ for every $y \in \mathfrak{g}$. Obviously $Z(f) \subset Z$ where Z is the centre of the Lie algebra $\mathfrak{g}$.

LEMMA 2.2. *If* $\dim_k \mathfrak{g} \geqslant 2$ *and* $Z(f) = 0$ *then there exists a nilpotent Lie algebra* $\mathfrak{g}'$ *with* $\dim_k \mathfrak{g} = \dim_k \mathfrak{g}'+2$ *and 2-cocycle* f' *on* $\mathfrak{g}'$ *such that*

$$\mathfrak{g}(f) \simeq \mathfrak{g}'(f') \underset{k}{\otimes} A_1(k),$$

where $A_1(k)$ *is the Weyl algebra* $k[X_1, \partial/\partial X_1]$ *of index* 1 *with coefficients in* k.

Proof. Since $\mathfrak{g}$ is nilpotent, $Z(f) = 0$ implies that there exists $x \in Z$, $y \in \mathfrak{g}$ such that $f(y, x) = 1$. We can choose a k-basis $(x_1, x_2, \ldots, x_n)$ of $\mathfrak{g}$ such that (I) $x_1 = x$, (II) $x_n = y$, (III) $f(x_1, x_i) = 0$ for $1 \leqslant i \leqslant n-1$. Since for any $z, w \in \mathfrak{g}$ we have

$$f(x, [z, w]) = f(x, [z, w])+f(w, [x, z])+f(z, [w, x]) = 0,$$

we see that the subspace h spanned by $(x_1, x_2, \ldots, x_{n-1})$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}(f)$ is the Ore-extension of $h(f)$ with respect to the locally nilpotent derivation d of $h(f)$ induced by x_n. Since x_1 is an element of the centre of $h(f)$ and $d(x_1) = 1$, by the lemma of [**11**; p. 78] there exists an isomorphism

$$\phi_1 : \mathfrak{g}(f) \to h(f)/(x_1) \underset{k}{\otimes} A_1(k),$$

$$\phi_1(x_i) = \overline{x_i} \otimes 1+\overline{d(x_i)} \otimes X_1+\overline{d^2(x_i)} \otimes \frac{X_1{}^2}{2!} +\ldots \quad \text{for} \quad 1 \leqslant i \leqslant n-1,$$

$$\phi_1(x_n) = 1 \otimes \partial/\partial X_1.$$

Since $f(x_1, y) = 0$ for all $y \in h$, f induces a 2-cocycle f' on $\mathfrak{g}' = h/(x_1)$ such that $h(f)/(x_1) \simeq \mathfrak{g}'(f')$. Since $\dim_k \mathfrak{g}' = \dim_k h/(x_1) = \dim_k h-1 = \dim_k \mathfrak{g}-2$ we get the required result.

We now show that for each positive integer m there exists a nilpotent Lie algebra $\mathfrak{g}_1$ and 2-cocycle f_1 on $\mathfrak{g}_1$ such that

(I) $\dim_k \mathfrak{g}_1 = \dim_k \mathfrak{g}+2m$

(II) $\mathfrak{g}(f) \underset{k}{\otimes} A_m(k) = \mathfrak{g}_1(f_1)$

(III) $Z(f) = Z(f_1)$,

where $A_m(k)$ is the Weyl algebra $k[X_1, X_2, \ldots, X_m, \partial/\partial X_1, \partial/\partial X_2, \ldots, \partial/\partial X_m]$ of index m with coefficients in k.

Let η be an abelian Lie algebra of dim $2m$ over k generated by $X_1, X_2, \ldots, X_m$, $\partial/\partial X_1, \partial/\partial X_2, \ldots, \partial/\partial X_m$. Let f' be the 2-cocycle on η defined by (1) $f'(X_i, X_j) = 0$, (2) $f'(\partial/\partial X_i, \partial/\partial X_j) = 0$, (3) $f'(\partial/\partial X_i, X_j) = \delta_{ij}$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m$. Write $\mathfrak{g}_1 = \mathfrak{g} \oplus \eta, f_1 = f \oplus f'$. Then one can see that $\mathfrak{g}_1$ is a nilpotent Lie algebra over k, f_1 a 2-cocycle on the "standard complex" for $\mathfrak{g}_1$ such that $\mathfrak{g}(f) \otimes_k A_m(k) = \mathfrak{g}_1(f_1)$, $Z(f) = Z(f_1)$ and $\dim_k \mathfrak{g}_1 = \dim_k \mathfrak{g}+2m$.

LEMMA 2.3. *Let* $\mathfrak{g}(f) \otimes_k A_1(k) = \mathfrak{g}_1(f_1)$, *where* $\mathfrak{g}_1$ *and* f_1 *are as defined above. Let* $\theta \in \mathrm{Hom}_k(\mathfrak{g}_1, k)$. *Then there exists a subalgebra* h *of* $\mathfrak{g}_1$ *such that* (I) $h \subset \mathfrak{g} \oplus k \cdot X_1$, (II) $X_1 \in h$, (III) h *is* f_1*-subordinate to* θ *and of maximal dimension with such property.*

Proof. Let h_1 be a subalgebra of $\mathfrak{g}_1$ which is f_1-subordinate to θ and of maximal dimension with such property. Let $h = h_1 \cap \mathfrak{g}'+k \cdot X_1$, where $\mathfrak{g}' = \mathfrak{g} \oplus k \cdot X_1$.

Then $h \subset \mathfrak{g} \oplus k \cdot X_1$. Since h_1 is f_1-subordinate to θ, $h_1 \cap \mathfrak{g}'$ is f_1-subordinate to θ. Since $k \cdot X_1$ is an ideal of $\mathfrak{g}'$ and $f(X_1, y) = 0$ for all $y \in \mathfrak{g}'$, $h_1 \cap \mathfrak{g}'$ being f_1-subordinate to θ implies that h is also f_1-subordinate to θ.

To prove that h is of maximal dimension with such property it is enough to prove that $\dim_k h_1 = \dim_k h$. Since $\mathfrak{g}'$ is an ideal of $\mathfrak{g}_1$ of codimension 1, we prove that $\dim_k h_1 = \dim_k h$ by proving that $h_1 \subset \mathfrak{g}'$ if and only if $X_1 \in h_1$.

If $X_1 \in h_1$, then h_1 being f_1-subordinate to θ implies that, for every $y \in h_1$, $\theta[y, X_1] + f(y, X_1) = 0$. Since $[y, X_1] = 0$ for every $y \in h_1$, $f(y, X_1) = 0$ for every $y \in h_1$. This shows that $h_1 \subset \mathfrak{g}'$.

Conversely, if $h_1 \subset \mathfrak{g}'$ then $h_1 = h_1 \cap \mathfrak{g}' \subset h$. Since both h and h_1 are f_1-subordinate to θ, h_1 being of maximal dimension with such property implies that $h_1 = h$, i.e. $X_1 \in h_1$.

Hence the result follows.

LEMMA 2.4. *Suppose* $\dim_k \mathfrak{g} \geqslant 2$ *and* $Z(f) = 0$. *Let*

$$\phi_1 : \mathfrak{g}(f) \to h(f)/(x_1) \underset{k}{\otimes} A_1(k)(=\mathfrak{g}_1(f_1))$$

be the isomorphism as defined in Lemma 2.2. *Let* h_1 *a subalgebra of* $\mathfrak{g}_1$, *and* $\theta_1 \in \mathrm{Hom}_k(\mathfrak{g}_1, k)$ *be such that*

(I) h_1 *is* f_1*-subordinate to* θ_1,

(II) $\mathrm{hd}\, \mathfrak{g}_1(f_1) \underset{h_1(f_1)}{\otimes} k(\theta_1, h_1) = 1.\mathrm{gl.dim}\, \mathfrak{g}_1(f_1)$.

Then there exists a subalgebra h' *of* $\mathfrak{g}$ *and an element* θ *of* $\mathrm{Hom}_k(\mathfrak{g}, k)$ *such that*

(I) h' *is* f*-subordinate to* θ,

(II) $\mathrm{hd}\, \mathfrak{g}(f) \underset{h'(f)}{\otimes} k(\theta, h') = 1.\mathrm{gl.dim}\, \mathfrak{g}(f)$.

Proof. From our earlier remark, it follows that $1.\mathrm{gl.dim}\, \mathfrak{g}_1(f_1) = \dim_k h_1$, and h_1 is f_1-subordinate to θ_1 and of maximal dimension with such property. Therefore, from Lemma 2.3 it follows that we can assume h_1 to be contained in $h/(x_1) \oplus k \cdot X_1$. This implies that $X_1 \in h_1$. Let $h_1 = h'' \oplus k \cdot X_1$, where h'' is a subalgebra of $h/(x_1)$. If $\theta_1(X_1) = \lambda \neq 0$, then let $\theta' \in \mathrm{Hom}_k(\mathfrak{g}_1, k)$ be such that $\theta_1|h'' = \theta'|h''$ and $\theta'(X_1) = 0$. h_1 is obviously f_1-subordinate to θ'.

Since h'' is a subalgebra of $h/(x_1)$, there exists a subalgebra h' of $\mathfrak{g}$ such that $x_1 \in h'$ and $h'/(x_1) = h''$. Let θ be an element of $\mathrm{Hom}_k(\mathfrak{g}, k)$ such that $\theta(x_1) = 0$ and $\bar{\theta}|h'/(x_1) = \theta'|h''$ where $\bar{\theta} : \mathfrak{g}/(x_1) \to k$ is the map induced by θ. Since

$$\dim_k h' = \dim_k h'' + 1 = \dim_k h_1,$$

we have

$$\mathrm{hd}\, \mathfrak{g}(f) \underset{h'(f)}{\otimes} k(\theta, h') = \dim_k h' = \dim_k h_1 = 1.\mathrm{gl.dim}\, \mathfrak{g}_1(f_1) = 1.\mathrm{gl.dim}\, \mathfrak{g}(f).$$

Thus the proof of Lemma 2.4 is complete.

Proof of Theorem 2.1. We shall prove the result by induction on $\dim_k \mathfrak{g}$. Let $\dim_k \mathfrak{g} = 1$. Then $\mathfrak{g}(f)$ is a polynomial ring $k[z]$ in one variable over k. Let $\theta : \mathfrak{g} \to k$ be the map given by $\theta(z) = 0$ where $\mathfrak{g} = k \cdot z$. Then it is easy to see that $\mathfrak{g}$ is f-subordinate to θ and $\mathrm{hd}\, \mathfrak{g}(f) \otimes_{\mathfrak{g}(f)} k(\theta, \mathfrak{g}) = \mathrm{hd}_{k[z]}\, k = 1 = 1.\mathrm{gl.dim}\, \mathfrak{g}(f)$.

Assume the result for $\dim_k \mathfrak{g} \leqslant n-1$. Let $\dim_k \mathfrak{g} = n \geqslant 2$.

If $Z(f) = 0$, then by Lemma 2.2 we get a nilpotent Lie algebra $\mathfrak{g}'$ with $\dim_k \mathfrak{g}' = \dim_k \mathfrak{g} - 2$ and a 2-cocycle f' on $\mathfrak{g}'$ such that $\mathfrak{g}(f) \simeq \mathfrak{g}'(f') \otimes_k A_1(k)$. If $Z(f') = 0$ then we apply Lemma 2.2 to $\mathfrak{g}'(f')$. Since $\dim_k \mathfrak{g}' < \dim_k \mathfrak{g}$, after a finite number of steps we get a nilpotent Lie algebra $\mathfrak{g}''$, k-valued 2-cocycle f'' on $\mathfrak{g}''$ with the property that either $\mathfrak{g}'' = 0$ or $Z(f'') \neq 0$ and an integer m such that $\dim_k \mathfrak{g}'' = \dim_k \mathfrak{g} - 2m$ and $\mathfrak{g}(f) \simeq \mathfrak{g}''(f'') \otimes_k A_m(k)$. But we have shown that there exists a nilpotent Lie algebra $\mathfrak{g}_1$ and 2-cocycle f_1 on $\mathfrak{g}_1$ such that

$$\mathfrak{g}''(f'') \underset{k}{\otimes} A_m(k) = \mathfrak{g}_1(f_1), \quad Z(f'') = Z(f_1) \quad \text{and} \quad \dim_k \mathfrak{g}_1 = \dim_k \mathfrak{g}'' + 2m.$$

Therefore $\mathfrak{g}(f) \simeq \mathfrak{g}_1(f_1)$, where $\mathfrak{g}_1$ and f_1 are such that $\dim_k \mathfrak{g}_1 = \dim_k \mathfrak{g}$ and either $Z(f_1) \neq 0$ or $\mathfrak{g}_1(f_1) \simeq A_m(k)$ for some positive integer m. Therefore from Lemma 2.4 it follows that it is sufficient to prove the theorem for $\dim_k \mathfrak{g} = n$ with either $\mathfrak{g}(f) \simeq A_m(k)$ or $Z(f) \neq 0$.

Case 1. $\mathfrak{g}(f) \simeq A_m(k)$. Then 0 is the only proper two-sided and therefore rational ideal of $\mathfrak{g}(f)$. Therefore from Proposition 1.2 it follows that there exists an element θ of $\mathrm{Hom}_k(\mathfrak{g}, k)$ and a subalgebra h of $\mathfrak{g}$ such that

(I) h is f-subordinate to θ,

(II) $M = \mathfrak{g}(f) \underset{h(f)}{\otimes} k(\theta, h)$ is a simple $\mathfrak{g}(f)$-module,

(III) $\mathrm{ann}\, M = 0$.

Let $\mathfrak{A}$ be the left ideal of $A_m(k)$ generated by $\partial/\partial X_1, \partial/\partial X_2, \dots, \partial/\partial X_m$. Then it is well known that $\mathfrak{A}$ is a maximal left ideal of $A_m(k)$ and $\mathrm{hd}\, A_m(k)/\mathfrak{A} = m = \mathrm{l.gl.dim}\, A_m(k)$. Let G be the group of k-algebra automorphisms of $A_m(k)$. Then obviously G acts on the set of simple modules of $A_m(k)$.

Let $\phi_m : \mathfrak{g}(f) \to A_m(k)$ be an isomorphism of k-algebras. Then by [5; p. 460] we have an element σ of G such that $\phi_m(M) = \sigma(M')$, where $M' = A_m(k)/\mathfrak{A}$.

Therefore $\mathrm{hd}\, M = m = \mathrm{l.gl.dim}\, \mathfrak{g}(f)$.

Case 2. $Z(f) \neq 0$. Let $0 \neq x \in Z(f)$. Let $(x_1, x_2, \dots, x_n)$ be a k-basis of $\mathfrak{g}$ such that $x = x_1$.

By Proposition 1.1, we have a simple module M such that $\mathrm{hd}\, M = \mathrm{l.gl.dim}\, \mathfrak{g}(f)$. Let $I = \mathrm{ann}\, M$. Then I is a primitive ideal of $\mathfrak{g}(f)$. Since k is algebraically closed, it follows from [6; Theorem 2] that I is a rational ideal of $\mathfrak{g}(f)$.

Since $k[x_1]$ is contained in the centre of $\mathfrak{g}(f)$ and the centre of $\mathfrak{g}(f)/I$ is k, we have $0 \neq I \cap k[x_1] = (x_1 - \lambda)$, where $\lambda \in k$.

Let $\mathfrak{g}' = \mathfrak{g}/(x_1)$. Let $\alpha : \mathfrak{g} \to k$ be the k-linear map given by

$$\alpha(x_1) = \lambda,$$

$$\alpha(x_i) = 0, \qquad 2 \leqslant i \leqslant n.$$

Let $f' : \mathfrak{g}' X \mathfrak{g}' \to k$ be the map defined by

$$f'(\bar{z}, \bar{w}) = f(z, w) + \alpha[z, w].$$

Then $\mathfrak{g}'$ is a nilpotent Lie algebra, f' a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}'$ such that $\mathfrak{g}'(f') = \mathfrak{g}(f)/(x_1 - \lambda)$.

Since $x_1 - \lambda \in I = \operatorname{ann} M$, we can regard M as a $\mathfrak{g}'(f')$-module. Since $x_1 - \lambda$ is an element of the centre of $\mathfrak{g}(f)$ which is neither a unit nor a divisor of zero and $\operatorname{hd} M_{\mathfrak{g}'(f')} \leqslant \text{l.gl.dim}\, \mathfrak{g}'(f') \leqslant \dim_k \mathfrak{g}' < \infty$, by Kaplansky's theorem [9; p. 172, Theorem 3], $\operatorname{hd} M_{\mathfrak{g}'(f')} = \operatorname{hd}_{\mathfrak{g}(f)} M - 1 = \text{l.gl.dim}\, \mathfrak{g}(f) - 1$. But since

$$\text{l.gl.dim}\, \mathfrak{g}'(f') \leqslant \dim_k \mathfrak{g}' < \infty,$$

we always have $\text{l.gl.dim}\, \mathfrak{g}'(f') \leqslant \text{l.gl.dim}\, \mathfrak{g}(f) - 1$ [9; p. 173, Theorem 4]. Therefore $\text{l.gl.dim}\, \mathfrak{g}'(f') = \text{l.gl.dim}\, \mathfrak{g}(f) - 1$.

Since $\dim_k \mathfrak{g}' = n-1$, by the induction hypothesis there exists a subalgebra h' of $\mathfrak{g}'$ and an element θ' of $\operatorname{Hom}_k(\mathfrak{g}', k)$ such that (I) h' is f'-subordinate to θ', (II) $\operatorname{hd} \mathfrak{g}'(f') \otimes_{h'(f')} k(\theta', h') = \text{l.gl.dim}\, \mathfrak{g}'(f')$. Therefore $\text{l.gl.dim}\, \mathfrak{g}'(f') = \dim_k h'$.

Let h be a subalgebra of $\mathfrak{g}$ such that $x_1 \in h$ and $h/(x_1) = h'$. Let θ be an element of $\operatorname{Hom}_k(\mathfrak{g}, k)$ such that $\theta(x_1) = \lambda$ and $\theta(x_i) = \theta'(\bar{x}_i)$ for $2 \leqslant i \leqslant n$, where $\bar{x}_i$ denotes the image of x_i in $\mathfrak{g}'(=\mathfrak{g}/(x_1))$ under the canonical mapping $\eta : \mathfrak{g} \to \mathfrak{g}'(=\mathfrak{g}/(x_1))$.

Then h' being f'-subordinate to θ' implies that h is f-subordinate to θ and $\operatorname{hd} \mathfrak{g}(f) \otimes_{h(f)} k(\theta, h) = \dim_k h = \dim_k h' + 1 = \text{l.gl.dim}\, \mathfrak{g}'(f') + 1 = \text{l.gl.dim}\, \mathfrak{g}(f)$.

Thus the theorem is proved for $\dim_k \mathfrak{g} = n$.

This completes the proof of Theorem 2.1.

THEOREM 2.5. *Let k be an algebraically closed field of characteristic 0. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over k. Let f be a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}$. Let $(h_j)_{j \in J}$ be the family of subalgebras of $\mathfrak{g}$ for which the restriction of f to $h_j \times h_j$ is a coboundary. Then* $\text{l.gl.dim}\, \mathfrak{g}(f) = \sup_{j \in J} \dim_k h_j$.

Proof. As we have remarked, if h is a subalgebra of $\mathfrak{g}$ such that the restriction of f to $h \times h$ is a coboundary then $\dim_k h = \text{l.gl.dim}\, h(f) \leqslant \text{l.gl.dim}\, \mathfrak{g}(f)$. Therefore we always have $\text{l.gl.dim}\, \mathfrak{g}(f) \geqslant \sup_{j \in J} \dim_k h_j$.

Proposition 2.1 shows that there exists a subalgebra h of $\mathfrak{g}$ and an element θ of $\operatorname{Hom}_k(\mathfrak{g}, k)$ such that h is f-subordinate to θ and

$$\text{l.gl.dim}\, \mathfrak{g}(f) = \operatorname{hd} \mathfrak{g}(f) \underset{h(f)}{\otimes} k(\theta, h).$$

But h being f-subordinate to θ implies that the restriction of f to $h \times h$ is a coboundary and $\dim_k h = \operatorname{hd}_{h(f)} k(\theta, h) = \operatorname{hd}_{\mathfrak{g}(f)} \mathfrak{g}(f) \otimes_{h(f)} k(\theta, h) = \text{l.gl.dim}\, \mathfrak{g}(f)$.

Therefore $\text{l.gl.dim}\, \mathfrak{g}(f) \leqslant \sup_{j \in J} \dim_k h_j$. Hence we have the equality.

This completes the proof of Theorem 2.5.

Let k be a commutative field of characteristic 0. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over k, f a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}$. For every field extension L of k let $\mathfrak{g}^L$ denote the nilpotent Lie algebra $\mathfrak{g} \otimes_k L$ over L and f_L denote the L-valued 2-cocycle $f \otimes_k I_L$ on the "standard complex" for $\mathfrak{g}^L$, where I_L is the identity automorphism of L.

THEOREM 2.6. *Let k be a commutative field of characteristic 0. Let Ω be an algebraic closure of k. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over k, f a k-valued 2-cocycle on the "standard complex" for $\mathfrak{g}$. Then with the above notation* $\text{l.gl.dim}\, \mathfrak{g}(f) = \text{l.gl.dim}\, \mathfrak{g}^{\Omega}(f_{\Omega})$.

Proof. Since Ω is algebraically closed, by Theorem 2.5 we get a subalgebra h' of $\mathfrak{g}^{\Omega}$ such that the restriction of $\mathfrak{f}_{\Omega}$ to $h' \times h'$ is a coboundary and $\text{l.gl.dim}\, \mathfrak{g}^{\Omega}(f_{\Omega}) = \dim_{\Omega} h'$. Since $\mathfrak{g}$ is finite-dimensional over k and f is completely determined by its values on a k-basis of $\mathfrak{g} \times \mathfrak{g}$, it follows that there exists a finite extension L of k and a subalgebra h of $\mathfrak{g}^L$ such that the restriction of f_L to $h \times h$ is a coboundary and $h \underset{L}{\otimes} \Omega = h'$. Therefore we have

$$\text{l.gl.dim}\, \mathfrak{g}^{\Omega}(f_{\Omega}) = \dim_{\Omega} h' = \dim_L h \leqslant \text{l.gl.dim}\, \mathfrak{g}^L(f_L).$$

But $\mathfrak{g}^L(f_L) \simeq \mathfrak{g}(f) \underset{k}{\otimes} L$. Therefore, by [7; p. 74], we have

$$\text{l.gl.dim}\, \mathfrak{g}^L(f_L) \leqslant \text{l.gl.dim}\, \mathfrak{g}(f) + \text{hd}_{L^e} L,$$

where $L^e = L \otimes_k L$. But since L is a finite extension of k and $\text{char.}\, k = 0$, L is separable over k, i.e. $\text{hd}_{L^e} L = 0$. Therefore

$$\text{l.gl.dim}\, \mathfrak{g}^{\Omega}(f_{\Omega}) \leqslant \text{l.gl.dim}\, \mathfrak{g}^L(f_L) \leqslant \text{l.gl.dim}\, \mathfrak{g}(f).$$

But $\mathfrak{g}^{\Omega}(f_{\Omega})$ is $\mathfrak{g}(f)$-free as a right as well as a left module and contains $\mathfrak{g}(f)$ as a direct summand. Therefore, by [**10**; Lemma 1], we have $\text{l.gl.dim}\, \mathfrak{g}(f) \leqslant \text{l.gl.dim}\, \mathfrak{g}^{\Omega}(f_{\Omega})$.

Hence the equality is proved.

This completes the proof of Theorem 2.5.

COROLLARY 2.7. *Let k, $\mathfrak{g}$, f be as given in Theorem 2.6. Let $A_m(k)$ be the Weyl algebra of index m with coefficients in k. Then*

$$\text{l.gl.dim}\, \mathfrak{g}(f) \underset{k}{\otimes} A_m(k) = \text{l.gl.dim}\, \mathfrak{g}(f) + m.$$

Proof. Since $A_m(k) \simeq A_{m-1}(k) \otimes_k A_1(k)$, it is enough to prove the result for $m = 1$. From Theorem 2.6, it follows that we can assume, without loss of generality, k to be algebraically closed. Let $\mathfrak{g}(f) \otimes_k A_1(k) = \mathfrak{g}_1(f_1)$, where $\mathfrak{g}_1$ and f_1 are as defined earlier.

Then by Theorem 2.5 and Lemma 2.3 we get a subalgebra h of $\mathfrak{g}$ such that the restriction of f_1 to $h_1 \times h_1$ is a coboundary, where $h_1 = h \oplus k \cdot X_1$ and

$$\text{l.gl.dim}\, \mathfrak{g}(f) \underset{k}{\otimes} A_1(k) = \dim_k h_1.$$

This implies that the restriction of f to $h \times h$ is a coboundary. Therefore

$$\text{l.gl.dim}\, \mathfrak{g}(f) \geqslant \dim_k h = \dim_k h_1 - 1 = \text{l.gl.dim}\, \mathfrak{g}(f) \underset{k}{\otimes} A_1(k) - 1.$$

But by [7; p. 74] we have

$$\text{l.gl.dim}\, \mathfrak{g}(f) + 1 = \text{l.gl.dim}\, \mathfrak{g}(f) + \text{l.gl.dim}\, A_1(k) \leqslant \text{l.gl.dim}\, \mathfrak{g}(f) \underset{k}{\otimes} A_1(k).$$

Hence the equality follows.

The proof of Corollary 2.7 is complete.

Remark. It has been proved in [**2**; Corollary 2.6] that if S is a commutative k-algebra then $\text{l.gl.dim}\, S \otimes_k A_m(k) = \text{gl.dim}\, S + m$. But R. Hart has given an example in [8; p. 344] of a division k-algebra D such that

$$\text{l.gl.dim}\, D \underset{k}{\otimes} A_1(k) = 2 > \text{l.gl.dim}\, D+1.$$

In this context Corollary 2.7 is interesting.

We refer to [12] for the definition of Krull dimension of a module over a (not necessarily commutative) ring. For a ring A let $\text{l.Kr.dim}\, A$ denote the Krull dimension of A when A is regarded as a left module over A.

We state a result which has been proved by J. E. Roos in [13].

THEOREM OF ROOS. *Let A be a filtered noetherian ring whose associated graded ring is a commutative regular noetherian ring. Then* $\text{l.Kr.dim}\, A \leqslant \text{l.gl.dim}\, A$.

As a consequence of the above theorem we get the following corollary.

COROLLARY 2.8. *Let* $\mathfrak{g}$, f, k *be as given in Theorem* 2.5. *Then*

$$\text{l.gl.dim}\, \mathfrak{g}(f) = \text{l.Kr.dim}\, \mathfrak{g}(f).$$

Proof. By the theorem of Roos, we have $\text{l.Kr.dim}\, \mathfrak{g}(f) \leqslant \text{l.gl.dim}\, \mathfrak{g}(f)$.

By Theorem 2.5 we get a subalgebra h of $\mathfrak{g}$ such that the restriction of f to $h \times h$ is a coboundary and $\text{l.gl.dim}\, \mathfrak{g}(f) = \dim_k h$. Since $h(f)$ is isomorphic to the usual enveloping algebra of the nilpotent Lie algebra h, by [12; p. 713, (9)] we have $\dim_k h \leqslant \text{l.Kr.dim}\, h(f)$.

But since $\mathfrak{g}(f)$ is $h(f)$-free as a right as well as a left module and contains $h(f)$ as a direct summand, it is easy to see that $\text{l.Kr.dim}\, h(f) \leqslant \text{l.Kr.dim}\, \mathfrak{g}(f)$. Therefore we have $\text{l.gl.dim}\, \mathfrak{g}(f) = \dim_k h \leqslant \text{l.Kr.dim}\, h(f) \leqslant \text{l.Kr.dim}\, \mathfrak{g}(f)$.

Hence the equality follows.

This completes the proof of Corollary 2.8.

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