Realizability of stationary spherically symmetric transonic accretion

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The spherically symmetric stationary transonic (Bondi) flow is considered a classic example of an accretion flow. This flow, however, is along a separatrix, which is usually not physically realizable. We demonstrate, using a pedagogical example, that it is the dynamics which selects the transonic flow.

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In astrophysics, the importance of accretion processes can hardly be overstated, especially in the context of the study of compact astrophysical objects and active galactic nuclei. Of such processes, a very important paradigm is that of the steady spherically symmetric flow, in which the motion of the accreting matter is steady and spherically symmetric, obeying the boundary condition that the bulk velocity is to fall off to zero at infinity, while the density asymptotically approaches a fixed value. Since this is the simplest situation in the class of accretion flows, it is the starting point in all relevant texts [1, 2]. Studied extensively, among others, by Bondi [3], about fifty years ago, it also gives one of the clearest examples of a transonic flow (with which, in spherical symmetry, Bondi's name is associated), i.e. a flow in which the velocity is subsonic far away from the star and becomes supersonic as the surface of the star is approached. It is this simplicity and easy comprehensibility of the model, that overshadows the fact that it is not encountered most often in practice. Accordingly, the spherically symmetric flow has been of continuing interest [4, 5, 6, 7]. We first argue that the transonic (Bondi) solution of the steady spherically symmetric flow, would not be physically realizable. This proposition is in contradiction to the position of Bondi himself, that "the case physically most likely to occur is that with the maximum rate of accretion" [3], which, in spherically symmetric accretion, is a case that is readily identified as the transonic solution [4]. We then argue that it is the dynamics which actually selects the transonic flow from among all possible trajectories.

The typical solutions for the spherically symmetric flow in the velocity-coordinate space are shown in Fig.1 (to be read without the arrows). The two dark solid curves labelled **A** and **W** refer to the accretion flow and the wind flow respectively. The intersection point is an equilibrium point. The problem with Fig.1 is that when it is read without the arrows it is slightly misleading. It does not show along what route an integration of dv/dr would proceed if we start with an initial condition $v = v_{\rm in}$ at $r = r_{\rm in}$ far away from the star. For a physically realizable flow, an initial condition infinitesimally close to a point on the accretion curve **A**, would trace out a curve infinitesimally close to **A** and in the limit would correctly reproduce **A**, evolving along it and passing through the equilibrium point (sonic point) as we integrate dv/dr, obtained from Euler's equation. We will show that the arrows on the integration route are as shown in Fig.1. The direction of the arrows indicates that the spherically symmetric transonic accretion flow should not be physically realizable.

Is this a result confined to the spherically symmetric flow? We have verified, as we shall show briefly, that this holds also for the axisymmetric rotating accretion flow for a thin disc [8] which is a situation of practical interest. The technique for assigning the arrows remains identical to that of the spherically symmetric case [9].

We wish to make our point with a fairly straightforward example. We consider the differential equation

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)} = \frac{x+y-2}{y-x} \tag{1}$$

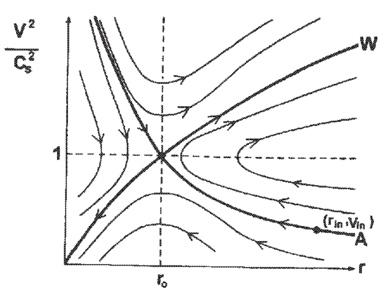

Fig.1

FIG. 1: Phase trajectories for spherically symmetric accretion onto a star. The bold solid lines, **A** and **W**, represent "accretion" and "wind" respectively. The fixed point is at $r = r_0$ and $v^2/{c_{\rm s}}^2 = 1$. Linear stability analysis indicates that the fixed point of the flow is a saddle point. The direction of the arrows along the line **A**, demonstrates that the transonic flow is not physically realizable.

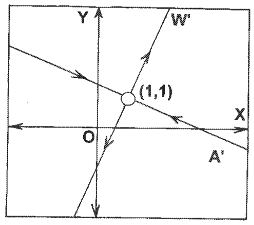

Fig. 2

FIG. 2: Integration of Eq.(1) gives a pair of straight lines, with the integration constant fixed by the intersection point $(1, 1)$. In the figure the lines are marked $\mathbf{A}'$ and $\mathbf{W}'$. Parametrizing Eq.(1) as Eq.(5), we get the concept of arrows. Linear stability analysis indicates that for $\mathbf{A}'$ and $\mathbf{W}'$, the intersection point $(1, 1)$ is actually a saddle point, for which the arrows are as shown above.

whose integral can be written down as

$$x^2 - y^2 - 4x + 2xy = -C \tag{2}$$

where C is a constant. If we want that particular solution which passes through the point where $Y(x, y) = X(x, y) = 0$, namely $x = y = 1$, then $C = 2$. The curve $x^2 - y^2 - 4x + 2xy = -2$ factorizes into a pair of straight lines : $y - x(\sqrt{2}+1) + \sqrt{2} = 0$ and $y + x(\sqrt{2}-1) - \sqrt{2} = 0$. This pair is shown in Fig.2 (to be read again without the arrows) as the lines marked $\mathbf{A}'$ and $\mathbf{W}'$.

We want to explore the process of drawing the line $\mathbf{A}'$ from a given initial condition. On line $\mathbf{A}'$, $y = 0$ at $x = 2 + \sqrt{2}$. Let us begin with the initial condition $y = 0$ at $x = 2 + \sqrt{2} - \epsilon$, where $0 < \epsilon \ll 1$. The given initial condition fixes the constant C as $C = 2[1 + \sqrt{2}\epsilon - \epsilon^2/2]$. Using this value of C, we can plot the curve given by Eq.(2). For a given x, a value of y is given by the relevant root of the quadratic equation thus obtained. The two roots are

$$y = x \pm \sqrt{2}\left[(x-1)^2 + \sqrt{2}\epsilon - \frac{\epsilon^2}{2}\right]^{1/2} \tag{3}$$

from which it is clear that to satisfy $y = 0$ at $x = 2 + \sqrt{2} - \epsilon$, the negative sign has to be chosen to let us have

$$y = x - \sqrt{2}\left[(x-1)^2 + \sqrt{2}\epsilon - \frac{\epsilon^2}{2}\right]^{1/2} \tag{4}$$

At $x = 0$, we then get $y = -\sqrt{2}(1 + \sqrt{2}\epsilon - \epsilon^2/2)^{1/2}$, which is very different from $y = \sqrt{2}$, that one gets on line $\mathbf{A}'$. In the limit of $\epsilon \longrightarrow 0$, one generates a part of $\mathbf{A}'$ $(x \geqslant 1)$ and a part of $\mathbf{W}'$ $(x \leqslant 1)$, instead of the entire line $\mathbf{A}'$. Another way of stating this is the sensitivity to initial conditions in the drawing of $\mathbf{A}'$. If we make an error of an infinitesimal amount ϵ in prescribing the initial condition on $\mathbf{A}'$, i.e. if we prescribe $y = 0$ at $x = 2 + \sqrt{2} - \epsilon$ instead of $y = 0$ at $x = 2 + \sqrt{2}$, then the "error" made at $x = 0$ relative to $\mathbf{A}'$ is $2\sqrt{2}$ which is $\mathcal{O}(1)$. An infinitesimal separation at one point leads to a finite separation at a point a short distance away. This is what we mean by saying that line $\mathbf{A}'$ (and similarly $\mathbf{W}'$) should not be physically realized.

The clearest and most direct understanding of the difficulty is achieved by writing Eq.(1) as a set of two parametrized differential equations

$$\frac{dy}{d\tau} = x + y - 2$$
$$\frac{dx}{d\tau} = y - x \tag{5}$$

where τ is some convenient parameter. The fixed point of this dynamical system is $(1, 1)$, namely the point where $Y(x, y)$ and $X(x, y)$ are simultaneously zero — the point through which both $\mathbf{A}'$ and $\mathbf{W}'$ pass. Linear stability analysis of this fixed point in τ space shows that it is a saddle. The critical solutions in this y — x space can now be drawn with arrows and the result is as shown in Fig.2. The distribution of the arrows (characteristic of a saddle [10]) implies $\mathbf{A}'$ and $\mathbf{W}'$ cannot be physically realized.

For the spherically symmetric flow, the relevant variables are the radial velocity v and the local density ρ. Ignoring viscosity, we write down Euler's equation for v as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial V}{\partial r} \tag{6}$$

where P is the local pressure and $V \equiv V(r)$ is the potential due to a gravitational attractor of mass M, i.e. $V(r) = -GM/r$. The pressure is related to the local density through the equation of state $P = K\rho^\gamma$ where K is a constant, and γ is the polytropic exponent with an admissible range given by $1 < \gamma < 5/3$. The local density evolves according to the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho v r^2\right) = 0 \tag{7}$$

The stationary solution implies $\partial v/\partial t = \partial \rho/\partial t = 0$. The local sound speed is given by ${c_{\mathrm{s}}}^2 = \partial P/\partial \rho = \gamma K \rho^{\gamma-1}$ and in the stationary situation, we can use Eq.(7) to write Eq.(6) as

$$\frac{d(v^2)}{dr} = \frac{2v^2}{r}\frac{\left(2{c_{\mathrm{s}}}^2 - GM/r\right)}{v^2 - {c_{\mathrm{s}}}^2} \tag{8}$$

whose integration with different initial conditions is supposed to generate the curves in Fig.1. To assign arrows we write Eq.(8) in a parametrized form

$$\frac{d(v^2)}{d\tau} = 2v^2\left(2{c_\mathrm{s}}^2 - \frac{GM}{r}\right)$$
$$\frac{dr}{d\tau} = r\left(v^2 - {c_\mathrm{s}}^2\right) \tag{9}$$

to find that the critical point is at $r = r_0$ and $v = v_0$, such that ${v_0}^2 = {c_{\mathrm{s}0}}^2$ and ${c_{\mathrm{s}0}}^2 = GM/2r_0$. It is immediately clear that the critical point is the so called sonic point, and it can be fixed in terms of the constants of the system with the help of suitable boundary conditions [1]. We now carry out a linear stability analysis around the fixed point by writing $v^2 = {v_0}^2(1+\delta_1)$ and $r = r_0(1+\delta_2)$. Linearizing in δ_1 and δ_2, followed by some straightforward algebra then yields

$$\frac{d\delta_1}{d\tau} = 2{c_{\mathrm{s}0}}^2\Big[-(\gamma-1)\delta_1 + (6-4\gamma)\delta_2\Big]$$
$$\frac{d\delta_2}{d\tau} = {c_{\mathrm{s}0}}^2\left[\frac{\gamma+1}{2}\delta_1 + 2(\gamma-1)\delta_2\right] \tag{10}$$

The solutions for $\delta_{1,2}$ are of the form $e^{\lambda\tau}$, where λ is to be found from the roots of

$$\begin{vmatrix} \lambda + 2(\gamma-1){c_{\mathrm{s}0}}^2 & 2(4\gamma-6){c_{\mathrm{s}0}}^2 \\ -[(\gamma+1)/2]{c_{\mathrm{s}0}}^2 & \lambda - 2(\gamma-1){c_{\mathrm{s}0}}^2 \end{vmatrix} = 0 \tag{11}$$

The eigenvalues λ are given by $\lambda = \pm {c_{\mathrm{s}0}}^2\sqrt{2(5-3\gamma)}$. For $\gamma < 5/3$, the fixed point is a saddle. Since the physical situations are restricted to this range, we find that the fixed point is always a saddle and hence the distribution of arrows will be as in Fig.1.

For the more realistic axisymmetric case (as in a thin accretion disc), using R as the radial distance in the plane, the static equations of the flow for an angular velocity Ω and for the disc thickness H, are

The Equation of Continuity :

$$\rho R H v = \text{constant} \tag{12}$$

The Momentum Balance Equation :

$$\frac{1}{2}\frac{d(v^2)}{dR} = R\Omega^2 - R{\Omega_\mathrm{K}}^2 - \frac{1}{\rho}\frac{dP}{dR} \tag{13}$$

The Angular Momentum Balance Equation :

$$v\frac{d}{dR}\left(\Omega R^2\right) = \frac{1}{\rho R H}\frac{d}{dR}\left(\frac{\alpha\rho {c_\mathrm{s}}^2 R^3 H}{\Omega_\mathrm{K}}\frac{d\Omega}{dR}\right) \tag{14}$$

where ${\Omega_\mathrm{K}}^2 = GM/R^3$ and α is the effective viscosity of Shakura and Sunyaev [8]. An integral of motion follows from Eq.(13) and we can obtain an equation for $d(v^2)/dR$ akin to Eq.(8), which we can write as the dynamical system

$$\frac{d(v^2)}{d\tau} = 2v^2\left[\frac{5{c_\mathrm{s}}^2}{1+\gamma} - R^2\left({\Omega_\mathrm{K}}^2 - \Omega^2\right)\right]$$
$$\frac{dR}{d\tau} = R\left[v^2 - \frac{2{c_\mathrm{s}}^2}{1+\gamma}\right] \tag{15}$$

from which, taking the inviscid limit (i.e. $\alpha = 0$), an analysis identical to the one following Eq.(9) leads to the eigenvalues λ of the stability matrix. Of these one can easily be shown to be appropriate for a saddle [9].

We now focus on the fact that the real physical problem is dynamic in nature as exhibited in Eqs.(6) and (7), even as we have tacitly assumed that we can study the problem directly in its stationary limit. As it turns out there are an infinite number of stationary solutions and if the static limit is taken directly, the Bondi solution becomes very sensitive to the process of integration to determine the stationary trajectory. Several studies in the linear stability analysis of the stationary flows that have been carried out in the past have not really helped clarify the situation as it was finally established that among the stationary solutions, the transonic Bondi solution as well as all the subsonic solutions are stable in the linear stability sense [4, 11]. Any selection mechanism then must be of a non-perturbative nature.

Accordingly, we return to our pedagogic example of Eq.(1) but now consider y as a field $y(x,t)$ with the evolution

$$\frac{\partial y}{\partial t} + (y-x)\frac{\partial y}{\partial x} = y + x - 2 \tag{16}$$

The stationary solutions $y(x)$ satisfy Eq.(1) and Fig.2 shows that their separatrices are given by $y - x\left(\sqrt{2}+1\right) + \sqrt{2} = 0$ and $y + x\left(\sqrt{2}-1\right) - \sqrt{2} = 0$. It should be easy to see that a linear stability analysis in t around the family of stationary solutions $y(x)$ would show an infinite number of them to be stable, and therefore each one of them could be a perfectly valid physical solution. We will now show that the non-perturbative dynamics actually selects the separatrices.

The general solution of Eq.(16) can be obtained by the method of characteristics [12]. The two independent characteristic curves of Eq.(16) are obtained from

$$\frac{dt}{1} = \frac{dx}{y-x} = \frac{dy}{y+x-2} \tag{17}$$

They are

$$y^2 - 2xy - x^2 + 4x = C$$
$$\left[x - 1 \pm \frac{1}{\sqrt{2}}(y - x)\right] e^{\mp\sqrt{2}t} = \tilde{C} \tag{18}$$

in which the latter is obtained by integrating the dx/dt integral with the help of the first solution. With the choice of the upper sign now (since we should want the evolution to proceed through a positive range of t), the solutions of Eq.(16) can be written as

$$y^2 - 2xy - x^2 + 4x = f\left(\left[x - 1 + \frac{1}{\sqrt{2}}(y - x)\right] e^{-\sqrt{2}t}\right) \tag{19}$$

where f is an arbitrary function, whose form is to be determined from initial conditions. In this case we choose the initial condition that $y(x) = 0$ at $t = 0$ for all x. This leads to $f\big(x\big[1 - 1/\sqrt{2}\big] - 1\big) = -x^2 + 4x$ to give a form for f as $f(z) = -2\big(2\sqrt{2} + 3\big)z^2 - 4\big(\sqrt{2} + 1\big)z + 2$. The solution to Eq.(16) is then given as

$$\begin{aligned}\Big[y - x\big(\sqrt{2} + 1\big) + \sqrt{2}\Big]\Big[y + x\big(\sqrt{2} - 1\big) - \sqrt{2}\Big] \\ = -2\big(2 + \sqrt{2}\big)\Big[x\big(\sqrt{2} - 1\big) + y - \sqrt{2}\Big] e^{-\sqrt{2}t} \\ -\big(3 + 2\sqrt{2}\big)\Big[x\big(\sqrt{2} - 1\big) + y - \sqrt{2}\Big]^2 e^{-2\sqrt{2}t}\end{aligned} \tag{20}$$

Clearly for $t \longrightarrow \infty$, the right hand side in Eq. (20) tends to zero and we approach one of the two separatrices shown in Fig.2. Of the two possible separatrices, the one which will be relevant will be determined by some other requirement. For the astrophysical flow the two separatrices are the Bondi accretion flow and the transonic wind solution. One chooses the proper sign of the velocity to get the flow in which one is interested.

The mechanism for the selection of the asymptotes in Fig.1 as the favoured trajectories is identical. This can be appreciated from a look at Eq.(6). In the "pressure-free" approximation, we have a set of stationary solutions $v^2 - 2GM/r = c^2$. Which of these would be selected by the dynamics? If we start from $v = 0$ at $t = 0$ for all r, an identical reasoning to the one given above shows that it is the path with $c = 0$ which is selected. The solution of the differential equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{GM}{r^2} \tag{21}$$

by the method of characteristics yields

$$\frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{r}\left(\frac{v}{c} + 1\right)^{-2} \exp\left(\frac{2rv}{cr_c} - \frac{2ct}{r_c}\right) \tag{22}$$

in which $r_c = 2GM/c^2$. Obviously for $t \longrightarrow \infty$ we will get the static solution $v^2 = 2GM/r$. This is also the choice according to the energy criterion. Corresponding to the initial condition given, the lowest possible total energy is $E = v^2/2 - GM/r = 0$ and the dynamics selects this particular stationary trajectory. This selection mechanism is entirely non-perturbative in character and provides the mathematical justification for Bondi's assertion that the energy criterion should select the stationary flow [3]. With the pressure term included, this argument leads to the selection of the transonic path i.e. gravity always wins over pressure at small radial distances.

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[1] S. K. Chakrabarti, *Theory of Transonic Astrophysical Flows* (World Scientific, Singapore, 1990).
[2] J. Frank, A. King, and D. Raine, *Accretion Power in Astrophysics* (Cambridge University Press, Cambridge, 1992).
[3] H. Bondi, Monthly Notices of the Royal Astronomical Society **112**, 195 (1952).
[4] J. A. Petterson, J. Silk, and J. P. Ostriker, Monthly Notices of the Royal Astronomical Society **191**, 571 (1980).
[5] T. Theuns and M. David, The Astrophysical Journal **384**, 587 (1992).
[6] M. Colpi, S. L. Shapiro, and I. Wasserman, The Astrophysical Journal **470**, 1075 (1996).
[7] Y. M. Toropin, O. D. Toropina, V. V. Savelyev, M. M. Romanova, V. M. Chechetkin, and R. V. E. Lovelace, The Astrophysical Journal **517**, 906 (1999).
[8] R. Narayan and I. Yi, The Astrophysical Journal **428**, L13 (1994).
[9] A. K. Ray and J. K. Bhattacharjee, astro-ph/0307447.
[10] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations* (Clarendon Press, Oxford, 1977).
[11] A. R. Garlick, Astronomy and Astrophysics **73**, 171 (1979).
[12] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers* (Birkhäuser, Boston, 1997).