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Corners of normal matrices

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To Kalyan Sinha on his sixtieth birthday

Abstract. We study various conditions on matrices B and C under which they can be the off-diagonal blocks of a partitioned normal matrix.

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The structure of general normal matrices is far more complicated than that of two special kinds — hermitian and unitary. There are many interesting theorems for hermitian and unitary matrices whose extensions to arbitrary normal matrices have proved to be extremely recalcitrant (see e.g., [1]). The problem whose study we initiate in this note is another one of this sort.

We consider normal matrices N of size $2n$, partitioned into blocks of size n as

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1}$$

Normality imposes some restrictions on the blocks. One such restriction is the equality

$$\|B\|_2 = \|C\|_2 \tag{2}$$

between the *Hilbert–Schmidt (Frobenius) norms* of the off-diagonal blocks B and C. If T is any $m \times m$ matrix with entries t_{ij}, then

$$\|T\|_2 = \left(\sum_{j=1}^{m} |t_{ij}|^2\right)^{1/2}.$$

The equality (2) is a consequence of the fact that the Euclidean norm of the jth column of a normal matrix is equal to the Euclidean norm of its jth row.

Replacing the Hilbert–Schmidt norm by another unitarily invariant norm, we may ask whether the equality (2) is replaced by interesting inequalities. Let $s_1(T) \geq \cdots \geq s_m(T)$ be the singular values of T. Every unitarily invariant norm $|||T|||$ is a symmetric gauge function of $\{s_j(T)\}$ (see chapter IV of [1] for properties of such norms). Much of our concern in this note is with the special norms

$$\|T\|_2 = (\operatorname{tr} T^*T)^{1/2} = \left(\sum_{j=1}^{m} s_j^2(T)\right)^{1/2}$$

and

$$\|T\| = s_1(T) = \sup_{x\in\mathbb{C}^m, \|x\|=1} \|Tx\|. \tag{3}$$

The latter is the norm of T as a linear operator on the Euclidean space $\mathbb{C}^m$. Clearly

$$\|T\| \leq \|T\|_2 \leq \sqrt{m}\|T\|, \tag{4}$$

for every $m \times m$ matrix T.

If the matrix N in (1) is hermitian, then $C = B^*$, and hence, $|||C||| = |||B|||$ for all unitarily invariant norms. If N is unitary, then $AA^* + BB^* = A^*A + C^*C = I$. Hence, the eigenvalues λ_j satisfy the relations

$$\begin{aligned}\lambda_j(BB^*) &= \lambda_j(I - AA^*) = 1 - \lambda_j(AA^*)\\ &= 1 - \lambda_j(A^*A) = \lambda_j(I - A^*A) = \lambda_j(C^*C).\end{aligned}$$

Thus B and C have the same singular values, and again $|||B||| = |||C|||$ for all unitarily invariant norms.

This equality of norms does not persist when we go to arbitrary normal matrices, as we will soon see. From (2) and (4) we get a simple inequality

$$||B|| \leq \sqrt{n}\, ||C||. \tag{5}$$

One may ask whether the two sides of (5) can be equal, and that is the first issue addressed in this note.

When $n = 2$, it is not too difficult to construct a normal matrix N of the form (1) in which $\|B\| = \sqrt{2}\|C\|$. One example of such a matrix is

$$N = \left[\begin{array}{cc|cc} 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right]. \tag{6}$$

When $n = 3$, examples seem harder to come by. One that preserves some of the features of (6) is given by the matrix

$$N = \left[\begin{array}{ccc|ccc} 0 & \sqrt{\frac{2}{\sqrt{3}} - 1} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} & 0 & 0 & 0 \\ \sqrt{\frac{2}{\sqrt{3}} + 1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & \sqrt{\frac{2}{\sqrt{3}} + 1} \\ 0 & 1 & 0 & \sqrt{\frac{2}{\sqrt{3}} - 1} & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} & 0 \end{array}\right]. \tag{7}$$

It can be seen that N is normal and plainly $\|B\| = \sqrt{3}$ while $\|C\| = 1$. When $n = 4$, it is impossible to find such a matrix, and that is our first theorem.

The following elementary lemma (which can be verified by induction on the integer k) is used repeatedly in our proof.

Lemma. Let V be an n-dimensional vector space and let $V_1, \ldots, V_k$ be subspaces of V the sum of whose dimensions is larger than $(k-1)n$; i.e.,

$$\sum_{j=1}^{k} \dim\ V_j \ > \ (k-1)n.$$

Then the intersection of these k subspaces is nonzero.

Theorem 1. *There exists a normal matrix N of the form* (1) *with*

$$\|B\| = \sqrt{n}\ \|C\| \tag{8}$$

if and only if $n \leq 3$.

Proof. Note first that if equalities (2) and (8) hold simultaneously, then rank B must be one and C must be unitary. So, after applying a unitary similarity by $\begin{bmatrix} C & O \\ O & I \end{bmatrix}$, we may assume that

$$N = \begin{bmatrix} A & B \\ I & D \end{bmatrix}. \tag{9}$$

The normality condition $N^*N = NN^*$ leads to two equations

$$A - D = A^*B - BD^*, \tag{10}$$

$$2I = AA^* - A^*A + BB^* + B^*B + D^*D - DD^*. \tag{11}$$

Since B is of rank one,

$$\dim(\ker\ B) = \dim(\ker\ B^*) = n - 1,$$

where $\dim\ X$ stands for the dimension of a space X. So, if $n \geq 3$, then the dimensions of $\ker\ B$ and $\ker\ B^*$ add up to more than n. Hence their intersection is nonzero, and we may choose a unit vector x in this intersection. For this vector, we obtain from (10)

$$(A - D)x = -BD^*x, \tag{12}$$

and

$$(A - D)^*x = B^*Ax. \tag{13}$$

Equation (11) leads to the condition

$$2 = \|A^*x\|^2 - \|Ax\|^2 + \|Dx\|^2 - \|D^*x\|^2. \tag{14}$$

The rest of the proof shows that if $n > 3$, then we can choose a vector $x \in (\ker\ B) \cap (\ker\ B^*)$ for which these conditions cannot be satisfied.

The two matrices BD^* and B^*A have rank at most 1, so their kernels have dimension at least $n-1$. Hence

$$\dim(\ker B) + \dim(\ker B^*) + \dim(\ker BD^*) + \dim(\ker B^*A) \geq 4n-4. \tag{15}$$

This is larger than $3n$ whenever $n > 4$. So, in this case the four kernel spaces involved in (15) have a nonzero intersection. Let x be a unit vector in this intersection. Then from (12) and (13) we find that

$$(A-D)x = 0 \quad \text{and} \quad (A-D)^*x = 0.$$

Hence, $\|Ax\| = \|Dx\|$ and $\|A^*x\| = \|D^*x\|$. This contradicts the condition (14).

Now consider the case $n = 4$. The spaces $\ker B$ and $\ker B^*$ have dimension 3 each, while the space $\ker B(A+D)^*$ has dimension at least 3. The three dimensions add up to more than 8. Hence, we can find a unit vector x in the intersection of these three spaces. For this vector we have

$$\begin{aligned} \|A^*x\|^2 - \|D^*x\|^2 &= \mathrm{Re}\,\langle (A+D)^*x, (A-D)^*x\rangle \\ &= \mathrm{Re}\,\langle (A+D)^*x, B^*Ax\rangle \\ &= \mathrm{Re}\,\langle B(A+D)^*x, Ax\rangle \\ &= 0. \end{aligned} \tag{16}$$

Here the second equality is a consequence of (13), and at the last step we have used the fact that $B(A+D)^*x = 0$.

Using (12) instead of (13) we get

$$\begin{aligned} \|Dx\|^2 - \|Ax\|^2 &= \mathrm{Re}\,\langle (A+D)x, (D-A)x\rangle \\ &= \mathrm{Re}\,\langle (A+D)x, BD^*x\rangle \\ &= \mathrm{Re}\,\langle B^*(A+D)x, D^*x\rangle. \end{aligned} \tag{17}$$

Since B is a matrix with rank equal to 1 and norm equal to 2, we have $B^*BB^* = 4B^*$. (Use the polar decomposition $B = UP$. In some orthonormal basis P is diagonal with only one nonzero entry 2 on the diagonal. So $B^*BB^* = P^3U^* = 4PU^* = 4B^*$.) Hence we have

$$\begin{aligned} 4B^*Ax &= B^*BB^*Ax \\ &= B^*B(A-D)^*x \quad (\text{using } (13)) \\ &= B^*B(A+D)^*x - 2B^*BD^*x \\ &= -2B^*BD^*x \\ &= 2B^*(A-D)x \quad (\text{using } (12)) \\ &= 4B^*Ax - 2B^*(A+D)x. \end{aligned}$$

This shows that $B^*(A+D)x = 0$, and we get from (17)

$$\|Dx\|^2 - \|Ax\|^2 = 0. \tag{18}$$

Clearly the relations (14), (16) and (18) cannot be simultaneously true.

We have shown that when $n \geq 4$, there cannot exist a $2n \times 2n$ normal matrix of the form (9) in which B is an $n \times n$ matrix of rank one. This proves the theorem. ∎

Our discussion leads to some natural questions.

Problem 1. For $n \geq 4$, evaluate the quantity

$$\alpha_n = \sup \left\{ \|B\|/\|C\| \colon \exists A, D \text{ for which } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ is normal} \right\}.$$

We have seen $\alpha_n < \sqrt{n}$ for $n \geq 4$. It would be of interest to know whether α_n is a bounded sequence.

Problem 2. What matrix pairs B, C can be the off-diagonal entries of a normal matrix N as in (1)? In other words, when does $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ have a normal completion?

Example 1. Consider the 2×2 matrices

$$B = \begin{bmatrix} 1 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

Then, $\|B\|_2 = \|C\|_2$. However, there do not exist any 2×2 matrices A and D for which $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is normal. We leave the verification of this statement to the reader. Thus the equality (2) is only a necessary condition for normality of the matrix (1).

We consider some special cases of the question raised in Problem 2. We assume either $B = C$, or $B = C^*$.

For every B, the matrix $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$ has a normal completion, and this completion may be chosen to be of the special type $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$. Indeed, if U is the unitary matrix $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, then

$$U \begin{bmatrix} A & B \\ B & A \end{bmatrix} U^* = \begin{bmatrix} A + B & 0 \\ 0 & A - B \end{bmatrix}.$$

So $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is normal if and only if $\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}$ is normal, and this is the case if and only if $A + B$ and $A - B$ both are normal. The most obvious choice of A that assures this is $A = B^*$. Thus

$$\widetilde{B} = \begin{bmatrix} B^* & B \\ B & B^* \end{bmatrix} \tag{19}$$

is a normal completion of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. We have the norm inequality

$$\|B\| \leq \|\widetilde{B}\| \leq 2\|B\|. \tag{20}$$

When $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\|\widetilde{B}\| = \|B\|$. On the other hand, if B is any hermitian matrix, then $\|\widetilde{B}\| = 2\|B\|$. In this case, and more generally when B is normal, $\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$ is normal and has norm equal to $\|B\|$. This raises the question of finding completions of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$ that are 'optimal' in various senses.

Problem 3. Given a matrix B find a matrix A such that

$$N = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is normal and has the least possible norm. This is equivalent to asking for a matrix A such that $A + B$ and $A - B$ are normal and the quantity $\max(\|A + B\|, \|A - B\|)$ is minimised. It might be difficult to find *all* solutions to this problem. The following considerations lead to *one* solution.

We assume that B is a contraction, i.e. $\|B\| \leq 1$ and ask for an A so that $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is unitary. This is a unitary completion of the matrix $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. Let $B = USV$ be the singular value decomposition of B. Then

$$\begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} U^*AV^* & S \\ S & U^*AV^* \end{bmatrix}.$$

So, the problem reduces to finding an A' such that $\begin{bmatrix} A' & S \\ S & A' \end{bmatrix}$ is unitary. A familiar idea from the theory of unitary dilations (p. 232 of [2]) suggests the choice $A' = i(I - S^2)^{1/2}$.

This tells us how to find for any matrix B one of the least-norm normal completions of $\begin{bmatrix} ? & B \\ B & ? \end{bmatrix}$. Assume $\|B\| = 1$ and find a unitary completion as proposed above.

Next we consider the case $B = C^*$, and ask for matrices A and D such that

$$N = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \tag{21}$$

is normal. A calculation shows that the matrices A and D must be normal and satisfy the equation

$$(A - A^*)B = B(D - D^*). \tag{22}$$

Let $A = H_1 + iK_1$ and $D = H_2 + iK_2$ be the Cartesian decompositions of A and D. Here (H_1, K_1) and (H_2, K_2) are two pairs of commuting hermitian matrices. Equation (22) is equivalent to $K_1B = BK_2$. This shows that

$$B^*BK_2 = B^*K_1B = (K_1B)^*B = (BK_2)^*B = K_2B^*B.$$

So K_2 commutes with B^*B, and hence with the factor P in the polar decomposition $B = UP$.

Thus the general solution to (22) is obtained as follows: Choose K_0 and K_2, both hermitian, satisfying the conditions

$$K_0P = PK_0, \quad K_2P = PK_2, \quad (K_0 - K_2)P = 0.$$

Let $K_1 = UK_0U^*$. This condition ensures

$$K_1B = UK_0U^*B = UK_0P = UK_2P = UPK_2 = BK_2.$$

Choose hermitian matrices H_1 and H_2 that commute with K_1 and K_2, respectively. Let $A = H_1 + iK_1$ and $D = H_2 + iK_2$. This leads to N in (21) being normal.

As before, we also consider the special case $\|B\| \leq 1$ and ask for A and D such that the matrix (21) is unitary. This can be solved as follows: Let $B = UP$ be any polar decomposition. Choose hermitian matrices K_0 and K_2 that commute with P and satisfy the inequalities

$$K_0^2 \leq I - P^2, \quad K_2^2 \leq I - P^2.$$

Then choose hermitian matrices H_0 and H_2 that commute with K_0 and K_2, respectively, and satisfy the conditions

$$H_0^2 + K_0^2 = H_2^2 + K_2^2 = I - P^2.$$

Let $A = U(H_0 + iK_0)U^*$ and $D = H_2 + iK_2$. Then the matrix (21) is unitary.

Example 1 shows that the equality $\|B\|_2 = \|C\|_2$ is not a sufficient condition for the existence of a normal completion of $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$.

Our next proposition shows that equality between all unitarily invariant norms is a sufficient condition.

PROPOSITION

Let B, C be $n \times n$ matrices with $|||B||| = |||C|||$ for every unitarily invariant norm. Then the matrix $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a completion that is a scalar multiple of a unitary matrix.

Proof. If $|||B||| = |||C|||$ for every unitarily invariant norm, then $s_j(B) = s_j(C)$ for all $j = 1, 2, \ldots, n$. Hence, there exist unitary matrices U_1, U_2, V_1, V_2 such that $B = U_1SU_2$, and $C = V_1SV_2$. Divide B and C by $||S||$, and thus assume $||S|| = 1$. Then $I - S^2$ is positive, and has a positive square root. It is easy to see that the matrix

$$\begin{bmatrix} (I - S^2)^{\frac{1}{2}} & S \\ S & -(I - S^2)^{\frac{1}{2}} \end{bmatrix}$$

is unitary. Multiply this matrix on the left by the unitary matrix $U_1 \oplus V_1$, and on the right by the unitary matrix $V_2 \oplus U_2$. This gives a unitary matrix whose off-diagonal blocks are B and C. ■

While the condition in the Proposition is not necessary, it is sensitive to small perturbations. The matrices B and C in Example 1 satisfy the conditions $\|B\|_2 = \|C\|_2$, $|||B||| = |||C||| + O(\varepsilon)$, but for $\varepsilon \neq 0$, there is no possible normal completion of $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$.

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