HOW AND WHY TO SOLVE THE OPERATOR EQUATION $AX - XB = Y$

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1. *Introduction*

The entities A, B, X, Y in the title are *operators*, by which we mean either linear transformations on a finite-dimensional vector space (matrices) or bounded (= continuous) linear transformations on a Banach space. (All scalars will be complex numbers.) The definitions and statements below are valid in both the finite-dimensional and the infinite-dimensional cases, unless the contrary is stated.

The simplest kind of equation involving operators is a linear equation of the form $AX = Y$ or of the form $XB = Y$. The condition that is both necessary and sufficient that such an equation have a unique solution is that A (or B) should be *invertible* (bijective). Let $\sigma(A)$ denote the *spectrum* of A; in the finite-dimensional case this is just the set of eigenvalues of A. In any case, invertibility of A is the statement $0 \notin \sigma(A)$. The equation $ax = y$ in numbers can always be solved when $a \neq 0$. The analogous condition for the operator equation $AX = Y$ is $0 \notin \sigma(A)$. This analogy provides guidance in formulating potential theorems and guessing solutions of $AX - XB = Y$.

When does $AX - XB = Y$ have a unique solution X for each given Y? In the scalar case of $ax - xb = y$, the answer is obvious: we must have $a - b \neq 0$. The answer is almost as obvious in another case: if the matrices A and B are both diagonal, with diagonal entries $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\}$, respectively, then the above equation has the solution $x_{ij} = y_{ij}/(\lambda_i - \mu_j)$ provided $\lambda_i - \mu_j \neq 0$ for all i, j. It is easy to see that this condition is necessary and sufficient for the existence of a unique solution X for each given Y. In terms of spectra, this condition says $\sigma(A) \cap \sigma(B) = \varnothing$, or $0 \notin \sigma(A) - \sigma(B)$ (the set of all differences). It is shown below that the same result is true for general operators A and B.

It is remarkable that simply knowing *when* solutions to $AX - XB = Y$ exist gives striking results on many topics, including similarity, commutativity, hyperinvariant subspaces, spectral operators and differential equations. Some of these are discussed below. We then obtain several different explicit forms of the solution, and show how these are useful in perturbation theory.

2. *The solvability of the equation*

The basic theorem was proven by Sylvester [**96**] in the matrix case. Several people discovered the extension to operators. The first may have been M. G. Krein, who apparently lectured on the theorem in the late 1940s. Dalecki [**15**] found the theorem independently, as did Rosenblum [**81**]. Rosenblum's paper made the operator case

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widely known, and presented an explicit solution (see Theorem 9.3 below). Among operator theorists it is known as Rosenblum's Theorem, and matrix theorists call the equation Sylvester's Equation. We have decided to give it the following name.

SYLVESTER–ROSENBLUM THEOREM. *If A and B are operators such that $\sigma(A) \cap \sigma(B) = \emptyset$, then the equation $AX - XB = Y$ has a unique solution X for every operator Y.*

The following proof is due to Lumer and Rosenblum [**67**]. Define the linear operator $\mathscr{T}$ on the space of operators by $\mathscr{T}(X) = AX - XB$. The conclusion of the theorem can then be rephrased: $\mathscr{T}$ is invertible if $\sigma(A) \cap \sigma(B) = \emptyset$. To see that this holds, consider the operators $\mathscr{A}$ and $\mathscr{B}$ defined on the space of operators by $\mathscr{A}(X) = AX$ and $\mathscr{B}(X) = XB$, respectively. Then $\mathscr{T} = \mathscr{A} - \mathscr{B}$, and $\mathscr{A}$ and $\mathscr{B}$ commute (regardless of whether or not A and B do). It is easy to see that $\sigma(\mathscr{A}) \subset \sigma(A)$ and $\sigma(\mathscr{B}) \subset \sigma(B)$. Thus the theorem is an immediate consequence of the following lemma.

LEMMA. *If $\mathscr{A}$ and $\mathscr{B}$ are commuting operators, then $\sigma(\mathscr{A} - \mathscr{B}) \subset \sigma(\mathscr{A}) - \sigma(\mathscr{B})$.*

In the finite-dimensional case, this lemma is easy to prove. Since $\mathscr{A}$ and $\mathscr{B}$ commute, there exists a basis in which $\mathscr{A}$ and $\mathscr{B}$ are both upper triangular. The lemma then follows from the fact that the spectrum of a triangular matrix is the set of numbers on the main diagonal.

This proof can be modified to cover the infinite-dimensional case by using a little Gelfand theory of commutative Banach algebras [**85**]. Imbed $\mathscr{A}$ and $\mathscr{B}$ in a maximal commutative subalgebra of the algebra of operators, and note that the spectrum of an operator is equal to its spectrum relative to a maximal commutative subalgebra. The spectrum of an element of a commutative Banach algebra with identity is the range of its Gelfand transform. This gives

$$\begin{aligned}\sigma(\mathscr{A} - \mathscr{B}) &= \{\phi(\mathscr{A} - \mathscr{B}) : \phi \text{ is a nonzero complex homomorphism}\}\\ &= \{\phi(\mathscr{A}) - \phi(\mathscr{B}) : \phi \text{ is a nonzero complex homomorphism}\}\\ &\subset \sigma(\mathscr{A}) - \sigma(\mathscr{B}).\end{aligned}$$

This proves the lemma, and the Sylvester–Rosenblum Theorem follows.

It should be noted, as Rosenblum [**81**] did, that the theorem holds, with the above proof, if A and B are elements of any (complex) Banach algebra. In another direction, the theorem is valid when A is an operator on a space $\mathscr{H}$ and B is an operator on a different space $\mathscr{K}$. In this case, the variables X and Y are operators from $\mathscr{K}$ into $\mathscr{H}$.

When A and B are operators on the same space and $\sigma(A) \cap \sigma(B) \neq \emptyset$, then the operator $\mathscr{T}$ is not invertible. This was shown by D. C. Kleinecke [**81**].

3. *An application to similarity*

Consider the 2×2 matrices $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ whose entries are operators. (These matrices represent operators on the direct sum of the spaces on which A and B operate.) When are these two matrices similar? Note that every operator of the form $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$, where the I are identity operators (possibly on different spaces) and X is any operator, is invertible: its inverse is $\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$.

Thus the given matrices will be similar by a similarity via this kind of matrix if we can find an X satisfying

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Multiplying out the matrices and equating corresponding entries gives four operator equations, of which only one is not automatically satisfied. That equation is $AX+C=XB$, or $AX-XB=-C$. The Sylvester–Rosenblum Theorem therefore gives the following result.

THEOREM. *If* $\sigma(A)\cap\sigma(B)=\varnothing$, *then for every* C *the operator* $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ *is similar to* $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

This was first observed by Roth [**84**], who went on to prove a much deeper result in the finite-dimensional case.

ROTH'S THEOREM [**84**]. *If* A, B *are operators on finite-dimensional spaces, then* $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ *is similar to* $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *if and only if the equation* $AX-XB=C$ *has a solution* X.

(Note that then $A(-X)-(-X)B=-C$.) Thus if the matrices $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are similar, the similarity can be implemented by a matrix of the form $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$.

In the finite-dimensional case, Roth's Theorem gives an interesting necessary and sufficient condition for the equation $AX-XB=C$ to have a solution. We do not include a proof of Roth's Theorem here; a nice proof was given by Flanders and Wimmer [**32**]. Roth's Theorem does not extend to infinite-dimensional cases: a counterexample was given by Rosenblum [**82**], who also showed that it does hold in the special case when A and B are self-adjoint operators on a Hilbert space. Schweinsberg [**87**] extended this affirmative result to the case where A and B are normal.

An easy induction using the first theorem of this section gives the following. If $\{A_i\}_{i=1}^n$ are operators such that $\sigma(A_i)\cap\sigma(A_j)=\varnothing$ whenever $i\neq j$, then every upper triangular matrix of the form

$$\begin{pmatrix} A_1 & A_{12} & \dots & A_{1n} \\ 0 & A_2 & & \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

is similar to the block-diagonal matrix

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix}.$$

In the scalar case this reduces to the familiar result that an $n \times n$ matrix with distinct eigenvalues is similar to a diagonal matrix.

4. *Embry's Theorem on commutativity*

If C commutes with $A+B$ and with AB, must C commute separately with A and with B? Certainly not, in general. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then every 2×2 matrix C commutes with $A+B$ and with AB.

The following beautiful result was discovered by M. Embry [**25**].

Embry's Theorem. *Let A and B be operators with $\sigma(A) \cap \sigma(B) = \varnothing$. Then every operator that commutes with $A+B$ and with AB also commutes with A and with B.*

The proof of this theorem is very simple. Let

$$(A+B)\,C = C(A+B) \quad \text{and} \quad (AB)\,C = C(AB).$$

Premultiplying the first equation by A, and then making use of the second, leads to

$$AAC + CAB = ACA + ACB,$$

or

$$A(AC - CA) = (AC - CA)\,B.$$

But then, by the Sylvester–Rosenblum Theorem, we must have $AC - CA = 0$. Thus C commutes with A, and hence also with B.

5. *Hyperinvariant subspaces*

One of the most famous unsolved problems in functional analysis is the *invariant subspace problem*. This is the question: does every operator on an infinite-dimensional Hilbert space have a non-trivial invariant subspace? Here, subspace means a closed linear manifold. A subspace is trivial if it is either $\{0\}$ or the entire space; it is invariant if it is mapped into itself by the operator.

There are Banach spaces on which some operators have only the trivial invariant subspaces [**27**, **79**]. But, in spite of much work and many partial results (see [**78**, **7**]), the solution to the Hilbert space problem remains elusive.

There are several variants of this problem which also remain unsolved, one of which is the following. A subspace is said to be *hyperinvariant* for the operator A if it is invariant under every operator which commutes with A. If A is a (scalar) multiple of the identity, then it clearly does not have hyperinvariant subspaces other than the trivial two. The *hyperinvariant subspace* problem is the question: on an infinite-dimensional Hilbert space, does every operator which is not a multiple of the identity have a non-trivial hyperinvariant subspace? In the finite-dimensional case, every non-scalar operator has a non-trivial hyperinvariant subspace, because a common eigenvector can be found for any commuting family of operators. A far-reaching generalization of this due to Lomonosov [**64**, **79**] says that every compact operator other than 0 has a non-trivial hyperinvariant subspace, but the general hyperinvariant subspace problem is unsolved.

It is easy to make examples of subspaces which are invariant but not hyperinvariant for an operator. However, a sufficient condition that an invariant subspace be hyperinvariant can be derived from the Sylvester–Rosenblum Theorem.

If a subspace $\mathscr{M}$ of the Hilbert space $\mathscr{H}$ is invariant under A, then with respect to the orthogonal decomposition $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$, the operator A decomposes as $\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$; the 0 in the bottom left corner expresses the invariance of $\mathscr{M}$.

THEOREM. *Let* $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ *and* $B = \begin{pmatrix} B_1 & B_2 \\ B_4 & B_3 \end{pmatrix}$. *If* $\sigma(A_1) \cap \sigma(A_3) = \varnothing$ *and* B *commutes with* A, *then* $B_4 = 0$.

Proof. The $(2, 1)$ entry of the equation $AB = BA$ reads $A_3 B_4 = B_4 A_1$. Therefore the Sylvester–Rosenblum Theorem implies $B_4 = 0$.

Thus $\mathscr{M}$ is a hyperinvariant subspace for A if $\sigma(A_1) \cap \sigma(A_3) = \varnothing$. There is a generalization.

THEOREM [**77**; **78**, Theorem 6.22]. *If A is a block upper triangular operator*

$$\begin{pmatrix} A_{11} & * & \dots & * \\ 0 & A_{22} & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix}$$

and $\sigma(A_{11}) \cap \sigma(A_{nn}) = \varnothing$, *then A has a non-trivial hyperinvariant subspace.*

Proof. Let x be any non-zero vector of the form $x = x_1 \oplus 0 \oplus 0 \oplus \dots \oplus 0$ with respect to the orthogonal decomposition of the space in which A is upper triangular. Let $\mathscr{M}$ be the closure of the linear manifold $\{Bx \colon AB = BA\}$. Then $\mathscr{M} \neq \{0\}$, and $\mathscr{M}$ is a hyperinvariant subspace for A. We shall show that $\mathscr{M}$ is not the entire space by showing that all vectors of the form $y = 0 \oplus \dots \oplus 0 \oplus y_n$ are in $\mathscr{M}^{\perp}$. Suppose that $B = (B_{ij})$ is any operator commuting with A. Equating the $(n, 1)$ entry of AB to that of BA gives $A_{nn} B_{n1} = B_{n1} A_{11}$. Thus the Sylvester–Rosenblum Theorem shows that $B_{n1} = 0$, so the nth component of Bx is 0 for all B commuting with A, giving the result.

An operator A is called *n-normal* if it can be expressed in an $n \times n$ block matrix form $A = (A_{ij})$ in which the A_{ij} are mutually commuting normal operators. The above theorem has the corollary that every n-normal operator which is not a multiple of the identity has a non-trivial hyperinvariant subspace (compare [**77**, **78**]); this result was first proven by other methods by Hoover [**50**].

6. *Spectral operators*

The Spectral Theorem is the assertion that every normal operator A on a Hilbert space can be expressed as an integral $A = \int \lambda \, dE(\lambda)$ where E is a *spectral measure* defined on Borel subsets of the complex plane $\mathbb{C}$. A well-known theorem of Fuglede states that every operator that commutes with a normal operator A also commutes with its adjoint, A^*. This is easily seen to be equivalent to saying that if E is the spectral measure associated with A, then for every Borel subset S of $\mathbb{C}$, the range of $E(S)$ is a hyperinvariant subspace for A.

The concept of a spectral operator is a generalization of that of a normal operator. Dunford [**24**] obtained a generalization of Fuglede's Theorem to spectral operators; a proof much simpler than Dunford's can be based on the Sylvester–Rosenblum Theorem.

To see this, we first recall the basic facts. Let $\mathscr{S}$ be the σ-algebra of all Borel subsets of $\mathbb{C}$. A *spectral measure* on a Banach space $\mathscr{X}$ is a mapping E from $\mathscr{S}$ into the space of operators on $\mathscr{X}$ with the following properties.

(1) $(E(S))^2 = E(S)$ for all $S \in \mathscr{S}$ (that is, the values of the measure are projection operators).
(2) $E(\varnothing) = 0$, $E(\mathbb{C}) = I$.
(3) E is bounded (that is, there is a K such that $\|E(S)\| \leqslant K$ for all S).
(4) E has compact support.
(5) $E(S_1 \cap S_2) = E(S_1)\,E(S_2)$ for all S_1, S_2.
(6) Whenever $\{S_j\}_{j=1}^{\infty}$ is a disjoint collection of subsets of $\mathbb{C}$,

$$E\left(\bigcup_{j=1}^{\infty} S_j\right)x = \sum_{j=1}^{\infty} E(S_j)\,x$$

for each $x \in \mathscr{X}$ (that is, $E(\cdot)$ is countably additive in the strong topology).

Note that every spectral measure is *regular*, in the sense that for every Borel set S, the space $\operatorname{ran} E(S) = E(S)\mathscr{X}$ is the closed linear span of $\{\operatorname{ran} E(K)\colon K \subset S,\ K$ compact$\}$.

EXAMPLE. Proto-typical spectral measures can be constructed as follows. Let (X, μ) be a finite measure space, and let $\phi \in L^\infty(X, \mu)$. For a Borel subset S of the plane, let χ_S denote the characteristic function of S. Then for each real $p \geqslant 1$, the operator $E(S)$ of multiplication by $\chi_S \circ \phi$ defines a spectral measure on the Banach space $L^p(X, \mu)$.

An operator A is called a *spectral operator* with spectral measure $E(\cdot)$ if, for every Borel set S,

(1) A commutes with $E(S)$, and
(2) the spectrum of $A|_{\operatorname{ran} E(S)}$ is contained in the closure of S.

FUGLEDE DUNFORD THEOREM. *If A is a spectral operator with spectral measure $E(\cdot)$, then for each Borel set S, the space* $\operatorname{ran} E(S)$ *is hyperinvariant for A.*

(Equivalent formulation: every operator that commutes with A commutes with all the projections $E(S)$.)

Proof [**77**]. Let $AB = BA$, and let S be any Borel set. The invariance of $\operatorname{ran} E(S)$ is equivalent to $(1 - E(S))\,BE(S) = 1$. Since $1 - E(S) = E(\mathbb{C}\backslash S)$, and since every spectral measure is regular, it suffices to show that

$$E(K_1)\,BE(K_2) = 0$$

whenever K_1, K_2 are disjoint compact subsets of $\mathbb{C}$.

Let K_1 and K_2 be any such sets. We have

$$E(K_1)\,ABE(K_2) = E(K_1)\,BAE(K_2).$$

Since each $E(S)$ is idempotent and commutes with A, this gives

$$E(K_1)\,AE(K_1)\cdot E(K_1)\,BE(K_2) = E(K_1)\,BE(K_2)\cdot E(K_2)\,AE(K_2).$$

Now $E(K_1)\,AE(K_1)$ and $E(K_2)\,AE(K_2)$ can be regarded as operators acting on $\operatorname{ran} E(K_1)$ and $\operatorname{ran} E(K_2)$, respectively, and their spectra are contained in K_1 and K_2, respectively. So, by the Sylvester–Rosenblum Theorem,

$$E(K_1)\,BE(K_2) = 0,$$

and the proof is complete.

7. *Lyapunov's equation and stability*

A century ago, Lyapunov [**68**] discovered a very interesting relationship between solutions of a matrix equation and stability of solutions of systems of linear differential equations, as well as deeper results in the non-linear case. The subject begins with the solution of the Lyapunov equation $AX + XA^* = -I$.

THEOREM 7.1. *If the spectrum of the Hilbert space operator A is contained in the open left half plane, then there is a unique positive invertible operator X satisfying $AX + XA^* = -I$.*

Proof. Since $\operatorname{Re}\sigma(A) < 0$, the same is true for $\operatorname{Re}\sigma(A^*)$, so $\sigma(A) \cap \sigma(-A^*) = \varnothing$. Thus the Sylvester–Rosenblum Theorem implies that there is a unique X such that $AX + XA^* = -I$. It remains to be shown that X is positive and invertible.

The standard approach to this is to use the explicit form of the solution X given in Theorem 9.2 below. Here we outline a simple alternative proof, due to Williams [**101**].

Note that taking adjoints of $AX + XA^* = -I$ yields $X^*A^* + AX^* = -I$, so the uniqueness of the solution implies that X is self-adjoint. Thus to show that X is positive, it suffices to show that $\sigma(X) > 0$.

As Williams [**101**] shows, without loss of generality it can be assumed that the numerical range of A is contained in the left half plane. Then $\sigma(X) > 0$ can be shown as follows. Suppose that $f \neq 0$ and Xf is equal to λf. Then

$$(-f, f) = ((AX + X^*A)f, f) = (AXf, f) + (Af, Xf) = 2\lambda\,(Af, f).$$

Since $\operatorname{Re}(Af, f)$ and $(-f, f)$ are both negative, λ must be positive.

This shows that all eigenvalues of X are positive, and finishes the proof in the finite-dimensional case. In the infinite-dimensional case, essentially the same argument shows that all approximate eigenvalues of X are positive, and hence so are all points in $\sigma(X)$.

Note that I could be replaced by any positive invertible matrix.

The most elementary application of the Lyapunov equation is the following.

THEOREM 7.2. *If A is an operator on a Hilbert space with spectrum contained in the open left half plane, then every solution of the vector differential equation* $\frac{dZ}{dt} = AZ$ *is stable* (*in the sense that* $\lim_{t\to\infty}\|Z(t)\| = 0$).

Proof. Let X be the positive solution of $A^*X + XA = -I$ given by Theorem 7.1, and define the real-valued, non-negative function F by $F(t) = (XZ(t), Z(t))$, where $(\cdot, \cdot)$ is the inner product on the Hilbert space. Then $F'(t) = (XZ'(t), Z(t)) + (XZ(t), Z'(t))$. But $Z'(t) = AZ(t)$, so

$$\begin{aligned} F'(t) &= (XAZ(t), Z(t)) + (XZ(t), AZ(t)) \\ &= ((XA + A^*X)\, Z(t), Z(t)) = -\|Z(t)\|^2. \end{aligned}$$

Choose any $\delta > 0$ such that $X \geqslant \delta I$. Then

$$F(t) = (XZ(t), Z(t)) \geqslant \delta\|Z(t)\|^2,$$

so

$$\frac{F'(t)}{F(t)} \leqslant \frac{-\|Z(t)\|^2}{\|X\|\,\|Z(t)\|^2} = -\frac{1}{\|X\|}.$$

Thus $\log F(t) \leqslant -(t/\|X\|) + c$ for some constant c, or $F(t) \leqslant e^c e^{-t/\|X\|}$. Therefore $\lim_{t\to\infty} F(t) = 0$. Since $F(t) \geqslant \delta\|Z(t)\|^2$, the theorem follows.

This is merely the beginning of the Lyapunov stability theory; see [**68**, **39**] and references given there for additional results.

8. *Existence without uniqueness*

On infinite-dimensional spaces, operators can be onto without being injective. This suggests the possibility of refining the Sylvester–Rosenblum Theorem to obtain a less restrictive sufficient condition for $AX - XB = Y$ to have a solution X for every Y.

DEFINITION. The *approximate defect spectrum* of A, denoted $\sigma_\delta(A)$, is the set $\{\lambda : A - \lambda I \text{ is not onto}\}$. The *approximate point spectrum* of B, denoted $\Pi(B)$, is the set $\{\lambda : \|(B - \lambda I) f_n\| \to 0 \text{ for some } \{f_n\} \text{ with } \|f_n\| = 1\}$. Clearly, $\sigma_\delta(A) \subset \sigma(A)$ and $\Pi(B) \subset \sigma(B)$.

THEOREM 8.1 [**19**]. *If $\sigma_\delta(A) \cap \Pi(B) = \varnothing$, then $AX - XB = Y$ has a solution X for every Y.*

The proof of this theorem (see [**19**]) consists of defining $\mathscr{A}X = AX$, $\mathscr{B}X = XB$ and $\mathscr{T} = \mathscr{A} - \mathscr{B}$, and showing:

(1) $\sigma_\delta(\mathscr{A}) = \sigma_\delta(A)$;
(2) $\sigma_\delta(\mathscr{B}) = \Pi(B)$;
(3) $\sigma_\delta(\mathscr{T}) \subset \sigma_\delta(\mathscr{A}) - \sigma_\delta(\mathscr{B})$.

There are some situations where this variant of the Sylvester–Rosenblum Theorem is useful (compare [**47**]). If A and B are operators in Hilbert spaces, then the converse holds (that is, if $AX - XB = Y$ has a solution for every Y, then $\sigma_\delta(A) \cap \Pi(B) = \varnothing$).

It might be conjectured that there is another variant of the Sylvester–Rosenblum Theorem giving a refined sufficient condition for uniqueness. However, the following examples show that no such result can hold.

EXAMPLE 8.2. On a Hilbert space $\mathscr{H}$, let S^* be the backward shift of infinite multiplicity, and let T be any operator with spectral radius less than 1. Then there is an injective X satisfying $S^*X = XT$.

Proof. Define $X: \mathscr{H} \to \sum_{j=1}^{\infty} \oplus \mathscr{H}_j$, with $\mathscr{H}_j = \mathscr{H}$ for all j, by $Xf = (f, Tf, T^2f, \ldots)$. Since the spectral radius of T is less than 1, $\sum_{n=0}^{\infty} \|T^n f\|^2$ is finite, so $Xf \in \sum_{j=1}^{\infty} \oplus \mathscr{H}_j$ for every f. By the same argument, X is bounded. Then for each f, $S^*Xf = S^*(f, Tf, T^2f, \ldots) = (Tf, T^2f, \ldots)$ and $XTf = (Tf, T^2f, \ldots)$. Therefore $S^*X = XT$.

Note that $\sigma(S^*) = \{z: |z| \leqslant 1\}$ and $\sigma(T) \subset \{z: |z| < 1\}$, so $\sigma(S^*) \cap \sigma(T) = \sigma(T)$ when the hypotheses of the above example are satisfied.

EXAMPLE 8.3. If S is the unilateral shift, and T has dense range, then the only solution of $SX = XT$ is $X = 0$.

Proof. If $SX = XT$, then $X^*S^* = T^*X^*$. If S shifts the orthonormal basis $\{e_n\}_{n=0}^{\infty}$, then S^* shifts the same basis backwards. Since T has dense range, T^* is injective. Thus $X^*S^*e_0 = T^*X^*e_0$ yields $0 = T^*X^*e_0$, so $X^*e_0 = 0$. Then $X^*S^*e_1 = T^*X^*e_1$ gives $X^*e_0 = T^*X^*e_1$, so $0 = T^*X^*e_1$ and $X^*e_1 = 0$. A trivial induction shows that $X^*e_n = 0$ for all n, so $X = 0$.

EXAMPLE 8.4 [**83**]. If A has no eigenvalues and 0 is not in the approximate point spectrum of A, and if B is compact, then the only solution of $AX = XB$ is $X = 0$.

See [**83**] for a proof.

9. *Constructing the solution*

Consider the scalar equation $ax - xb = y$, and, to exclude the trivial cases, assume $a \neq 0$, $b \neq 0$ and $a \neq b$. The solution to the equation can be written

$$x = a^{-1}\left(1 - \frac{b}{a}\right)^{-1} y.$$

Now, if $|b| < |a|$, the middle factor on the right can be expanded as a power series to give

$$x = a^{-1} \sum_{n=0}^{\infty} \left(\frac{b}{a}\right)^n y = \sum_{n=0}^{\infty} a^{-1-n}\, y b^n.$$

(The order of the factors is immaterial in the scalar case, but is a crucial consideration in the operator case.) This suggests the following result.

THEOREM 9.1. *Let A and B be operators such that $\sigma(B) \subset \{z: |z| < \rho\}$ and $\sigma(A) \subset \{z: |z| > \rho\}$ for some $\rho > 0$. Then the solution of the equation $AX - XB = Y$ is*

$$X = \sum_{n=0}^{\infty} A^{-n-1} Y B^n.$$

Proof. The only thing that needs to be proved is that the above series converges; it is then easy to check that it is a solution of the equation.

Choose $\rho_1 < \rho < \rho_2$ such that $\sigma(B)$ is contained in the disk $\{z : |z| < \rho_1\}$ and $\sigma(A)$ is outside the disk $\{z : |z| \leqslant \rho_2\}$. Then $\sigma(A^{-1})$ is inside the disk $\{z : |z| < \rho_2^{-1}\}$. By the spectral radius formula ($r(B) = \lim_{n\to\infty} \|B^n\|^{1/n}$—see [**85**] or any standard text on functional analysis), there exists a positive integer N such that for $n \geqslant N$, $\|B^n\| < \rho_1^n$ and $\|A^{-n}\| < \rho_2^{-n}$. Hence $\|A^{-n-1} Y B^n\| < (\rho_1/\rho_2)^n \|A^{-1} Y\|$, and the series is norm convergent.

If $\operatorname{Re} b < \operatorname{Re} a$, then the integral $\int_0^\infty e^{t(b-a)}\,dt$ is convergent and has the value $1/(a-b)$. Thus if $\operatorname{Re} b < \operatorname{Re} a$, the solution of the equation $ax - xb = y$ can be expressed as

$$x = \int_0^\infty e^{t(b-a)}\, y\, dt.$$

This suggests the following theorem, first proven by Heinz [**46**].

THEOREM 9.2 [**46**]. *Let A and B be operators whose spectra are contained in the open right half plane and the open left half plane, respectively. Then the solution of the equation $AX - XB = Y$ can be expressed as*

$$X = \int_0^\infty e^{-tA}\, Y e^{tB}\, dt.$$

We leave the proof of this to the reader (or see [**46**]). The conditions on A and B ensure that the integral converges.

It should be noted that translating both A and B by the same scalar does not change the equation $AX - XB = Y$. So the previous two theorems could be modified to yield solutions when the spectra of A and B are separated by an annulus or by a strip in the plane, respectively.

The next theorem gives an expression for the solution of $AX - XB = Y$ whenever $\sigma(A)$ and $\sigma(B)$ are disjoint, without any more special assumptions about the separation of the spectra.

THEOREM 9.3 (Rosenblum [**81**]). *Let Γ be a union of closed contours in the plane, with total winding numbers* 1 *around $\sigma(A)$ and* 0 *around $\sigma(B)$. Then the solution of the equation $AX - XB = Y$ can be expressed as*

$$X = \frac{1}{2\pi i}\int_\Gamma (A-\zeta)^{-1}\, Y (B-\zeta)^{-1}\, d\zeta.$$

Proof. If $AX - XB = Y$, then for every complex number ζ, we have $(A-\zeta)X - X(B-\zeta) = Y$. If $A - \zeta$ and $B - \zeta$ are invertible, this gives

$$X(B-\zeta)^{-1} - (A-\zeta)^{-1} X = (A-\zeta)^{-1}\, Y (B-\zeta)^{-1}.$$

The theorem now follows by integrating over Γ and noting that $\int_\Gamma (B-\zeta)^{-1}\, d\zeta = 0$ and $-\int_\Gamma (A-\zeta)^{-1}\, d\zeta = 2\pi i I$ for the Γ in question.

Rosenblum [**81**] discusses how the solution in Theorem 9.2 can be obtained from that in Theorem 9.3.

Another approach, popular among numerical analysts and engineers, is via the matrix sign function, as we now briefly explain. Let T be any matrix which has no purely imaginary eigenvalue. Let $T = SJS^{-1}$, where J is a matrix in Jordan canonical form with $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ and J_1 and J_2 having all their eigenvalues in the open right half plane and the open left half plane, respectively. Then the *sign* of T is the matrix

$$\operatorname{sgn}(T) = S\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} S^{-1},$$

where I denotes identity matrices (possibly of different sizes). This notion can be generalized to infinite dimensions. Let T be a bounded operator having no purely imaginary number in its spectrum. Let $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are subsets of the right half plane and the left half plane, respectively. Let Γ_1 and Γ_2 be two contours such that for $j = 1, 2$, Γ_j has winding number 1 around σ_j and winding number 0 around $\sigma(T)\backslash\sigma_j$. Let

$$T_j = \frac{1}{2\pi i}\int_{\Gamma_j} (\zeta - T)^{-1}\, d\zeta,$$

and define $\operatorname{sgn}(T) = T_1 - T_2 = 2T_1 - 1$. It is easy to see that in the case of matrices, this reduces to the earlier definition.

Now let $AX - XB = Y$. Then, as we saw in Section 3, we can write

$$\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

If $\sigma(A)$ and $\sigma(B)$ are contained in the open right half plane and the open left half plane, respectively, then

$$\operatorname{sgn}\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 2X \\ 0 & -I \end{pmatrix}.$$

Thus the solution X can be read off from the above equation, provided we know how to calculate $\operatorname{sgn}\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix}$. This can be done using the definitions above. There is also an integral representation

$$\operatorname{sgn}(T) = \frac{2}{\pi} T \int_0^\infty (\lambda^2 + T^2)^{-1}\, d\lambda.$$

More useful for computation is an iterative scheme analogous to Newton's method for the square root. Let $T_0 = T$, and $T_{k+1} = \frac{1}{2}(T_k + T_k^{-1})$. Then the sequence T_{k+1} converges to $\operatorname{sgn}(T)$, and the rate of convergence is quadratic.

This method of solving the equation $AX - XB = Y$ was introduced by Roberts [**80**]. An interesting recent paper is Higham [**49**].

In many situations we are interested in the solution of $AX - XB = Y$ when A and B are normal or, even more specially, Hermitian or unitary. The special nature of A and B can be exploited to obtain other forms of the solution, as discussed below.

If A and B are Hermitian, then iA and iB are skew-Hermitian, and hence have their spectra on the imaginary line. This is the opposite of the hypothesis of Theorem

9.2. The integral $\int_0^\infty e^{-itA} Y e^{itB}\, dt$ (which we might try as a solution if we were to imitate Theorem 9.2) is not convergent. One remedy would be to insert a function to serve as a convergence factor, so that the integral

$$X = \int_{-\infty}^{\infty} e^{-itA}\, Y e^{itB} f(t)\, dt$$

would converge. Since the exponentials occurring here are unitary operators for every t, this integral is convergent if $f \in L^1(\mathbb{R})$. Can one choose a suitable f so that this is a solution of the equation $AX - XB = Y$? In the scalar case one can see that this is so if $\hat{f}(a-b) = 1/(a-b)$, where $\hat{f}$ is the Fourier transform of f, defined as

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-its} f(t)\, dt.$$

The following theorem generalizes this.

THEOREM 9.4 [**12**]. *Let A and B be Hermitian operators with $\sigma(A) \cap \sigma(B) = \emptyset$. Let f be any function in $L^1(\mathbb{R})$ whose Fourier transform $\hat{f}$ has the property that $\hat{f}(s) = 1/s$ whenever $s \in \sigma(A) - \sigma(B)$. Then the solution of the equation $AX - XB = Y$ can be expressed as*

$$X = \int_{-\infty}^{\infty} e^{-itA}\, Y e^{itB} f(t)\, dt.$$

Proof. First consider the finite-dimensional case. Let α and β be eigenvalues of A and B with eigenvectors u and v, respectively. Then, using the fact that e^{itA} is unitary with $(e^{itA})^* = e^{-itA}$, we obtain the following:

$$\langle A e^{-itA}\, Y e^{itB} v, u\rangle = \langle Y e^{itB} v, e^{itA} A u\rangle = e^{it(\beta-\alpha)} \alpha \langle Yv, u\rangle.$$

A similar consideration shows

$$\langle e^{-itA}\, Y e^{itB} B v, u\rangle = e^{it(\beta-\alpha)} \beta \langle Yv, u\rangle.$$

Hence if X is given by the above integral, then

$$\langle (AX - XB) v, u\rangle = \hat{f}(\alpha-\beta)(\alpha-\beta)\langle Yv, u\rangle = \langle Yv, u\rangle.$$

Since eigenvectors of A and B both span the whole space, this shows that $AX - XB = Y$.

The same argument proves the theorem when the space is infinite-dimensional but both A and B have pure point spectra. The general case follows from this by a standard continuity argument. (Operators with pure point spectra are dense in the space of Hermitian operators.)

Using slight modifications of these arguments yields the following two theorems.

THEOREM 9.5 [**12**]. *Let A and B be normal operators with $\sigma(A) \cap \sigma(B) = \emptyset$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where A_1 and A_2 are commuting Hermitian operators, and so are B_1 and B_2. Let f be any function in $L^1(\mathbb{R}^2)$ whose Fourier transform $\hat{f}$ has the property that $\hat{f}(s_1, s_2) = 1/(s_1 + is_2)$ whenever $s_1 + is_2 \in \sigma(A) - \sigma(B)$. Then the solution of the equation $AX - XB = Y$ can be expressed as*

$$X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(t_1 A_1 + t_2 A_2)}\, Y e^{i(t_1 B_1 + t_2 B_2)} f(t_1, t_2)\, dt_1\, dt_2.$$

THEOREM 9.6 [**12**]. *Let A and B be unitary operators with $\sigma(A)\cap\sigma(B)=\varnothing$. Let $\{a_n\}_{-\infty}^{\infty}$ be any sequence in l_1 such that*

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} = \frac{1}{1-e^{i\theta}}$$

whenever $e^{i\theta}\in\sigma(A)\cdot(\sigma(B))^{-1}$. Then the solution of the equation $AX-XB=Y$ can be expressed as

$$X=\sum_{n=-\infty}^{\infty} a_n A^{-n-1} Y B^n.$$

Notice that this series solution has the same relation to the series solution of Theorem 9.1 as the integral in Theorem 9.4 has to the integral in Theorem 9.2. The series in Theorem 9.1 would not converge if $\|A\|=\|B\|=1$, but the a_n in Theorem 9.6 act as convergence factors.

10. *Estimating the size of the solution*

The problem we now consider is that of finding a bound for $\|X\|$ in terms of $\|Y\|$ and the separation between $\sigma(A)$ and $\sigma(B)$. Applications of such bounds will be discussed in the next section.

What kind of bounds should one expect? To see this, let us consider the finite-dimensional case. Let A and B be $n\times n$ diagonal matrices with $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$ on their diagonals. Then the solution of $AX-XB=Y$ is $x_{ij}=y_{ij}/(\lambda_i-\mu_j)$. Let $\|\cdot\|_2$ denote the *Hilbert–Schmidt norm* (or the *Frobenius norm*); this is defined as

$$\|T\|_2=(\operatorname{tr} T^*T)^{1/2}=\Big(\sum_{i,j}|t_{ij}|^2\Big)^{1/2}.$$

A direct computation gives

$$\|X\|_2\leqslant\frac{1}{\delta}\|Y\|_2,$$

where

$$\delta=\min_{i,j}|\lambda_i-\mu_j|=\operatorname{dist}(\sigma(A),\sigma(B)).$$

Now, more generally, let A and B be normal matrices with eigenvalues $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$, respectively. Then we can find unitary matrices U and V such that $A=UA'U^*$ and $B=VB'V^*$, where A' and B' are diagonal matrices. The equation $AX-XB=Y$ can be rewritten as

$$UA'U^*X-XVB'V^*=Y$$

and then as

$$A'(U^*XV)-(U^*XV)B'=U^*YV.$$

So we have the same type of equation as before, but now with diagonal A and B. Hence

$$\|U^*XV\|_2\leqslant\frac{1}{\delta}\|U^*YV\|_2.$$

But the norm $\|\cdot\|_2$ is invariant under multiplication by unitary matrices. Thus we have

$$\|X\|_2\leqslant\frac{1}{\delta}\|Y\|_2.$$

Now several questions arise. Does a similar result hold for non-normal A and B? Can $\|\cdot\|_2$ here be replaced by the usual operator norm $\|\cdot\|$? Are there infinite-dimensional results also? The following examples answer the first two questions negatively.

Consider the 2×2 matrices $A = Y = I$ and $B = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, for any fixed real t. Then the equation $AX - XB = Y$ has the solution $X = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. In this example, $\delta = 1$ and $\|Y\|_2 = \surd 2$, but $\|X\|_2$ can be made arbitrarily large by choosing t large. Therefore we cannot have even a weaker inequality $\|X\|_2 \leqslant c\|Y\|_2/\delta$ for some constant c in this case. We shall therefore have to restrict to the case of normal A and B.

Next consider the following example in which all the matrices involved are Hermitian:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Y = \begin{pmatrix} 6 & 2\surd(15) \\ 2\surd(15) & -6 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & \surd(15) \\ \surd(15) & 3 \end{pmatrix}.$$

Then $AX - XB = Y$. In this example, $\delta = 2$. But $\|X\| > \frac{1}{2}\|Y\|$. Thus, even in the Hermitian case, the norm inequality $\|X\| \leqslant \|Y\|/\delta$ can be violated.

However, in the Hermitian case and, more generally, for normal A and B, inequalities of the form $\|X\| \leqslant c\|Y\|/\delta$ are true for rather small constants c, independent of the dimension of the space. When the spectra of A and B are separated in special ways, these constants can be replaced by 1. This will now be discussed, using the different solutions described in Section 9 above.

Let A and B be normal operators and suppose that $\sigma(A)$ and $\sigma(B)$ are separated by an annulus of width δ; that is, $\sigma(B)$ is contained in a disk of radius ρ centred at some point a, and $\sigma(A)$ is outside a disk of radius $\rho + \delta$ centred at a. By applying a translation, we can assume $a = 0$. The conditions of Theorem 9.1 are met, so the solution is

$$X = \sum_{n=0}^{\infty} A^{-n-1} Y B^n.$$

Hence

$$\begin{aligned} \|X\| &\leqslant \sum_{n=0}^{\infty} \|A^{-1}\|^{n+1} \|Y\| \|B\|^n \\ &\leqslant \|Y\| \sum_{n=0}^{\infty} (\rho+\delta)^{-n-1} \rho^n \\ &= \frac{1}{\delta}\|Y\| \end{aligned}$$

(summing the geometric series). (The equality of norm and spectral radius for normal operators was used in obtaining the second inequality above.)

Either by taking a limit as $\rho \to \infty$ in the above argument, or by using the solution given in Theorem 9.2, we can see that the same inequality holds when A and B are normal operators with $\sigma(A)$ and $\sigma(B)$ lying in half planes separated by a strip of width δ.

In both these cases, the above inequality was found by Davis and Kahan [**18**] for Hermitian A and B, while for normal A and B it was noted in [**12**].

Now let A and B be Hermitian operators with $\operatorname{dist}(\sigma(A), \sigma(B)) = \delta > 0$ but with no other restriction. Then, using Theorem 9.4, we see that

$$\|X\| \leqslant \left(\int_{-\infty}^{\infty} |f(t)|\, dt\right) \|Y\|,$$

where f is any integrable function on $\mathbb{R}$ such that $\hat{f}(t) = 1/t$ whenever $|t| \geqslant \delta$. A change of variables leads to the inequality

$$\|X\| \leqslant \frac{c_1}{\delta} \|Y\|,$$

where

$$c_1 = \inf\left\{ \|f\|_{L_1} : f \in L^1(\mathbb{R}),\ \hat{f}(t) = \frac{1}{t} \text{ when } |t| \geqslant 1 \right\}.$$

When A and B are normal with $\operatorname{dist}(\sigma(A), \sigma(B)) = \delta > 0$, the same considerations (but using Theorem 9.5) lead to the inequality

$$\|X\| \leqslant \frac{c_2}{\delta} \|Y\|,$$

where

$$c_2 = \inf\left\{ \|f\|_{L_1} : f \in L^1(\mathbb{R}^2),\ \hat{f}(t_1, t_2) = \frac{1}{t_1 + it_2} \text{ when } t_1^2 + t_2^2 \geqslant 1 \right\}.$$

Both these inequalities were derived in [**12**]. The constant c_1 is related to several problems in number theory (see the interesting survey by Vaaler [**98**]), and was precisely calculated by Sz.-Nagy [**97**]. We have, somewhat miraculously, $c_1 = \pi/2$. The exact value of c_2 is not yet known. But it has been shown [**11**] that $c_2 < 2.91$. The problem of minimizing the L^1 norm of a function over the class of functions whose Fourier transforms are specified over some set is called a *minimal extrapolation problem*.

11. *Perturbation of eigenspaces*

A type of question of wide interest in physics, engineering and numerical analysis is: when an operator is changed slightly, how do various objects associated with it (eigenvalues, eigenvectors, canonical forms, etc.) change?

In the finite-dimensional case, the eigenvalues vary continuously with the operator, and many precise estimates of the change are known (see [**8**]). Eigenvectors, however, behave in a far more complicated way. The following simple example is illustrative of the situation. Let A and B be Hermitian operators on an n-dimensional space of the forms

$$A = \begin{pmatrix} \alpha + \varepsilon & 0 \\ 0 & \alpha - \varepsilon \end{pmatrix} \oplus H \quad \text{and} \quad B = \begin{pmatrix} \alpha & \varepsilon \\ \varepsilon & \alpha \end{pmatrix} \oplus H,$$

where H is any Hermitian matrix of size $n-2$. Both A and B have $\alpha + \varepsilon$ and $\alpha - \varepsilon$ in their spectra. The unit eigenvectors corresponding to these eigenalues are $(1, 0, 0, \ldots, 0)$

and $(0, 1, 0, \ldots, 0)$ in the case of A, and $\frac{1}{\sqrt{2}}(1, \pm 1, 0, \ldots, 0)$ in the case of B. Thus, no matter how small $\varepsilon > 0$ is compared to α and, hence, how close B is to A, these two eigenvectors of B remain far apart from those of A. However, observe that the spaces spanned by these two eigenvectors are the same for A and B. This phenomenon has long been recognized, and attempts have been made to quantify it. We shall explain one of the more successful and best known results, following Davis and Kahan [**18**].

Let E and F be two (orthogonal) projection operators on a Hilbert space. A good measure of separation between the spaces $\operatorname{ran} E$ and $\operatorname{ran} F$ is $\|EF\|$. When E, F are orthogonal, then $\|EF\| = 0$; when $\operatorname{ran} E \cap \operatorname{ran} F \neq \{0\}$, then $\|EF\| = 1$. In all cases, $\|EF\| \leqslant 1$. When the space is 2-dimensional and $\operatorname{ran} E$ and $\operatorname{ran} F$ are 1-dimensional, $\|EF\| = \cos\theta$, where θ is the angle between these subspaces. In the general case, $\|EF\|$ also has an interpretation as an angle between these two spaces (see Davis and Kahan [**18**], Stewart and Sun [**95**]).

Now let A be a normal operator with spectral resolution $A = \int \lambda \, dP_A(\lambda)$, where P_A is the spectral measure associated with A. Let S_1 and S_2 be two Borel subsets of $\mathbb{C}$, and let $E = P_A(S_1)$ and $F = P_A(S_2)$ be the spectral projections of A corresponding to them. (In the finite-dimensional case these are just the projections onto the subspaces spanned by the eigenvectors of A corresponding to those eigenvalues that are in S_1 and S_2, respectively.) If S_1 and S_2 are disjoint, then E and F are orthogonal to each other. If B were another normal operator with spectral resolution $B = \int \lambda \, dP_B(\lambda)$ and we let $F = P_B(S_2)$ and $E = P_A(S_1)$, we could expect F to be nearly orthogonal to E if B is close to A and S_1 and S_2 are far apart. The following theorem is one of several along these lines by Davis and Kahan [**18**]; it has become well known among numerical analysts as the '$\sin\theta$ theorem'.

DAVIS–KAHAN $\sin\theta$ THEOREM [**18**]. *Let A and B be Hermitian operators with spectral measures P_A and P_B, respectively. Let S_1 be any interval $[a, b]$, let δ be greater than* 0, *and let S_2 be the complement (in $\mathbb{R}$) of the interval $(a-\delta, b+\delta)$. Let $E = P_A(S_1)$ and $F = P_B(S_2)$. Then*

$$\|EF\| \leqslant \frac{1}{\delta}\|A-B\|.$$

The name '$\sin\theta$ theorem' comes from the interpretation of $\|EF\|$ as the sine of the angle between $\operatorname{ran} E$ and $\operatorname{ran} F^{\perp}$.

Proof. Davis and Kahan observed that a stronger inequality holds:

$$\|EF\| \leqslant \frac{1}{\delta}\|E(A-B)F\|.$$

To prove this, first note that since A commutes with its spectral projection E, and B with its spectral projection F, the above can be rewritten as

$$\|EF\| \leqslant \frac{1}{\delta}\|AEF-EFB\|.$$

Now let $EF = X$, regarded as an operator from $\operatorname{ran} F$ to $\operatorname{ran} E$. Restricted to these spaces, the operators B and A have their spectra inside S_2 and S_1, respectively. Thus the above inequality follows from the statement

$$\|X\| \leqslant \frac{1}{\delta}\|AX-XB\|$$

if $\sigma(A) \subset S_1$ and $\sigma(B) \subset S_2$, which, in turn, follows from the annulus inequality proved in Section 10.

Note that the Davis–Kahan Theorem has a straightforward generalization to normal operators, where S_1 is a disk and S_2 the complement of a concentric disk with $\operatorname{dist}(S_1, S_2) = \delta$.

If there is no 'annular separation', we still have the same inequality except for a constant factor. Let A and B be normal operators with spectral measures P_A and P_B, respectively, and let S_1 and S_2 be any two Borel sets in $\mathbb{C}$ with $\operatorname{dist}(S_1, S_2) = \delta$. Let $E = P_A(S_1)$ and $F = P_B(S_2)$. Then

$$\|EF\| \leqslant \frac{c}{\delta}\|A-B\|.$$

As mentioned in Section 10, the constant c occurring here is bounded by 2.91; in the special case when A and B are Hermitian, $c \leqslant \pi/2$. These results were first established in [**12**].

12. *Perturbation of the polar decomposition*

Let A and B be invertible operators with polar decompositions $A = UP$ and $B = VQ$, where P and Q are positive operators and U and V are unitary operators. We want to know how far apart the polar factors can be if A and B are close. (Such information is useful in numerical analysis and in physics.)

Note that

$$\|A-B\| = \|UP-VQ\| = \|P-U^*VQ\|,$$

and, by symmetry,

$$\|A-B\| = \|Q-V^*UP\|.$$

Let

$$Y = P-U^*VQ \quad \text{and} \quad Z = Q-V^*UP.$$

Then

$$Y+Z^* = P(I-U^*V)+(I-U^*V)\,Q.$$

Note that $\sigma(P)$ is bounded below by $\|A^{-1}\|^{-1}$ and $\sigma(Q)$ by $\|B^{-1}\|^{-1}$. So $\operatorname{dist}(\sigma(P), \sigma(-Q)) \geqslant \|A^{-1}\|^{-1}+\|B^{-1}\|^{-1}$. Hence, since $\sigma(P)$ and $\sigma(-Q)$ are separated by an annulus of width $\|A^{-1}\|^{-1}+\|B^{-1}\|^{-1}$, the annular separation result of Section 10 gives

$$\|I-U^*V\| \leqslant \frac{1}{\|A^{-1}\|^{-1}+\|B^{-1}\|^{-1}}\|Y+Z^*\|.$$

Since $\|Y\| = \|Z\| = \|A-B\|$ and $\|I-U^*V\| = \|U-V\|$, this gives

$$\|U-V\| \leqslant \frac{2}{\|A^{-1}\|^{-1}+\|B^{-1}\|^{-1}}\|A-B\|.$$

This inequality was proven by Li [**62**].

If f is a (Fréchet) differentiable map on the space of operators, denote its derivative at A by $Df(A)$. Then D is a real linear map whose action is given by

$$Df(A)(B) = \frac{d}{dt}\bigg|_{t=0} f(A+tB).$$

Example: if $f(A) = A^2$, then $Df(A)(B) = AB+BA$. If f is more complicated, the derivative may not be easy to calculate. Let g be the map defined on positive operators by $g(A) = A^{1/2}$. To calculate Dg, one might calculate the derivative of its inverse map $f(A) = A^2$, and then use the relation $Dg(A) = [Df(g(A))]^{-1}$. This consideration shows that if

$$Dg(A)(B) = X,$$

then

$$B = A^{1/2}X + XA^{1/2}.$$

This is exactly the kind of equation we have been considering. We obtain from this that

$$\|X\| \leqslant \tfrac{1}{2}\|A^{-1/2}\|\,\|B\|,$$

by the same methods as in the first part of this section. Hence

$$\|Dg(A)\| = \sup_{\|B\|=1} \|Dg(A)(B)\| \leqslant \tfrac{1}{2}\|A^{-1/2}\|.$$

Now let $h(A) = A^*A$ for every operator A. The derivative of this map is easy to calculate:

$$Dh(A)(B) = A^*B + B^*A.$$

This yields

$$\|Dh(A)\| \leqslant 2\|A\|.$$

Finally, let $\phi(A) = g(h(A)) = (A^*A)^{1/2}$. By the chain rule for differentiation, $D\phi(A) = Dg(h(A)) \circ Dh(A)$. Then combining the above two inequalities gives

$$\|D\phi(A)\| \leqslant \|A^{-1}\|\,\|A\|.$$

The number $\|A^{-1}\|\,\|A\|$ is called the *condition number* of A. Now, using Taylor's Theorem, we can obtain the following first-order perturbation bound: if A is an invertible operator with polar decomposition $A = UP$, and B is an operator near A with polar decomposition VQ, then

$$\|P-Q\| \leqslant \|A^{-1}\|\,\|A\|\,\|A-B\| + O(\|A-B\|^2).$$

This result was obtained in [**10**], which also contains other perturbation bounds of this type, and references to related papers.

13. *Conclusion*

There are many other situations in which the Sylvester–Rosenblum equation arises. There have also been numerical-analytic studies of rates of convergence of approximations to its solution. The following bibliography is fairly extensive, including many papers in addition to those cited above. The interested reader should peruse these (and their bibliographies) for further information.

There has also been work on more general equations. An *elementary operator* is an operator of the form $\mathscr{T}(X) = \sum_{i=1}^{n} A_i X B_i$; the operator $\mathscr{T}$ in Section 2 of this paper is a particular kind of elementary operator. In the case where each of $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ is a commutative set, an obvious extension of the method of Section 2 yields $\sigma(\mathscr{T}) \subset \sum_{i=1}^{n} \sigma(A_i)\,\sigma(B_i)$. There are many known results on elementary operators—see 'Elementary operators and applications', *Proc. Int. Workshop Blaubeuren* (World Scientific, Singapore, 1992).

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