CLASSICAL THEORY OF ELECTRONS

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Dirac (1938) has shown that by considering the conservation of energy and momentum, equations can be derived for the motion of a point charge in an electromagnetic field which are the same as those derived by Lorentz for the approximate motion in an external field of an electron whose charge is distributed in a finite volume. As derived by Dirac, the mass appears in these equations as an arbitrary constant, and the term containing it is not uniquely determined in its form. There are an infinite number of possibilities which fulfil the conditions required of this term, of which Dirac himself has given two. Only one of these, namely the simplest, leads to the equations of Lorentz. The purpose of this note is to show that there are other conditions which this term has to satisfy which have not been considered by Dirac, and which very drastically cut down the possible choice for this term. Nevertheless there is an infinite choice still possible. If however it is demanded that the equations shall not contain higher derivatives of the velocity than the second explicitly, then there is only one set of equations possible, namely those of Lorentz-Dirac. If the third derivative of the velocity is allowed to appear explicitly but not higher ones, then again an infinite number of equations becomes possible. But being equations in which the third derivative of the velocity appears explicitly, three data are necessary before the trajectory is properly determined, and it does not seem possible to give these in any natural way. There is, however, one exception, namely an equation much more complicated than the Lorentz-Dirac equation which leads to a motion of the point charge quite unlike the known behaviour of electrons.

In physics, besides the energy and momentum, the angular momentum is also conserved, so that in finding new equations for the motion of a point charge in an electromagnetic field, the conservation of this quantity has also to be taken into consideration. Now in fields without singularities the conservation of energy and momentum at every point of the field necessarily leads to the conservation of angular momentum, but this is no longer so when point charges are present in the field, and equations of motion for the point charge can be found such that energy and momentum but not the angular momentum are conserved for the whole system consisting of the point charge and the field. When point charges are present, therefore, the conservation of angular momentum has to be demanded explicitly.

Our method in this paper follows that of Dirac closely. We assume that the Maxwell equations hold exactly everywhere in space. Corresponding to the energy momentum density tensor of the field we now introduce a tensor of the third rank to represent the angular momentum density of the field. By means of this we calculate the flow of angular momentum out of a tube surrounding the world-line of the point charge. The conservation of angular momentum then demands that this flow shall only depend on conditions at the two ends of the tube. This cannot of course lead to a new set of equations for the motion of the point charge, since this is only described by one set of co-ordinates whose change is determined by the equations derived from the conservation of energy and momentum. But it will be shown that only for some equations describing the motion of the point charge does the conservation of energy and momentum automatically lead to the conservation of angular momentum of the system as a whole.

In the case of a point charge *with a spin attached to it,* the conservation of momentum and angular momentum lead to different sets of equations. The conservation of energy and momentum leads to equations for the motion of the point charge as a whole, while the conservation of angular momentum leads to equations for the rotation of the spin. This problem can be solved, but is very much more complicated than that of a point charge, and will be dealt with in a separate paper.

The Equations of Motion

The co-ordinates of a point will be denoted by x_μ, and the metric tensor will be assumed to be given by $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, the other components vanishing. The electromagnetic field strength at any point will be denoted by $F_{\mu\nu}$, which is an antesymmetrical tensor. The energy-momentum density tensor is given as usual by

$$4\pi\, T_{\mu\nu} = F_{\mu\sigma} F^{\sigma}{}_{\nu} + \tfrac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \tag{1}$$

which satisfies the equation of conservation in empty space

$$\frac{\partial}{\partial x_\nu} T_{\mu\nu} = 0. \tag{2}$$

We now introduce a tensor of the third rank $M_{\lambda\mu\nu}$, antesymmetrical in λ and μ, defined by

$$M_{\lambda\mu\nu} = x_\lambda T_{\mu\nu} - x_\mu T_{\lambda\nu}. \tag{3}$$

It also satisfies the law of conservation, since

$$\frac{\partial}{\partial x_\nu} M_{\lambda\mu\nu} = T_{\mu\lambda} + x_\lambda \frac{\partial}{\partial x_\nu} T_{\mu\nu} - T_{\lambda\mu} - x_\mu \frac{\partial}{\partial x_\nu} T_{\lambda\nu} = 0 \tag{4}$$

on account of (2) and the symmetry of $T_{\mu\lambda}$ in μ and λ. This tensor may therefore be taken to describe the angular momentum density of the field, for its space components M_{klo} correspond to the ordinary idea of angular momentum.

We denote the co-ordinates of the point charge e by z_μ (τ), which are functions of the proper time τ measured along the world line from some point on it. A dot over a symbol will be used to denote differentiation with respect to the proper time. The velocity $\dot{z}_\mu$ of the point charge will be denoted by v_μ. The 4-velocity v_μ and its derivatives satisfy the relations

$$\left.\begin{aligned} v^2 &= 1 \\ (v\,\dot{v}) &= 0 \\ (v\,\ddot{v}) + \dot{v}^2 &= 0 \\ (v\,\dddot{v}) + 3\,(\dot{v}\,\ddot{v}) &= 0 \end{aligned}\right\} \tag{5}$$

For brevity (XY) will be used to denote $X_\mu Y^\mu$ and X^2 to denote $X_\mu X^\mu$, where X_μ and Y_μ are any two 4-vectors. The retarded field produced by this point charge at any point is given by (see for example Dirac, 1938)

$$F^{\text{ret}}_{\mu\nu} = -e\left\{\frac{s_\mu v_\nu - s_\nu v_\mu}{\kappa^3}(1-\kappa') + \frac{s_\mu \dot{v}_\nu - s_\nu \dot{v}_\mu}{\kappa^2}\right\} \tag{6}$$

where

$$\begin{aligned} \kappa &= s_\mu v^\mu \\ \kappa' &= s_\mu \dot{v}^\mu, \end{aligned} \tag{7}$$

and

$$s_\mu = x_\mu - z_\mu(\tau_0) \tag{8}$$

is the distance from the point x_μ to the "retarded point" z_μ (τ_0), *i.e.*, the point on the world line such that

$$s_\mu s^\mu = 0, \; s_0 > 0. \tag{9}$$

The theory is quite symmetrical between retarded and advanced potentials so that for brevity we will restrict ourselves to the retarded potentials. The incoming field $F^{\text{in}}_{\mu\nu}$ at a point may then be defined following Dirac as the actual field $F^{\text{act}}_{\mu\nu}$ at the point minus the retarded field.

$$F^{\text{in}}_{\mu\nu} = F^{\text{act}}_{\mu\nu} - F^{\text{ret}}_{\mu\nu}. \tag{10}$$

We now assume the world line between the points τ_1 and τ_2 to be given and surround it by a tube. For convenience we take the tube to be defined by

$$\kappa \equiv s_\mu v^\mu = \epsilon \tag{11}$$

where ϵ is a small constant which in the end will be made to tend to zero. If dS_μ, denote an element of the three dimensional surface of this tube, then the flow of energy and momentum out of the tube is given by

$$\int T_{\mu\nu}\, dS^\nu \tag{12}$$

integrated over the surface of the tube, while the flow of angular momentum out of the tube is given by

$$\int M_{\lambda\mu\nu}\, dS^\nu . \tag{13}$$

Using (1) and (6), (12) can be calculated at once, as has been done in a previous paper (Bhabha, 1939). Omitting terms which vanish with ϵ it is equal to

$$-\int_{\tau_1}^{\tau_2} d\tau \left[e^2 \left(\tfrac{1}{2} \frac{\dot{v}_\mu}{\epsilon} - \tfrac{2}{3} v_\mu \dot{v}^2 \right) + e \overset{\text{in}}{F}_{\mu\sigma} v^\sigma \right].$$

For conservation this must only depend on conditions at the two ends of the tube, so that the integrand must be a perfect differential. As before we may put it equal to $\dot{A}_\mu - e^2 (\tfrac{1}{2} \frac{\dot{v}_\mu}{\epsilon} + \tfrac{2}{3} \ddot{v}_\mu)$ so that

$$- e^2 \left(\tfrac{1}{2} \frac{\dot{v}_\mu}{\epsilon} - \tfrac{2}{3} v_\mu \dot{v}^2 \right) - e \overset{\text{in}}{F}_{\mu\sigma} v^\sigma = \dot{A}_\mu - e^2 \left(\tfrac{1}{2} \frac{\dot{v}_\mu}{\epsilon} + \tfrac{2}{3} \ddot{v}_\mu \right), \tag{14}$$

that is

$$e^2 (\tfrac{2}{3} v_\mu \dot{v}^2 + \tfrac{2}{3} \ddot{v}_\mu) - e \overset{\text{in}}{F}_{\mu\sigma} v^\sigma = \dot{A}_\mu . \tag{15}$$

We have to find A_μ. Contracting (15) with v_μ we see that $\dot{A}_\mu$ must satisfy the condition

$$v^\mu \dot{A}_\mu = 0. \tag{16}$$

Now consider (13). By (3) and (8) it is equal to

$$\int (x_\lambda T_{\mu\nu} - x_\mu T_{\lambda\nu})\, dS^\nu = \int (s_\lambda T_{\mu\nu} - s_\mu T_{\lambda\nu})\, dS^\nu + \int (z_\lambda T_{\mu\nu} - z_\mu T_{\lambda\nu}) dS^\nu . \tag{17}$$

The first term on the right-hand side of (17) is evaluated in the appendix. Omitting terms which vanish with ϵ, it is equal to

$$-\tfrac{2}{3} e^2 \int_{\tau_1}^{\tau_2} d\tau\, (v_\lambda \dot{v}_\mu - v_\mu \dot{v}_\lambda). \tag{18}$$

Consider the second term. An integral over the tube can always be split into an integration over a two-dimensional section of the tube such that all points on this section correspond to the same retarded point, and then an integral along the world line with respect to the retarded points. In carrying out the first integration over the two-dimensional section, $z_\lambda(\tau)$ remains constant, so that this integration is the same as in (12). The second term may therefore be written at once

$$\int_{\tau_1}^{\tau_2} d\tau\, z_\lambda(\tau)\left[-e^2\left(\tfrac{1}{2}\frac{\dot{v}_\mu}{\epsilon}-\tfrac{2}{3}v_\mu\,\dot{v}^2\right)-e\,\overset{\text{in}}{F}_{\mu\sigma}\,v^\sigma\right]$$

$$-\int_{\tau_1}^{\tau_3} d\tau\, z_\mu(\tau)\left[-e^2\left(\tfrac{1}{2}\frac{\dot{v}_\lambda}{\epsilon}-\tfrac{2}{3}v_\lambda\,\dot{v}^2\right)-e\,\overset{\text{in}}{F}_{\lambda\sigma}\,v^\sigma\right],$$

which using (14) reduces at once to

$$\int_{\tau_1}^{\tau_2} d\tau\left[z_\lambda\left\{\dot{A}_\mu-e^2\left(\tfrac{1}{2}\frac{\dot{v}_\mu}{\epsilon}+\tfrac{2}{3}\ddot{v}_\mu\right)\right\}-z_\mu\left\{\dot{A}_\lambda-e^2\left(\tfrac{1}{2}\frac{\dot{v}_\lambda}{\epsilon}+\tfrac{2}{3}\ddot{v}_\lambda\right)\right\}\right]. \quad (19)$$

The flow of angular momentum out of the world tube is equal to the sum of (18) and (19). For conservation, this must only depend on conditions at the two ends of the tube, so that the integrand must be a perfect differential as before. This integrand is

$$z_\lambda\left\{\dot{A}_\mu-e^2\left(\tfrac{1}{2}\frac{\dot{v}_\mu}{\epsilon}+\tfrac{2}{3}\ddot{v}_\mu\right)\right\}-z_\mu\left\{\dot{A}_\lambda-e^2\left(\tfrac{1}{2}\frac{\dot{v}_\lambda}{\epsilon}+\tfrac{2}{3}\ddot{v}_\lambda\right)\right\}$$

$$-\tfrac{2}{3}e^2\left(v_\lambda\,\dot{v}_\mu-v_\mu\,\dot{v}_\lambda\right)$$

$$=\frac{\delta}{\delta\tau}\left[z_\lambda\left\{A_\mu-e^2\left(\tfrac{1}{2}\frac{v_\mu}{\epsilon}+\tfrac{2}{3}\dot{v}_\mu\right)\right\}-z_\mu\left\{A_\lambda-e^2\left(\tfrac{1}{2}\frac{v_\lambda}{\epsilon}+\tfrac{2}{3}\dot{v}_\lambda\right)\right\}\right]$$

$$-\left(v_\lambda\,A_\mu-v_\mu\,A_\lambda\right). \quad (20)$$

The first term is a perfect differential, so that the second term

$$v_\lambda\,A_\mu-v_\mu\,A_\lambda \quad (21)$$

has to be a perfect differential. This is a further restriction on the choice of A_μ.*

It is clear that the demand for conservation of angular momentum cannot lead to new equations of motion for a point charge, unlike the demand for the conservation of momentum, since the incoming field does not appear either in (18) or in (19). This is because the highest singularities in the retarded field are only of order ϵ^{-2}. For a point dipole, the highest singularities in the retarded field are of order ϵ^{-3}, so that the incoming field also appears explicitly in the flow of angular momentum out of the tube and leads to equations for the rotation of the dipole. In passing it should be noticed that whereas the direct flow of angular momentum out of the tube given by

* One choice for A_μ given by Dirac is $m\{v_\mu\,\dot{v}^4+4\,\dot{v}_\mu\,(v\,\dot{v})\}$, which is a particular case of $m\,[v_\mu\,\dot{v}^4\,C+\dot{v}_\mu\,\{4\,(\dot{v}\,\ddot{v})\,C+\dot{v}^2\,\dot{C}\}]$ with C an arbitrary scalar function fo τ. All these satisfy (16) but do not make (21) a perfect differential. They would therefore lead to the conservation of momentum but not of angular momentum,

(18) does not contain any singularities, the total flow of angular momentum is nevertheless infinite due to the infinite flow of momentum out of the tube as shown by (19).

We now come to the choice of an expression for A_μ. We notice at once that A_μ cannot contain z_μ, for as shown by Pryce (1938), A_μ can be interpreted as the " mechanical energy-momentum " of the point charge. This cannot depend on the choice of the axis of co-ordinates and hence on z_μ. If we therefore assume that A_μ does not contain higher derivatives of v_μ than the second explicitly, the most general substitution is

$$A_\mu = v_\mu \mathrm{B} + \dot{v}_\mu \mathrm{C} + \ddot{v}_\mu \mathrm{D}, \tag{22}$$

where B, C and D are invariant functions of the velocity and its higher derivatives. Using (5) the condition (16) then becomes

$$\dot{\mathrm{B}} - \dot{v}^2 \mathrm{C} - \dot{v}^2 \dot{\mathrm{D}} - 3 (\dot{v}\,\ddot{v}) \mathrm{D} = 0. \tag{23}$$

Substituting in (21) we get

$$(v_\lambda \dot{v}_\mu - v_\mu \dot{v}_\lambda) \mathrm{C} + (v_\lambda \ddot{v}_\mu - v_\mu \ddot{v}_\lambda) \mathrm{D} \tag{24}$$

Now due to its antesymmetrical properties, $(v_\lambda \dot{v}_\mu - v_\mu \dot{v}_\lambda)$ cannot be the derivative of a function of τ not containing z_μ explicitly. Its derivative is just the coefficient of D in (24). Thus, if (24) is to be a perfect differential,

$$\mathrm{C} = \dot{\mathrm{D}}. \tag{25}$$

With the help of this (23) becomes

$$\begin{aligned} \dot{\mathrm{B}} &= 2\dot{v}^2 \dot{\mathrm{D}} + 3 (\dot{v}\,\ddot{v}) \mathrm{D} \\ &= 2 \frac{\partial}{\partial \tau} (\dot{v}^2 \mathrm{D}) - (\dot{v}\,\ddot{v}) \mathrm{D}. \end{aligned} \tag{26}$$

$(\dot{v}\,\ddot{v})$ D must therefore be a perfect differential, and since we are restricting ourselves to derivatives of v not higher than the second, D must be a power of $\dot{v}^2$.

There are an infinite number of solutions which satisfy (25) and (26) given by

$$\begin{aligned} \mathrm{B} &= (4n + 3) (- \dot{v}^2)^{n+1} m \\ \mathrm{C} &= 4 (n + 1)n (\dot{v}\,\ddot{v}) (- \dot{v}^2)^{n-1} m \\ \mathrm{D} &= - 2 (n + 1) (- \dot{v}^2)^n m \end{aligned} \tag{27}$$

with arbitrary n and m. However, (27) is only of interest when $n \geqslant \frac{1}{2}$, for otherwise C at least becomes infinite when $\dot{v}_\mu = 0$, and hence the equations become very singular. They cannot be used for describing the behaviour of a point charge. For $n > \frac{1}{2}$, $\dot{A}_\mu$ vanishes when $\dot{v}_\mu = 0$, so that in the absence of an external field $\ddot{v}_\mu = 0$ follows from this by (15). Hence, for all these cases if the initial acceleration is zero the charge will continue in a state of uniform motion, and all the higher derivatives of the velocity will also vanish. A closer investigation shows that this is also true for $n = \frac{1}{2}$. Since however the third derivative of the velocity also appears explicitly in the equations (15) through $\dot{A}_\mu$ it is possible in general to find a solution which satisfies three arbitrary conditions. Thus the trajectory is not completely determined by the initial velocity and the stipulation that the final acceleration shall be zero as in the case investigated by Dirac. Yet another datum is required to determine the trajectory and this extra datum cannot be the condition that the final derivative of the acceleration shall vanish since this automatically follows if the final acceleration vanishes, as shown above. The finding of another initial condition to determine the trajectory seems to be rather artificial, and hence I believe that the set (27) in general cannot be used for describing the motion of a point charge.

An exception to the above argument is the case $n = 0$. Then

$$B = -3\, m\, \dot{v}^2$$
$$C = 0$$
$$D = -2m.$$

In this case it no longer follows in the absence of an external field from equation (15) that when $\dot{v}_\mu = 0$, $\ddot{v}_\mu$ also vanishes. But if $\ddot{v}_\mu$ also vanishes, then $\dddot{v}_\mu$ must vanish by (15), so that the particle will continue in a state of uniform motion. Thus in this case the trajectory is completely determined if the initial velocity is given, and the conditions are imposed that the final acceleration $\dot{v}_\mu$ and its derivative $\ddot{v}_\mu$ should vanish. There does not seem to be any reason for excluding (28) as a possible substitution for A_μ in the equations describing the motion of a point charge. It should be noticed that the arbitrary constant m must now have the dimensions of a mass times a length squared. The behaviour of the point charge would be quite unlike the known behaviour of an electron.

If we demand that $\dddot{v}_\mu$ should not appear explicitly in the equation (15), then the possible choice for A_μ is unique. For we must now put $D = 0$ in (22). It then follows by (25) that $C = 0$, and by (26) that $\dot{B} = 0$. B can therefore only be an arbitrary constant. In fact, in this case

$$A_\mu = m\, v_\mu\,, \tag{29}$$

which is the substitution which leads to the equations of Lorentz. Thus we have shown that *if it is demanded that the equations of motion of a point charge do not contain higher derivatives of the acceleration than the first, then the set of equations possible, consistent with the conservation laws, is unique.*

Appendix

We wish to calculate

$$\int (s_\lambda \, T_{\mu\nu} - s_\mu \, T_{\lambda\nu}) \, dS^\nu \tag{30}$$

If we fix our attention on any point of the world line in the Lorentz frame in which the electron is instantaneously at rest, then a sphere about this point of radius ϵ, taken at a time ϵ later is a section of this tube. Denoting by $d\omega$ an element of solid angle of this sphere, then, as has been shown in a previous paper (Bhabha, 1939) the surface element of the tube is given by

$$dS^\nu = \{s^\nu (1 - \kappa') - v^\nu \epsilon\} \, \epsilon \, d\omega \, d\tau, \tag{31}$$

which is of order ϵ^2. The retarded field given by (6) is of order ϵ^{-2}. Thus no non-vanishing terms containing the ingoing field will occur in (30).

Using (1) and (6), the first term of (30) becomes

$$\frac{1}{4\pi} \int s_\lambda \left\{ \overset{\text{ret}}{F_{\mu\sigma}} \; \overset{\sigma\,\text{ret}}{F_\nu} + \tfrac{1}{4} g_{\mu\nu} \overset{\text{ret}}{F_{\rho\sigma}} \; F^{\rho\sigma\,\text{ret}} \right\} dS^\nu$$

$$= \frac{e^2}{4\pi} \int s_\lambda \left\{ \frac{s_\mu - \epsilon v_\mu}{\epsilon^3} - 2 \, \frac{s_\mu - \epsilon v_\mu}{\epsilon^3} \kappa' - \frac{v_\mu}{\epsilon^2} \kappa' - \frac{\dot{v}_\mu}{\epsilon} \right\} d\omega \; d\tau$$

$$- \frac{e^2}{8\pi} \int \frac{s_\lambda}{\epsilon^3} \left\{ s_\mu (1 - \kappa') - v_\mu \, \epsilon \right\} d\omega \, d\tau.$$

The terms symmetrical in λ and μ vanish, and (30) reduces to

$$\frac{e^2}{4\pi} \int \left\{ - \tfrac{1}{2} \, \frac{s_\lambda v_\mu - s_\mu v_\lambda}{\epsilon^2} + \frac{s_\lambda v_\mu - s_\mu v_\lambda}{\epsilon^2} \kappa' - \frac{s_\lambda \dot{v}_\mu - s_\mu \dot{v}_\lambda}{\epsilon} \right\} d\omega \, d\tau,$$

which may be evaluated as in the previous paper by using the relations*

$$\frac{1}{4\pi} \int \frac{s_\mu}{\epsilon} \, d\omega = v_\mu$$

$$\frac{1}{4\pi} \int \frac{s_\mu}{\epsilon^2} \, s_\nu \, A^\nu \, d\omega = - \tfrac{1}{3} A_\mu + \tfrac{4}{3} v_\mu \, v_\nu \, A^\nu,$$

which hold for any arbitrary vector A_μ which is not a function of position on the sphere. The result is given in the text by (18).

Summary

It is shown that when a point charge is present in an electromagnetic field, the conservation of energy and momentum does not in general lead to conservation of angular momentum for the system as a whole. The conservation laws impose stringent restrictions on the possible equations which may describe the motion of the point charge. *If it is required that higher derivatives of the velocity than the second should not appear explicitly in these*

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* In the second and third of equations (58) of the previous paper, factors ϵ^{-1} and ϵ^{-2} respectively have been omitted on the right-hand side by mistake.

equations, then the choice is unique and the only possible equations are those originally derived by Lorentz. If the third derivative is allowed to appear explicitly in the equations, but not higher ones, then it is possible to give one other system of equations for describing the behaviour of a point singularity which can be used without entirely artificial initial and final conditions.

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