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A softer, stronger Lidskii theorem

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Abstract. We provide a new approach to Lidskii's theorem relating the eigenvalues of the difference $A-B$ of two self-adjoint matrices to the eigenvalues of A and B respectively. This approach combines our earlier work on the spectral matching of matrices joined by a normal path with some familiar techniques of functional analysis. It is based, therefore, on general principles and has the additional advantage of extending Lidskii's result to certain pairs of normal matrices. We are also able to treat some related results on spectral variation stemming from the work of Sunder, Halmos and Bouldin.

Keywords. Lidskii theorem; spectral variation; functional analysis; normal path inequality.

1. Introduction

To discuss the classical Lidskii theorem we first recall the notion of majorization. For real or complex vectors v, w ($\in \mathbb{C}^n$), we define majorization (of v by w), written $v \ll w$, to mean that v is a convex combination of the permutations σw of w (σw denotes the vector obtained from w by permuting the components according to the permutation σ of $\{1,\ldots,n\}$). For our purposes it will be convenient to define also "soft" majorization (of v by w), written $v \ll_s w$, to mean that $v = \Sigma z_k \sigma_k w$ (finite sum) where the $\sigma_k w$ are permutations of w and the z_k are complex numbers such that $\Sigma |z_k| \leqslant 1$. It is clear that $\Sigma_1^n v_j = \Sigma_1^n w_j$ when $v \ll w$; on the other hand if these quantities are equal and non-zero then $v \ll w$ follows from $v \ll_s w$, because Σz_k must be 1.

For any n-by-n matrix T, we write $\operatorname{Eig} T$ to indicate the n-vector of eigenvalues of T, including multiplicity and ordered arbitrarily. In other contexts $\operatorname{Eig} T$ may stand for the related diagonal matrix with eigenvalues of T on its diagonal. Lidskii's theorem (see, for example, Kato [13, §6.5 of chap. 2]) says that if A and B are self-adjoint, α_* denotes the version of $\operatorname{Eig} A$ with eigenvalues in decreasing order, and β_* is the same for B, then

$$\alpha_* - \beta_* \ll \operatorname{Eig}(A-B). \tag{1.1}$$

For the early history of this theorem see Bhatia [5, §9 and the notes and references for chap. III]. Many of the earlier proofs require smoothness results on the eigenvalues of the intermediate matrices $(1-t)A + tB$; a recent proof due to Hiai and Nakamura [12] is not encumbered in that way, but rests on a rather intricate interpolation method.

Our approach stresses the relation between (1.1) and norm inequalities. It has long been recognized that (1.1) implies

$$\mu(\operatorname{diag}(\alpha_*) - \operatorname{diag}(\beta_*)) \leqslant \mu(A - B), \tag{1.2}$$

for every strongly unitarily invariant (sui) norm μ (as in [8], we say that a norm μ on the space of matrices is sui if it satisfies $\mu(UTV) = \mu(T)$ for every matrix T and unitary U and V). This implication is traditionally based on the well-known relation between sui norms, symmetric gauge functions and majorization. In fact, however, a comment of Ando (see [1, Theorem 7.1]) makes it clear that we may deduce (1.2) directly for the more general weakly unitarily invariant (wui) norms. A wui norm τ is a norm on the space of matrices that satisfies $\tau(U^* TU) = \tau(T)$ for every T and unitary U. Ando pointed out that if S and T are self-adjoint and $\operatorname{Eig} S \ll \operatorname{Eig} T$ then there are unitary U_k such that S is a convex combination of the $U_k^* T U_k$. It follows that

$$\tau(\operatorname{diag}(\alpha_*) - \operatorname{diag}(\beta_*)) \leqslant \tau(A - B), \tag{1.3}$$

for every wui norm τ.

Our strategy will be to prove (1.3), and more, using our general "normal path inequality" (see §2), then reverse the implications discussed in the last paragraph (via "soft" functional analysis – see §4) to obtain (1.1), and more.

We shall use $\mathbb{N}(n)$ to denote the set of normal operators on complex n-space $\mathbb{C}^n$.

2. The normal path inequality

Given any wui norm τ and matrices T and S, we define the τ-spectral distance between T and S, denoted by $\tau(\operatorname{Eig} T, \operatorname{Eig} S)$, by setting

$$\tau(\operatorname{Eig} T, \operatorname{Eig} S) = \min\{\tau(\operatorname{Eig} T - \operatorname{Eig} S)\}, \tag{2.1}$$

where the minimum is taken over all orderings of the diagonal matrix $\operatorname{Eig} S$; because τ is wui (and permutation matrices are unitary), $\tau(\operatorname{Eig} T - \operatorname{Eig} S)$ depends only on the relative ordering of the two diagonal matrices and the spectral distance is a pseudometric.

In [8] we showed that the spectral distance between any two normal operators, measured by a fixed wui norm τ, is bounded by the τ-length of any normal path joining the operators. For our present purposes it is important to observe that, for a fixed path, the matching of eigenvalues may be determined by "following the path" so that the matching can be the same for all wui norms τ. To see this we first recall a key lemma from [8].

Lemma 1 (Proposition 5.2 of [8]). *For fixed* wui *norm* τ, *normal* N_0, *and* $\varepsilon > 0$,

$$\tau(\operatorname{Eig} N_1, \operatorname{Eig} N_0) \leqslant (1 + \varepsilon)\tau(N_1 - N_0) \tag{2.2}$$

whenever N_1 *is normal and sufficiently close to* N_0.

Remark. In [8] we discussed certain situations where $\varepsilon = 0$ is appropriate in the foregoing lemma, relating this phenomenon to some work of Halmos and Bouldin; in

§7 below, we identify a broad class of norms exhibiting this behaviour.

In the following proposition $\tau(N(\cdot))$ denotes the arc-length of a curve $N(t)$ (t in some parametric interval) measured via the metric induced by the (wui) norm τ. In cases of interest this will be finite (i.e. the curve will be τ-rectifiable). In view of the proposition it is important to determine how short this arc-length can be made by a suitable choice of normal path; in [7] there are results of this type.

PROPOSITION 2

If $N(\cdot)$ *is a path defined on* $[0,1]$ *and with values in* $\mathbb{N}(n)$ *then there is a fixed ordering of* $\mathrm{Eig}\,N(0)$ *and* $\mathrm{Eig}\,N(1)$ *such that, for all* wui *norms* τ

$$\tau(\mathrm{Eig}\,N(1) - \mathrm{Eig}\,N(0)) \leqslant \tau(N(\cdot)). \tag{2.3}$$

Proof. Since $N(t)$ varies continuously with t, standard results on spectral continuity (see, for example, [13, § 5.2 of chap. 2]) ensure that there are continuous functions $\mu_k(t)$ such that $\{\mu_1(t),\ldots,\mu_n(t)\}$ is the spectrum (with multiplicity) of $N(t)$. We claim that, for any such system of functions, (2.3) is satisfied when we choose $\mu_1(1),\ldots,\mu_n(1)$ as the ordering for the eigenvalues in $\mathrm{Eig}\,N(1)$ and $\mu_1(0),\ldots,\mu_n(0)$ as the ordering for $\mathrm{Eig}\,N(0)$.

Fix a wui norm τ and $\varepsilon>0$, and let G be the set of $t\in[0,1]$ such that $\tau(D(t)-D(0))\leqslant(1+\varepsilon)\tau(N[0,t])$, where $D(t)$ denotes the diagonal matrix with $\mu_k(t)$ as the kth diagonal entry and $N[s,t]$ denotes the part of the path $N(\cdot)$ defined on $[s,t]$. By continuity it is clear that G includes its supremum (maximum) g. We wish to show that $g=1$. If this is not the case. Lemma 1 ensures that for $g'>g$ and sufficiently close to g there is an ordering of $\mathrm{Eig}\,N(g')$ such that

$$\tau(D(g)-\mathrm{Eig}\,N(g'))\leqslant(1+\varepsilon)\tau(N(g)-N(g')). \tag{2.4}$$

Let $\alpha_1,\ldots,\alpha_m$ be the distinct eigenvalues of $D(g)$. There is some permutation σ such that $\mathrm{Eig}\,N(g')=\sigma D(g')$ (or, more properly, the unitary similarity corresponding to σ applied to $D(g')$). Since both sides of (2.4) tend to 0 as g' approaches g, the continuity of the functions $\mu_k(\cdot)$ forces $\tau(\mathrm{Eig}\,N(g')-D(g'))$ to be small relative to the minimum distance between the α_k, for g' sufficiently close to g. Thus, if $\sigma(i)=j$, both i and j must index eigenvalues of $D(g')$ close to the same α_k; in other words, $\sigma D(g)=D(g)$ $(=\sigma^{-1}D(g))$. Since τ is wui,

$$\tau(D(g)-D(g'))=\tau(\sigma^{-1}D(g)-D(g'))=\tau(D(g)-\sigma D(g')), \tag{2.5}$$

so that (2.4) yields

$$\tau(D(g)-D(g'))\leqslant(1+\varepsilon)\tau(N(g)-N(g')). \tag{2.6}$$

Combining this inequality with

$$\tau(D(0)-D(g))\leqslant(1+\varepsilon)\tau(N[0,g]), \tag{2.7}$$

we obtain

$$\tau(D(0)-D(g'))\leqslant(1+\varepsilon)\tau(N[0,g']), \tag{2.8}$$

so that $g' \in G$; this contradiction shows that $g = 1$. Since the inequality

$$\tau(D(0) - D(1)) \leqslant (1 + \varepsilon)\tau(N(\cdot)) \tag{2.9}$$

holds for all $\varepsilon > 0$ and all wui norms τ, our claim is established.

q.e.d.

Remark. The significance of the fixed matching of eigenvalues in the result above should perhaps be stressed. In general the matching that minimizes $\tau(\mathrm{Eig}\, T - \mathrm{Eig}\, S)$ depends on τ; examples of this phenomenon (where T is self-adjoint and S skew-adjoint) are discussed in [2].

3. A Lidskii theorem for normal operators

We have already discussed in § 1 the general strategy for proving the following theorem; the details of the proof may be found in § 5 below.

Theorem 3. *For any $T, S \in \mathbb{N}(n)$ such that $T - S$ is also normal, there is an ordering of Eig T and Eig S such that*

$$\mathrm{Eig}\, T - \mathrm{Eig}\, S \ll \mathrm{Eig}\,(T - S). \tag{3.1}$$

The restrictions imposed by the hypotheses of this theorem are discussed in § 6 below.

COROLLARY 4 (*Lidskii's theorem*)

If A and B are self-adjoint then

$$\alpha_* - \beta_* \ll \mathrm{Eig}\,(A - B), \tag{3.2}$$

where α_* is the version of Eig A *with the eigenvalues arranged in decreasing order, and β_* has the same relationship to B.*

Proof. The difference $A - B$ is again self-adjoint, hence normal. It is a well-known fact (see [15, chap. 6, § A]) about the partial ordering $\ll$ that for any real vectors α and β, $\alpha_* - \beta_* \ll \alpha - \beta$.

q.e.d.

The following result was proved by Sunder [17] for the case of sui norms and by a different, rather intricate, argument.

COROLLARY 5

If $T, S, T - S \in \mathbb{N}(n)$ then for each wui *norm τ there is an ordering of* Eig T *and* Eig S *(which may depend on τ) such that*

$$\tau(T - S) \leqslant \tau(\mathrm{Eig}\, T - \mathrm{Eig}\, S). \tag{3.3}$$

Proof. Apply the theorem with T replaced by $T - S$ and S by $-S$ to get an ordering such that

$$\mathrm{Eig}(T-S)+\mathrm{Eig}(S)=\Sigma t_k\sigma_k(\mathrm{Eig}\,T), \tag{3.4}$$

where the σ_k are permutations and $t_k \geqslant 0$ with $\Sigma t_k = 1$. Then

$$\mathrm{Eig}(T-S)=\Sigma t_k(\sigma_k(\mathrm{Eig}\,T)-\mathrm{Eig}\,S) \tag{3.5}$$

so that $\tau(T-S)\leqslant \Sigma t_k\tau(\sigma_k(\mathrm{Eig}\,T)-\mathrm{Eig}\,S)\leqslant \max_k \tau(\sigma_k(\mathrm{Eig}\,T)-\mathrm{Eig}\,S)$.

q.e.d.

Remark. For self-adjoint matrices the reasons for the inequality (3.3) may be identified more precisely. If A and B are self-adjoint, α_* is as in Corollary 4, and β^* puts Eig B in increasing order, then

$$\mathrm{Eig}(A-B)\ll \alpha_*-\beta^*, \tag{3.6}$$

as has been explicitly noted in [6]. By well-known properties of sui norms τ, it follows that

$$\tau(A-B)\leqslant \tau(\alpha_*-\beta^*); \tag{3.7}$$

actually, Ando's observation (see § 1) shows directly that (3.7) holds more generally for wui norms.

4. Software

PROPOSITION 6

For any operators X and Y on $\mathbb{C}^n$, the statements

$$\tau(X)\leqslant\tau(Y) \textit{ for every wui norm } \tau \tag{4.1}$$

and

$$X=\Sigma z_k U_k^* Y U_k \textit{ (finite sum) for some unitary operators } U_k \textit{ and complex } z_k \textit{ with } \Sigma|z_k|\leqslant 1 \tag{4.2}$$

are equivalent.

Proof. It is immediate from the definition of wui norms that (4.1) follows from (4.2). For the converse, suppose that (4.2) fails, i.e. X is not in the set K consisting of all finite sums of the type described in (4.2). Then for some $\varepsilon>0$ X is also outside the set K_ε defined by $K_\varepsilon = K+\{Z: \|Z\|\leqslant\varepsilon\}$. Let τ be the Minkowski functional corresponding to K_ε; K_ε is convex, absorbing, bounded, circled, and invariant under unitary similarities, so that τ is a wui norm. By construction $\tau(X)>1\geqslant\tau(Y)$.

q.e.d.

COROLLARY 7

For any operators X and Y on $\mathbb{C}^n, \tau(X)=\tau(Y)$ for all wui *norms τ exactly when $X=\exp(i\theta)U^*YU$ for some unitary U and real θ.*

Proof. If $\tau(X)=\tau(Y)$ for all wui norms τ, Proposition 6 ensures that X may be expressed in the form (4.2). Collect together any linearly dependent summands (dependent $U_k^* Y U_k$ and $U_j^* Y U_j$ would have to differ by a factor $\exp(i\alpha)$). Certain wui norms are strictly convex (e.g. the Hilbert–Schmidt (or Frobenius) norm is sui and induces an Euclidian metric on the space of matrices). For such a norm we cannot have $\tau(X)=\tau(Y)$ unless there is only a single summand in the expression for X. Clearly we must also have $|z_1|=1$.

q.e.d.

Remark. This corollary makes it clear that wui norms can behave quite differently from the more familiar sui norms. For example, $\tau(T^*)$ may differ from $\tau(T)$ for a wui norm τ (even when T is normal). Let T be a unitary of dimension 3 or more. The eigenvalues of T^* are the complex conjugates of those for T so that one spectrum cannot be obtained from the other by a rotation of the unit circle (excluding certain very special geometries for the spectra). We cannot, therefore, have $T^* = \exp(i\theta)U^*TU$ as in the corollary, so there is some wui norm τ with different values at T and T^*. Another such phenomenon occurs with positive definite matrices. If τ is a sui norm it is not hard to see that $\tau(A)\geqslant\tau(B)$ whenever $A\geqslant B\geqslant 0$. This is not the case for wui norms, in general; consider the wui norm τ defined in terms of the numerical radius $w(T)$ and numerical range $W(T)$ (both invariant under unitary similarities) by $\tau(T)=w(T)+\operatorname{diam}(W(T))$. If the spectrum of B is more dispersed than that of A it can certainly happen that $\tau(B)$ exceeds $\tau(A)$.

Remark. We have recently seen the preprint by Li and Tsing [14], who also developed "software" similar to Proposition 6 and Corollary 7.

PROPOSITION 8

For any $N, M \in \mathbb{N}(n)$, the statements

$$\tau(N)\leqslant\tau(M) \quad \textit{for every wui norm } \tau \tag{4.3}$$

and

$$\operatorname{Eig} N \ll_s \operatorname{Eig} M \tag{4.4}$$

are equivalent.

Proof. Since N and M are normal, they are unitarily equivalent to the matrices $\operatorname{Eig} N$ and $\operatorname{Eig} M$ respectively. Thus Proposition 6 tells us that (4.3) is equivalent to

$$\operatorname{Eig} N = \Sigma z_k U_k^* \operatorname{Eig} M U_k \text{ (finite sum) for some unitary operators } U_k \text{ and complex } z_k \text{ with } \Sigma|z_k|\leqslant 1. \tag{4.5}$$

It is easy to check that the diagonal entries of $U^*\operatorname{diag}(\beta)U$, where β is any complex n-vector and U is a unitary matrix $[u_{ij}]$, are given by the vector $D\beta$ with $D=[|u_{ji}|^2]$. Since U is unitary, the corresponding D is doubly stochastic. Thus by equating diagonals in (4.5) we obtain

$$\operatorname{Eig} N = \Sigma z_k D_k \operatorname{Eig} M \text{ (finite sum) for some doubly stochastic matrices } D_k \text{ and complex } z_k \text{ with } \Sigma|z_k|\leqslant 1; \tag{4.6}$$

note that in (4.6) we regard $\operatorname{Eig} N$ and $\operatorname{Eig} M$ as vectors. By a well-known theorem of Garrett Birkhoff (see, e.g. [1] or [15]) each doubly stochastic matrix is a convex combination of permutation matrices so that (4.6) is equivalent to (4.4). On the other hand, (4.4) directly implies (4.5) with permutation matrices as the U_k.

q.e.d.

5. Proof (soft) of Theorem 3

It is easy to check that if $T, S, T-S \in \mathbb{N}(n)$ then the direct path $N(t) = T + t(S-T)$ lies entirely in $\mathbb{N}(n)$. Applying Proposition 2 to this path we see that there is an ordering for $\operatorname{Eig} T$ and $\operatorname{Eig} S$ such that $\tau(\operatorname{Eig} T - \operatorname{Eig} S) \leqslant \tau(T-S)$ $(= \tau(N(\cdot)))$ for all wui norms τ. Applying Proposition 8 with $N = \operatorname{Eig} T - \operatorname{Eig} S$ and $M = T - S$ we conclude that $\operatorname{Eig} T - \operatorname{Eig} S \ll_s \operatorname{Eig}(T-S)$. By wiggling (e.g. replacing T by $T + \varepsilon I$ for small ε) we many assume that T and S have different traces. Then since the components of the vectors $\operatorname{Eig} T - \operatorname{Eig} S$ and $\operatorname{Eig}(T-S)$ have the same non-zero sum, we must have majorization rather than soft majorization.

q.e.d.

6. The condition $T-S$ normal; relation to the classical Lidskii theorem

It does not seem easy to clarify the domain of Theorem 3, i.e. to understand the structure of those pairs T and S such that T, S, and $T-S \in \mathbb{N}(n)$. Certainly they include pairs of the form

$$T = \oplus(z_k A_k + w_k), \quad S = \oplus(z_k B_k + v_k), \tag{6.1}$$

where the space is decomposed into a finite sum of orthogonal subspaces H_k. The operators A_k, B_k are self-adjoint on H_k and z_k, w_k, v_k are complex scalars. Note that it would be possible to obtain Theorem 3 for pairs of the form (6.1) by repeated applications of the classical Lidskii theorem (for self-adjoints). In dimensions three and up, examples show that Theorem 3 applies to a wider class of pairs, so that we have a more substantial extension of Lidskii's result as well as a new approach. In the two-dimensional case, however, a calculation shows that T, S, and $T-S$ are normal only when a representation of the form (6.1) exists.

7. Spectral variation in Q-norms

Following the terminology in [3] we shall say that a sui norm τ is a Q-norm if there exists another sui norm τ' such that for every A we have

$$\tau(A) = [\tau'(|A|^2)]^{1/2}, \tag{7.1}$$

where $|A|^2 = A^*A$. A Schatten p-norm is a Q-norm iff $p \geqslant 2$. The class of Q-norms, however, includes other interesting norms as well. For instance, if $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ is an enumeration of the singular values of an n-by-n matrix A, then for each $k = 1, 2, \ldots, n$ and $1 \leqslant p < \infty$ the expression

$$\|A\|_{k,p} = \left[\sum_{j=1}^{k} (s_j(A))^p \right]^{1/p} \tag{7.2}$$

defines a sui norm on matrices (see, e.g. [16]). If $p \geqslant 2$, then for each $k = 1, 2, \ldots, n$ the norm defined by (7.2) is a Q-norm. To see this simply note that for such p we have

$$\|A\|_{k,p} = (\| |A|^2 \|_{k,p/2})^{1/2} \tag{7.3}$$

so that the requirement (7.1) is satisfied. The Schatten p-norms are a special case of (7.2) for $k = n$.

In some recent work it has been observed that in the derivation of several operator inequalities the "quadratic character" (7.1) of Q-norms plays a special role. See [3], [2] for material directly related to our present discussion and [9], [4] for other operator inequalities involving Q-norms.

The purpose of this section is to point out the following; the result of Halmos [11] on spectral approximants of normal operators, established by him for the operator norm, was extended to the class of Schatten p-norms, $p \geqslant 2$, by Bouldin [10]; in [3] this result has been extended further to the wider class of Q-norms. Our results in [8, §4] made use of the above mentioned results of Halmos and Bouldin. So, by much the same arguments, they can now be extended to the class of Q-norms. In particular, we have the following result on spectral variation.

PROPOSITION 9

Let $A, B \in \mathbb{N}(n)$ and let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be the respective eigenvalues of A and B. Suppose there is a permutation σ such that

$$|\alpha_i - \beta_{\sigma(i)}| \leqslant |\alpha_i - \beta_{\sigma(j)}| \tag{7.4}$$

for all i, j. Then for every Q-norm $\|\cdot\|_Q$ we have

$$\|\operatorname{diag}(\alpha_i - \beta_{\sigma(i)}\|_Q \leqslant \|A - B\|_Q. \tag{7.5}$$

Proof. Follow the model in [8, § 4].

As noted in [8, Proposition 4.2], if the normal matrices A and B are sufficiently close their eigenvalues do meet the condition (7.4). Thus when τ is a Q-norm, Lemma 1 in § 2 can be strengthened; its conclusion is true even when $\varepsilon = 0$. The proof of Proposition 2 can therefore be simplified somewhat in the case of a Q-norm.

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