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Loewner Matrices and Operator Convexity

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Abstract Let f be a function from $\mathbb{R}_+$ into itself. A classic theorem of K. Löwner says that f is operator monotone if and only if all matrices of the form $\left[\frac{f(p_i)-f(p_j)}{p_i-p_j}\right]$ are positive semidefinite. We show that f is operator convex if and only if all such matrices are conditionally negative definite and that $f(t) = tg(t)$ for some operator convex function g if and only if these matrices are conditionally positive definite. Elementary proofs are given for the most interesting special cases $f(t) = t^r$, and $f(t) = t \log t$. Several consequences are derived.

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1 Introduction

Let f be a continuously differentiable function from the interval $(0, \infty)$ into itself, with the additional property $f(0) = \lim_{t \to 0^+} f(t) = 0$. Given any n distinct points $p_1, \dots, p_n$ in $(0, \infty)$, let $L_f(p_1, \dots, p_n)$ be the $n \times n$ matrix defined as

$$L_f(p_1, \dots, p_n) = \left[\frac{f(p_i) - f(p_j)}{p_i - p_j} \right], \tag{1.1}$$

with the understanding that when $i = j$ the quotient in (1.1) is interpreted as $f'(p_i)$. (We use the notation $[a_{ij}]$ to mean a matrix whose entries are a_{ij}.) Such a matrix is called a *Loewner matrix* associated with f.

Of particular interest to us is the function $f(t) = t^r$ where $r > 0$. We use the symbol L_r for a Loewner matrix associated with this function. Thus

$$L_r = \left[\frac{p_i^r - p_j^r}{p_i - p_j} \right], \tag{1.2}$$

assuming that the n distinct points $p_1, \dots, p_n$ have been chosen and fixed.

The function f is said to be *operator monotone* on $[0, \infty)$ if for two positive semidefinite matrices A and B (of any size n) the inequality $A \geqq B$ implies $f(A) \geqq f(B)$. Here, as usual, $A \geqq B$ means that $A - B$ is positive semidefinite (p.s.d. for short).

In 1934 Karl Löwner (later Charles Loewner) wrote one of the most fundamental papers in matrix analysis [22]. One principal result of this paper is that f is operator monotone if and only if for all n, and all $p_1, \dots, p_n$, the Loewner matrices $L_f(p_1, \dots, p_n)$ are p.s.d. Another major result is that the function $f(t) = t^r$ is operator monotone if and only if $0 < r \leqq 1$. Consequently, if $0 < r \leqq 1$, then the matrix (1.2) is p.s.d., and therefore all its eigenvalues are non-negative.

Closely related to operator monotone functions are operator convex functions. Assume that f is a C^2 function from $(0, \infty)$ into itself, $f(0) = 0$ and $f'(0) = 0$. We say that f is *operator convex* if

$$f\left((1 - t)A + tB\right) \leqq (1 - t)f(A) + tf(B), \quad 0 \leqq t \leqq 1,$$

for all p.s.d. matrices A and B (of any size n).

Following Loewner's seminal work there have been several studies of these two classes of functions; see in particular [1, 7, 8, 15, 16, 17, 19, 20]. The emphasis of the present paper is on Loewner matrices, their spectral properties, and their role in characterising operator convexity.

Along with p.s.d. matrices we consider conditionally positive definite and conditionally negative definite matrices. Let H^n be the subspace of $\mathbb{C}^n$ consisting of all $x = (x_1, \dots, x_n)$ for which $\sum_{i=1}^{n} x_i = 0$. An $n \times n$ Hermitian matrix A is said to be *conditionally positive definite* (c.p.d. for short) or *almost positive* if

$$\langle x, Ax \rangle \geqq 0 \quad \text{for all} \quad x \in H^n,$$

and *conditionally negative definite* (c.n.d. for short) if $-A$ is c.p.d. We refer the reader to [5, 16, 19] for properties of these matrices.

We will prove the following:

Theorem 1.1. Let f be an operator convex function. Then all Loewner matrices associated with f are conditionally negative definite.

One of the interesting relations between operator monotone and convex functions is that $f(t)$ is operator convex on $[0, \infty)$ if and only if $g(t) = f(t)/t$ is operator monotone on $(0, \infty)$. This plays an important role in the analysis of [17]. The class of functions $f(t) = tg(t)$ where g is operator convex seems equally interesting in this context, as evidenced by our next theorem.

Theorem 1.2. Let $f(t) = tg(t)$ where g is an operator convex function. Then all Loewner matrices associated with f are conditionally positive definite.

Since the function $f(t) = t^r, 1 \leqq r \leqq 2$ is operator convex, part (i) of the following theorem is a corollary of Theorems 1.1 and 1.2. We state it as a separate theorem because all the essential ideas of the proof are contained in this special case.

Theorem 1.3. Let L_r be the $n \times n$ Loewner matrix (1.2) associated with distinct points $p_1, \ldots, p_n$. Then
(i) L_r is conditionally negative definite for $1 \leqq r \leqq 2$, and conditionally positive definite for $2 \leqq r \leqq 3$.
(ii) L_r is nonsingular for $1 < r < 2$ and for $2 < r < 3$.
(iii) As a consequence, for $1 < r < 2$ the matrix L_r has one positive and $n-1$ negative eigenvalues, and for $2 < r < 3$ it has one negative and $n-1$ positive eigenvalues.

Part (iii) of this theorem extends a theorem of Bhatia and Holbrook [11] and reveals the underlying cause for it. Motivated by some questions in perturbation analysis [14] they examined the matrices $L_r, 1 \leqq r \leqq 2$, and showed that in this case L_r has exactly one positive eigenvalue. Their proof was based on the fact that the function $f(t) = t^r$ is operator convex for r in this range, and they noted that their result is valid for the matrices L_f associated with all such functions. Theorem 1.3 makes the "why" of this apparent.

It is natural to ask whether the converse of Theorems 1.1 and 1.2 is true. It is, and we prove:

Theorem 1.4. Let f be a C^2 function from $(0, \infty)$ into itself with $f(0) = f'(0) = 0$. Suppose all Loewner matrices L_f are conditionally negative definite. Then f is operator convex.

Theorem 1.5. Let f be a C^3 function from $(0, \infty)$ into itself with $f(0) = f'(0) = f''(0) = 0$. Suppose all Loewner matrices L_f are conditionally positive definite. Then there exists an operator convex function g such that $f(t) = tg(t)$.

Löwner showed that f is operator monotone if and only if it has an analytic continuation mapping the upper half plane into itself. Conditions for this continuation to be a one-to-one map were found by R. Horn [18]. Theorems 1.2 and 1.5 are just a step away from some results in that paper. The emphasis there is on complex mapping properties and the connection with operator convex functions is not made.

In Section 2 we give a proof of Theorem 1.3 that is elementary and independent of the general theory of operator convex functions. We then show how it can be extended to the general case of Theorems 1.1 and 1.2. A few applications and connections with some important theorems of matrix theory are given in Section 3.

2 Proofs

Proof of Theorem 1.3. For $t > 0$ and $0 < r < 1$ we have the well known formula [8, p.116]

$$t^r = \frac{\sin r\pi}{\pi} \int_0^\infty \frac{t}{\lambda + t}\, \lambda^{r-1}\, d\lambda. \tag{2.1}$$

For our purpose it is convenient to abbreviate this as

$$t^r = \int_0^\infty \frac{t}{\lambda + t}\, d\mu(\lambda), \quad 0 < r < 1, \tag{2.2}$$

where μ is a positive measure on $(0, \infty)$. For each $\lambda > 0$ let

$$h_\lambda(t) = \frac{t}{\lambda + t}.$$

An $n \times n$ Loewner matrix corresponding to this function has as its (i, j) entry

$$\frac{\frac{p_i}{\lambda + p_i} - \frac{p_j}{\lambda + p_j}}{p_i - p_j},$$

which on simplification reduces to

$$\frac{\lambda}{(\lambda + p_i)(\lambda + p_j)}.$$

If E is the matrix with all its entries equal to 1, and D_λ is the diagonal matrix diag $\left(\frac{1}{\lambda + p_1}, \ldots, \frac{1}{\lambda + p_n}\right)$, then we can express the Loewner matrix above as

$$L_{h_\lambda}(p_1, \ldots, p_n) = \lambda D_\lambda E D_\lambda.$$

Clearly this matrix is p.s.d. The integral in (2.2) is a limit of positive linear combinations of h_λ, and hence the Loewner matrix $L_r(p_1, \ldots, p_n)$ is p.s.d. for every r in $(0, 1)$.

The simple idea behind this computation can be carried further. From (2.2) we obtain the formula

$$t^r = \int_0^\infty \frac{t^2}{\lambda + t}\, d\mu(\lambda), \quad 1 < r < 2. \tag{2.3}$$

Now for each $\lambda > 0$, let

$$g_\lambda(t) = \frac{t^2}{\lambda + t}.$$

Using the identity

$$\frac{1}{a-b}\left(\frac{a^2}{\lambda+a} - \frac{b^2}{\lambda+b}\right) = 1 - \frac{\lambda^2}{(\lambda+a)(\lambda+b)},$$

one can see that

$$L_{g_\lambda}(p_1, \ldots, p_n) = E - \lambda^2 D_\lambda E D_\lambda,$$

where D_λ is the diagonal matrix defined earlier. If x is any vector in the space H^n, then $Ex = 0$. The matrix $D_\lambda E D_\lambda$ is p.s.d., and hence $\langle x, D_\lambda E D_\lambda x\rangle \geqq 0$ for every x. This shows that $\langle x, L_{g_\lambda} x\rangle \leqq 0$ for $x \in H^n$, and L_{g_λ} is c.n.d. The integral in (2.3) is a limit of positive linear combinations of g_λ, and hence the Loewner matrix L_r is c.n.d. for $1 < r < 2$.

The next case is slightly more intricate. We have

$$t^r = \int_0^\infty \frac{t^3}{\lambda + t}\, d\mu(\lambda), \quad 2 < r < 3. \tag{2.4}$$

For each $\lambda > 0$, let

$$f_\lambda(t) = \frac{t^3}{\lambda + t}.$$

Simple algebraic manipulations show that

$$\begin{aligned} \frac{1}{a-b}\left(\frac{a^3}{\lambda+a} - \frac{b^3}{\lambda+b}\right) &= \frac{\lambda(a^2+ab+b^2) + ab(a+b)}{(\lambda+a)(\lambda+b)} \\ &= a + b - \frac{\lambda^2(a+b) + \lambda ab}{(\lambda+a)(\lambda+b)} \\ &= a + b - \lambda + \frac{\lambda^3}{(\lambda+a)(\lambda+b)}. \end{aligned}$$

Using this one sees that the Loewner matrix for f_λ can be expressed as

$$L_{f_\lambda}(p_1, \ldots, p_n) = DE + ED - \lambda E + \lambda^3 D_\lambda E D_\lambda,$$

where $D = \operatorname{diag}(p_1, \ldots, p_n)$, and D_λ is the diagonal matrix defined earlier. The matrix $D_\lambda E D_\lambda$ is p.s.d., and for all $x \in H^n$ we have $Ex = 0$ and $\langle x, EDx\rangle = \langle Ex, Dx\rangle = 0$. Thus $\langle x, L_{f_\lambda} x\rangle \geqq 0$ for all $x \in H^n$, and L_{f_λ} is a c.p.d. matrix. As before, it follows that L_r is c.p.d. for $2 < r < 3$.

Note that $L_1 = E$, $L_2 = [p_i + p_j] = DE + ED$, and $L_3 = [p_i^2 + p_i p_j + p_j^2] = D^2E + DED + ED^2$. This shows that L_1 is both p.s.d. and c.n.d. Since $\langle x, L_2 x\rangle = 0$ for all $x \in H^n$, the matrix L_2 is both c.n.d. and c.p.d. Similarly, $\langle x, L_3 x\rangle \geqq 0$ for all $x \in H^n$, and hence L_3 is a c.p.d. matrix.

This completes the proof of part (i). For part (ii) we again use the integrals (2.2)-(2.4). We include the case $0 < r < 1$ in our discussion.

Let x be any element of $\mathbb{C}^n$. Then our analysis for the case $0 < r < 1$ shows that

$$\begin{aligned}
\langle x, L_r x\rangle &= \int_0^\infty \langle x, D_\lambda E D_\lambda x\rangle \, d\mu(\lambda) \\
&= \int_0^\infty \sum_{i,j=1}^n \frac{\lambda}{(\lambda+p_i)(\lambda+p_j)} x_i \overline{x}_j \, d\mu(\lambda) \\
&= \int_0^\infty \lambda \left| \sum_{i=1}^n \frac{x_i}{\lambda+p_i} \right|^2 d\mu(\lambda).
\end{aligned}$$

This expression is equal to zero if and only if

$$\sum_{i=1}^n \frac{x_i}{\lambda+p_i} = 0$$

for almost every $\lambda > 0$. Since the functions $\varphi_i(\lambda) = \dfrac{1}{\lambda+p_i}, 1 \leqq i \leqq n$ on $[0,\infty)$ are linearly independent, this can happen if and only if $x_i = 0$ for all i. This means that L_r is nonsingular.

When $r > 1$ the matrix L_r is not p.s.d. Since all its entries are positive it cannot be negative semidefinite either. Once we know that L_r is c.n.d. or c.p.d., then the idea of the preceding paragraph works. According to Lemma 4.3.5 in [5] if A is a matrix which is c.p.d. but not p.s.d., then A is nonsingular if and only if for $x \in H^n$ we have $\langle x, Ax\rangle = 0$ only when $x = 0$. For $1 < r < 2$ we have seen that

$$L_r = \int_0^\infty \left(E - \lambda^2 D_\lambda E D_\lambda\right) \, d\mu(\lambda).$$

So for $x \in H^n$ we have

$$\begin{aligned}
\langle x, L_r x\rangle &= -\int_0^\infty \lambda^2 \langle x, D_\lambda E D_\lambda x\rangle \, d\mu(\lambda) \\
&= -\int_0^\infty \lambda^2 \left| \sum_{i=1}^n \frac{x_i}{\lambda+p_i} \right|^2 d\mu(\lambda).
\end{aligned}$$

If this is 0, then we must have $x = 0$ by the same argument as we used for the case $0 < r < 1$. Thus L_r is nonsingular.

For $2 < r < 3$ we have shown that

$$L_r = \int_0^\infty \left(DE + ED - \lambda E + \lambda^3 D_\lambda E D_\lambda\right) \, d\mu(\lambda).$$

Hence for $x \in H^n$ we have

$$\langle x, L_r x\rangle = \int_0^\infty \lambda^3 \left| \sum_{i=1}^n \frac{x_i}{\lambda+p_i} \right|^2 d\mu(\lambda).$$

This is 0 if and only if $x = 0$. Again the conclusion is that L_r is nonsingular. This proves part (ii).

To prove part (iii) we observe that for $1 < r < 2$, the matrix L_r is c.n.d., nonsingular, and has positive entries. Hence, it must have one positive and $n-1$ negative eigenvalues. See e.g. [5] Corollary 4.1.5. For $2 < r < 3$ the role of negative and positive is reversed. This completes the proof of Theorem 1.3. ■

We emphasize that we have used nothing from the general theory of operator monotone and convex functions in the proof above. The integral (2.1) is a standard one derived using contour integration. If we drop the conditions $f(0) = f'(0) = 0$ and $f(t) > 0$ from the definition, the general theory tells us that an operator convex function on $[0, \infty)$ has the form

$$f(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{t^2}{\lambda + t}\, d\nu(\lambda), \tag{2.5}$$

where α, β, γ are real numbers with $\gamma \geqq 0$, and ν is a positive measure on $(0, \infty)$. See [8, p.147]. The Loewner matrix corresponding to such an f is also c.n.d. This is seen by observing that the Loewner matrix corresponding to the function $g(t) = \alpha + \beta t + \gamma t^2$ is $\beta E + \gamma(DE + ED)$, and this is c.n.d. In particular, this proves Theorem 1.1.

In the same way if $f(t) = tg(t)$ where g is operator convex on $[0, \infty)$ with no special conditions imposed on its values at 0, then f has the form

$$f(t) = \alpha t + \beta t^2 + \gamma t^3 + \int_0^\infty \frac{t^3}{\lambda + t}\, d\nu(\lambda), \tag{2.6}$$

where $\gamma \geqq 0$. The Loewner matrix corresponding to $g(t) = \alpha t + \beta t^2 + \gamma t^3$ is $\alpha E + \beta(DE + ED) + \gamma(D^2E + DED + ED^2)$. This is a c.p.d. matrix. Hence every Loewner matrix corresponding to the function f in (2.6) is also c.p.d. In particular, this proves Theorem 1.2.

The function $f(t) = t \log t$ is especially important because of its connections with classical and quantum entropy. It is known to be operator convex on $[0, \infty)$, and the corresponding Loewner matrices L_f are c.n.d. This fact can also be derived in a simple way from Theorem 1.3. For each $x \in H^n$, and $1 < r \leqq 2$ we have

$$\langle x, (L_r - L_1)x \rangle \leqq 0.$$

Divide by $r - 1$, let $r \to 1^+$, and use the fact that

$$\lim_{r \to 1^+} \frac{t^r - t}{r - 1} = t \log t,$$

to get from the inequality above

$$\langle x, L_f\, x \rangle \leqq 0 \quad \text{for all} \ \ x \in H^n.$$

In other words L_f is c.n.d.

Proof of Theorem 1.4. We exploit the connection between c.n.d. and p.s.d. matrices and that between operator convex and monotone functions. If A is an $n \times n$ c.p.d. matrix, then the $(n-1)\times(n-1)$ matrix B with entries

$$b_{ij} = a_{ij} - a_{in} - a_{nj} + a_{nn} \tag{2.7}$$

is p.s.d. See [5, p.193] or [16, p.134]. Choose $p_1,\ldots,p_n$ in $(0,\infty)$ with $p_n = \varepsilon > 0$. Then the matrix $B(\varepsilon)$ associated with $L_f(p_1,\ldots,p_n)$ via the prescription (2.7) has entries

$$b_{ij}(\varepsilon) = \frac{f(p_i)-f(p_j)}{p_i-p_j} - \frac{f(p_i)-f(\varepsilon)}{p_i-\varepsilon} - \frac{f(p_j)-f(\varepsilon)}{p_j-\varepsilon} + f'(\varepsilon).$$

Since L_f is c.n.d. the matrix $-B(\varepsilon)$ is p.s.d. Let $\varepsilon \downarrow 0$; then $B(\varepsilon)$ converges to the matrix B with entries

$$b_{ij} = \frac{p_j^2 f(p_i) - p_i^2 f(p_j)}{p_i(p_i-p_j)p_j}, \tag{2.8}$$

and $-B$ is p.s.d. Let D be the diagonal matrix with entries $p_i/f(p_i)$, $1 \leqq i \leqq n-1$ on its diagonal. Then the matrix

$$-DBD = \left[\frac{p_i^2/f(p_i) - p_j^2/f(p_j)}{p_i-p_j}\right]$$

is p.s.d. But this is a Loewner matrix associated with the function $g(t) = t^2/f(t)$. The positive definiteness of all such matrices implies that the function $g(t)$ is operator monotone on $(0,\infty)$. Since g is operator monotone on $(0,\infty)$, so is the function $t/g(t) = f(t)/t$. See [17] Corollary 2.6. This, in turn implies that f is operator convex, by a theorem of Bendat and Sherman [7]; see also [17] Theorem 2.4. ■

Proof of Theorem 1.5. The argument is similar to that in the preceding proof. If the Loewner matrices L_f are c.p.d. then every $(n-1)\times(n-1)$ matrix B given in (2.8) is p.s.d. Let D be the diagonal matrix with entries $1/p_i$, $1 \leqq i \leqq n-1$ on its diagonal. Then the matrix

$$DBD = \left[\frac{f(p_i)/p_i^2 - f(p_j)/p_j^2}{p_i-p_j}\right]$$

is p.s.d. In other words all Loewner matrices associated with the function $h(t) = f(t)/t^2$ are p.s.d. and consequently this function is operator monotone on $(0,\infty)$. Again by [17], the function $g(t) = th(t)$ is operator convex, and we have $f(t) = tg(t)$. ■

Remark. Theorems 1.1, 1.2, 1.4 and 1.5 together say the following. Let f be a C^3 function from $(0,\infty)$ into itself with $f(0) = 0$. Let $g(t) = tf(t), h(t) = t^2 f(t)$. Then the following three conditions are equivalent.
(i) All Loewner matrices L_f are p.s.d.
(ii) All Loewner matrices L_g are c.n.d.
(iii) All Loewner matrices L_h are c.p.d.

3 Applications

A matrix A with nonnegative entries a_{ij} is said to be *infinitely divisible* if for every $r > 0$ the *Hadamard power* $A^{\circ r} = [a_{ij}^r]$ is p.s.d. Such matrices have been studied in various contexts in probability and harmonic analysis. See [9], [10, Chapter 5] for several examples of such matrices. According to Loewner's theory all matrices L_f are p.s.d. if and only if f has an analytic continuation mapping the upper half plane into itself. Horn [18] has proved that this continuation is a univalent map on the upper half plane if and only if the matrices L_f are infinitely divisible. Thus the Loewner matrices L_r are infinitely divisible for $0 < r \leqq 1$. Alternate proofs of this fact have been given in [12]. One more can be based on Theorem 1.3.

Theorem 3.1. Every Loewner matrix L_r, $0 < r \leqq 1$, is infinitely divisible.

Proof. By a theorem of R. Bapat [4] (see also [5, Section 4.4], [19, p.458]) if a symmetric matrix A has positive entries and exactly one positive eigenvalue, then the Hadamard reciprocal matrix $[1/a_{ij}]$ is infinitely divisible. So, together our Theorem 1.3 (iii) and Bapat's theorem imply that for $1 \leqq r \leqq 2$, the matrix

$$\left[\frac{p_i - p_j}{p_i^r - p_j^r}\right],$$

is infinitely divisible. This is the same as saying that for $\frac{1}{2} \leqq r \leqq 1$, the matrix

$$\left[\frac{p_i^r - p_j^r}{p_i - p_j}\right],$$

is infinitely divisible. Now suppose $\frac{1}{4} \leqq r \leqq \frac{1}{2}$. We have

$$\left[\frac{p_i^r - p_j^r}{p_i - p_j}\right] = \left[\frac{1}{p_i^r + p_j^r}\right] \circ \left[\frac{p_i^{2r} - p_j^{2r}}{p_i - p_j}\right], \tag{3.1}$$

where $\circ$ stands for the Hadamard (entrywise) product of matrices. The first matrix on the right hand side of (3.1) is a Cauchy matrix and is infinitely divisible [9], [10, p.24]. The second matrix is infinitely divisible since $\frac{1}{2} \leqq 2r \leqq 1$. The Hadamard product of two infinitely divisible matrices is infinitely divisible. Hence the matrix (3.1) is infinitely divisible. The argument can be repeated to complete the proof. ■

Remark. Our method gives one more proof of the infinite divisibility of the Cauchy matrix. We have $[p_i + p_j] = DE + ED$, which is a c.n.d. matrix, and hence by Bapat's theorem $\left[\frac{1}{p_i + p_j}\right]$ is infinitely divisible. The same idea leads to a simple proof of the infinite divisibility of some nice functions. A complex valued function f on $\mathbb{R}$ is said to be *positive definite* if for all n and for all $x_1, \ldots, x_n$ in $\mathbb{R}$ the matrix $[f(x_i - x_j)]$ is p.s.d. If f takes only nonnegative values, then f is called *infinitely divisible* if every matrix $[f(x_i - x_j)]$ is infinitely divisible.

Theorem 3.2. For every $b > 0$ the function

$$g(x) = \frac{1}{b + |x| + |x|^2}$$

on the real line is infinitely divisible.

Proof. Consider the matrix

$$[\varphi(x_i - x_j)] = [b + |x_i - x_j| + |x_i - x_j|^2].$$

If all x_i are replaced by $x_i + \alpha$, then this matrix does not change. So we may assume all x_i are positive. Write the last expression as

$$\varphi(x_i - x_j) = b + x_i + x_j - 2\min(x_i, x_j) + x_i^2 - 2x_i x_j + x_j^2,$$

to obtain

$$[\varphi(x_i - x_j)] = bE + DE + ED - 2M + D^2E - 2DED + ED^2,$$

where $M = [\min(x_i, x_j)]$. This matrix is p.s.d.[9], and so is DED. All the other matrices on the right hand side are c.n.d. So $[\varphi(x_i - x_j)]$ is c.n.d., and therefore $[g(x_i - x_j)]$ is infinitely divisible by Bapat's theorem. ■

Remark. The infinite divisibility of g can be proved in another way. The function $e^{-|x|^a}$ is positive definite for $0 < a \leqq 2$. See [10, p.151]. Hence for every $r > 0$ the function $\left(e^{-|x|^a}\right)^r = e^{-|r^{1/a}x|^a}$ is also positive definite. In other words $e^{-|x|^a}$ is infinitely divisible. Hence by Theorem 6.3.13 in [18] the function $|x|^a$ is c.n.d. for $0 < a \leqq 2$. This shows that if α_j are positive numbers and $0 \leqq a_j \leqq 2$, then

$$g(x) = \frac{1}{\alpha_0 + \alpha_1|x|^{a_1} + \cdots + \alpha_k|x|^{a_k}}$$

is an infinitely divisible function on $\mathbb{R}$. In fact more is true. A famous theorem of Schoenberg [23] says that if $x_1, \ldots, x_n$ are vectors in $\mathbb{R}^d$, $\|\cdot\|$ is the Euclidean norm, and $0 < a \leqq 2$, then the matrix $[\|x_i - x_j\|^a]$ is c.n.d. So the function g defined above with $|\cdot|$ replaced by $\|\cdot\|$ is infinitely divisible. Baxter [6, Lemma 2.9] shows that if $0 < p < 2$ then the matrix $[\|x_i - x_j\|_p^p]$ is c.n.d. So our argument shows that for all positive numbers α_j and $0 \leqq p_j \leqq 2$, the function

$$g(x) = \frac{1}{\alpha_0 + \alpha_1\|x\|_{p_1}^{p_1} + \cdots + \alpha_k\|x\|_{p_k}^{p_k}}$$

on $\mathbb{R}^d$ is infinitely divisible.

Another consequence of our discussion is the following.

Theorem 3.3. Let f be a C^1 function on $(0, \infty)$ and suppose $f'(t) > 0$. If for all $p_1, \ldots, p_n$ the Loewner matrix $L_f(p_1, \ldots, p_n)$ has exactly one positive eigenvalue, then the inverse function $g = f^{-1}$ is operator monotone.

Proof. By the theorem of Bapat mentioned above the matrix

$$\left[\frac{p_i - p_j}{f(p_i) - f(p_j)}\right]$$

is p.s.d. Putting $q_i = f(p_i)$ we see that the matrix

$$\left[\frac{g(q_i) - g(q_j)}{q_i - q_j}\right]$$

is p.s.d. Loewner's theorem then implies that g is operator monotone. ■

Corollary 3.4. Let f be a map from $(0, \infty)$ into itself such that $f(0) = 0$ and f is operator convex. Then the inverse function $g = f^{-1}$ is operator monotone.

Proof. By Theorem 1.1, every Loewner matrix L_f is c.n.d. and, therefore, has exactly one positive eigenvalue. Since $f'(t) > 0$ the assertion follows from Theorem 3.3. ■

Theorem 3.3 in a slightly different form has been proved by Horn [18, Theorem 5]. Corollary 3.4 has also been proved by Ando [2] using a completely different argument. See also [3]. He shows that for every nonnegative operator monotone function g on $(0, \infty)$ the inverse function of $tg(t)$ is operator monotone. By the theorem of Bendat-Sherman that we have used in Section 2 a nonnegative function f on $[0, \infty)$ with $f(0) = 0$ is operator convex if and only if $f(t)/t$ is operator monotone. So Ando's result and our Corollary 3.4 can be derived from each other.

Besides Loewner matrices the matrices

$$K_f(p_1, \ldots, p_n) = \left[\frac{f(p_i) + f(p_j)}{p_i + p_j}\right] \tag{3.2}$$

also have been of some interest. Kwong [21] has shown that if a function f from $[0, \infty)$ into itself is operator monotone, then all K_f are p.s.d. The arguments introduced in the proof of Theorem 1.3 lead to a simple proof of this. If $h_\lambda(t) = t/(\lambda + t)$, then

$$\begin{aligned} K_{h_\lambda}(p_1, p_2, \ldots, p_n) &= \left[\frac{p_i/(\lambda + p_i) + p_j/(\lambda + p_j)}{p_i + p_j}\right] \\ &= \left[\frac{\lambda(p_i + p_j) + 2p_ip_j}{(\lambda + p_i)(p_i + p_j)(\lambda + p_j)}\right] \\ &= \lambda D_\lambda E D_\lambda + 2DD_\lambda C D_\lambda D. \end{aligned}$$

This matrix is p.s.d. for every $\lambda > 0$, and hence so is the matrix K_f for every operator monotone function f. (This has been pointed out earlier in [10, p. 195].) For the functions considered in Theorems 1.1 and 1.2 there is a bit of surprise: the matrices K_f associated with both the classes are c.n.d.

Theorem 3.5. Let f be a real valued function on $[0, \infty)$. Suppose either (i) f is operator convex and $f(0) \leqq 0$, or (ii) $f(t) = tg(t)$ where g is operator convex and $f''(0) \geqq 0$. Then all matrices K_f associated with f are conditionally negative definite.

Proof. We use the integral representations (2.5) and (2.6). Let $g_\lambda(t) = t^2/(\lambda + t)$. Then

$$K_{g_\lambda}(p_1, p_2, \ldots, p_n) = \left[\frac{p_i^2/(\lambda + p_i) + p_j^2/(\lambda + p_j)}{p_i + p_j} \right].$$

Using the identity

$$\frac{1}{a+b}\left(\frac{a^2}{\lambda+a} + \frac{b^2}{\lambda+b}\right) = 1 - \frac{\lambda^2}{(\lambda+a)(\lambda+b)} - \frac{2\lambda ab}{(\lambda+a)(a+b)(\lambda+b)}$$

we can express K_{g_λ} as

$$K_{g_\lambda}(p_1, \ldots, p_n) = E - \lambda^2 D_\lambda E D_\lambda - 2\lambda D D_\lambda C D_\lambda D,$$

where $D = \mathrm{diag}\,(p_1, \ldots, p_n)$, $D_\lambda = \mathrm{diag}\left(\frac{1}{\lambda+p_1}, \ldots, \frac{1}{\lambda+p_n}\right)$ and C is the Cauchy matrix $\left[\frac{1}{p_i+p_j}\right]$. This shows that K_{g_λ} is c.n.d. Hence the matrices K_g corresponding to the function g represented by the integral in (2.5) are c.n.d. Let $h(t) = \alpha + \beta t + \gamma t^2$. A simple calculation shows that

$$K_h(p_1, \ldots, p_n) = 2\alpha C + \beta E + \gamma(DE + ED) - 2\gamma DCD.$$

Since $\alpha = f(0) \leqq 0$ and $\gamma \geqq 0$, this matrix is c.n.d. Thus each matrix K_f corresponding to an operator convex function f with $f(0) \leqq 0$ is c.n.d.

Now consider f given by (2.6). The identity

$$\frac{1}{a+b}\left(\frac{a^3}{\lambda+a} + \frac{b^3}{\lambda+b}\right) = \frac{a^2}{\lambda+a} + \frac{b^2}{\lambda+b} - \frac{\lambda ab}{(\lambda+a)(\lambda+b)} - \frac{2a^2b^2}{(\lambda+a)(a+b)(\lambda+b)}$$

can be easily verified. Using this one sees that for the function $h_\lambda(t) = t^3/(\lambda+t)$ we have

$$K_{h_\lambda}(p_1, \ldots, p_n) = D^2 D_\lambda E + E D_\lambda D^2 - \lambda D D_\lambda E D_\lambda D - 2D^2 D_\lambda C D_\lambda D^2.$$

It follows from arguments given before that this matrix is c.n.d. We have already seen that every matrix K_φ corresponding to the function $\varphi(t) = \alpha t + \beta t^2$ is c.n.d. whenever $\beta \geqq 0$. This condition on β in (2.6) translates to the hypothesis $f''(0) \geqq 0$ in the statement of our theorem. Finally, if $h(t) = \gamma t^3$, then

$$K_h(p_1, \ldots, p_n) = \gamma(D^2 E + ED^2 - DED),$$

and this matrix is c.n.d. if $\gamma \geqq 0$. Combining all these observations we see that K_f is c.n.d. ■

Remark. Again for the special functions $f(t) = t^r, r > 0$ we use the notation

$$K_r(p_1, \ldots, p_n) = \left[\frac{p_i^r + p_j^r}{p_i + p_j}\right].$$

A special case of Theorem 3.5 says that for $1 \leqq r \leqq 3$ all these matrices are c.n.d. Compare this with Theorem 1.3 (i). The arguments in the proof of parts (ii) and (iii) of that theorem can be modified to serve for the matrices K_r. Let $1 < r < 2$, or $2 < r < 3$. It is easy to see that a vector x in H^n satisfies $\langle x, K_r x\rangle = 0$ only if $x = 0$. We already know that K_r is c.n.d. All of its entries are positive and so it cannot be negative definite. Hence by Lemma 4.3.5 and Corollary 4.1.5 in [5] this matrix is nonsingular and admits just one positive eigenvalue.

Similarly we can modify the arguments of our Theorem 3.1 to show that for $0 < r \leqq 1$ the matrices K_r are infinitely divisible. This has been proved in [12] using entirely different arguments.

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