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H. J. Bhabha

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General classical theory of spinning particles in a meson field

By H. J. Bhabha, Ph.D.
Department of Physics, Indian Institute of Science, Bangalore

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An exact classical theory of the motion of a point dipole in a meson field is given which takes into account the effects of the reaction of the emitted meson field. The meson field is characterized by a constant $\chi = \mu/\hbar$ of the dimensions of a reciprocal length, μ being the meson mass, and as $\chi \to 0$ the theory of this paper goes over continuously into the theory of the preceding paper for the motion of a spinning particle in a Maxwell field. The mass of the particle and the spin angular momentum are arbitrary mechanical constants. The field contributes a small finite addition to the mass, and a negative moment of inertia about an axis perpendicular to the spin axis.

A cross-section (formula (88*a*)) is given for the scattering of *transversely* polarized neutral mesons by the rotation of the spin of the neutron or proton which should be valid up to energies of 10^9 eV. For low energies E it agrees completely with the old quantum cross-section, having a dependence on energy proportional to p^4/E^2 (p being the meson momentum). At higher energies it deviates completely from the quantum cross-section, which it supersedes by taking into account the effects of radiation reaction on the rotation of the spin. The cross-section is a maximum at $E \sim 3{\cdot}5\mu$, its value at this point being 3×10^{-26} cm.^{2}, after which it decreases rapidly, becoming proportional to E^{-2} at high energies. Thus the quantum theory of the interaction of neutrons with mesons goes wrong for $E \gtrsim 3\mu$. The scattering of longitudinally polarized mesons is due to the translational but not the rotational motion of the dipole and is at least twenty thousand times smaller.

With the assumption previously made by the present author that the heavy partilesc may exist in states of any integral charge, and in particular that protons of charge $2e$ and $-e$ may occur in nature, the above results can be applied to *charged* mesons. Thus transversely polarised mesons should undergo a very big scattering and consequent absorption at energies near $3 \cdot 5\mu$. Hence the energy spectrum of transversely polarized mesons should fall off rapidly for energies below about 3μ. Scattering plays a relatively unimportant part in the absorption of longitudinally polarized mesons, and they are therefore much more penetrating.

The theory does *not* lead to Heisenberg explosions and multiple processes.

It has been shown in the preceding paper† with Corben that it is possible to give a complete classical theory of the motion of a point dipole in a Maxwell field. This theory is free from the infinite energies associated in the usual theory with a point charge or a point dipole, and it was shown that the appearance of these infinities in the usual theory is fictitious and due to an incorrect definition of field energy when singularities are present in the field. It appears that the effects of radiation reaction are even more important for a point dipole than for a point charge, and this makes the scattering of light decrease as ω^{-2} for high frequencies ω instead of increasing as ω^2, as would be the case if radiation reaction were neglected.

The work of this paper is an extension to the meson field of the work of the preceding paper with Corben. It will constantly be necessary to refer to the results of that paper. The underlying ideas and assumptions are exactly the same, as is also the method of procedure. It must be understood that whenever the charge g_1 and the dipole moment g_2 are referred to in this paper, these are quantities which interact with and create the meson field and not the Maxwell field. In other words, the charge is not an electric charge, nor the dipole moment an electromagnetic moment. When need arises for specifically distinguishing these quantities from the corresponding electromagnetic quantities, I shall refer to them as a mesic charge and a mesic dipole moment, and correspondingly I shall talk of an electromesic force and a magnetomesic force.

The extension of the theory to the case of spinning particles moving in a meson field is of both theoretical and practical interest, for it is known that protons and neutrons have an explicit spin interaction with the meson field involving the interaction constant g_2, whereas this term appears to be absent in the interaction of electrons or protons with the Maxwell field. The meson field has a characteristic constant χ which is connected, when the field is quantized, with the rest mass μ of the meson by the relation $\chi = \mu/\hbar$ (the velocity of light is taken to be unity). In the classical theory neither μ nor $\hbar$ appear explicitly, but only in the combination χ which has the dimensions of a reciprocal length. Maxwell's theory is the particular case of meson theory

† Bhabha and Corben (1941), referred to in this paper as C.

in which $\chi = 0$. Now it was shown in a previous paper (Bhabha 1939, referred to in this paper as A) that the Green's function for the meson field is just the Green's function for the Maxwell field plus a non-singular part which has a plain discontinuity on the light cone. This additional part tends to zero continuously as $\chi \to 0$. It follows from this that in any given case, the highest singularities in the meson field are the same as those in the Maxwell field, and all additional terms are singular to a lesser degree. One would expect from this that it would be possible to extend the theory of spinning dipoles of the previous paper to a meson field. This is done in the present paper. Exact equations are derived for the motion of a point dipole in a meson field which contain the constant χ explicitly, and go over continuously into the equations of the previous paper as $\chi \to 0$. Even for the simplest type of point dipole, namely one whose mechanical properties are those of a pure gyroscope, there are three constants having the dimensions of a reciprocal length which enter into the equations of motion. These are $3M/2g_1^2$, $(3I/2g_2^2)^{\frac{1}{2}}$, and χ, where M and I denote the mass and spin angular momentum of the particle, and g_1 and g_2 the mesic charge and dipole moment respectively. If these constants are given the values which they have in nuclear theory, it appears that the first and second are respectively about 450 and 3 times the third. As stated above, Maxwell's theory corresponds to the limit $\chi = 0$, so that it follows from the magnitude of the constants that all processes which involve only the first and third constants, as, for example, the scattering of transverse waves by a point charge, will differ very slightly in the Maxwell and meson theories. This is confirmed by the results of a previous paper (A). On the other hand, in processes which involve the second constant also, the differences in the magnitudes of corresponding processes in the Maxwell and meson theories may be expected to be considerable, since the second constant is only about three times the third. The results of this paper confirm this. For frequencies ω_0 large compared with χ, the difference between the two theories is small and gets continuously smaller with increasing ω_0.

A cross-section is given for the scattering of meson waves due to the g_2 spin interaction by a particle which is free to rotate but not to translate. This cross-section goes over continuously as $\chi \to 0$ into the corresponding cross-section for the scattering of light. It has the same general dependence on frequency (formula (88) and the curve marked $\beta = 0$ in the figure). For low frequencies it increases as $(\omega_0^2 - \chi^2)^2/\omega_0^2$, and, if the angular momentum of the spin of the particle is put equal to $\frac{1}{2}\hbar$, it agrees completely† with the quantum cross-section for the scattering of neutral mesons by neutrons.

† See, however, second footnote on p. 341.

However, it differs from the behaviour of the quantum cross-section, which continues to increase as ω_0^2 to the highest energies, in that it reaches a maximum at $\omega_0 \sim 3{\cdot}5\chi$, and finally decreases for high frequencies as ω_0^{-2}. This decrease is entirely due to the effect of radiation reaction on the rotation of the spin, which is completely neglected in the quantum theory. Moreover, this behaviour at high frequencies is similar to the scattering by a point charge. For scattering by a point charge the cross-section decreases but slowly up to a certain frequency, after which the decrease becomes more rapid, finally becoming proportional to ω_0^{-2}. Since the scattering by the translational motion of a point charge and the rotation of a dipole are very different processes, it seems plausible to suppose that *this decrease of the scattering at high frequencies proportionally to* ω_0^{-2} *is a fundamental property of radiation fields, both Maxwellian and mesic.*

A non-relativistic attempt to avoid the increase of the quantum cross-section for the scattering of mesons due to the *spin* of the heavy particles has recently been made by Heitler (1940) in which it is assumed that the heavy particles can exist in states of spin $\frac{3}{2}\hbar$, $\frac{5}{2}\hbar$, etc. There are a number of objections to this assumption. First, it is known that the effect of the reaction of the emitted radiation is neglected in the quantum theory, so that a failure of the quantum formulae at high energies is in any case to be expected. The classical theory of this paper then shows that the effect of this radiation reaction is to make all cross-sections decrease at high frequencies like ω_0^{-2}, and hence it completely removes all necessity for assuming that the heavy particles may exist in higher spin states. Secondly, the assumption that the heavy particles can exist in higher spin states is an assumption about the *mechanical properties of the heavy particles,* and would only alter the scattering of mesons by heavy particles, whereas the difficulty is equally pronounced in the scattering of light by an electron *having an explicit spin interaction* with the Maxwell field, as is theoretically possible, although for the actual electron which occurs in nature this spin interaction happens to be zero. *There is complete parallelism between the two cases,* and the theories of this paper and the preceding paper with Corben deal with both on the same footing. To apply Heitler's idea to the scattering of light in the above hypothetical case we should have to assume that the electron could also exist in states of higher spin, and there is no evidence whatever for this. It is not unreasonable to suppose that all elementary particles with a spin $\frac{1}{2}\hbar$ in their normal state have the *same mechanical characteristics* and there is *only one relativistic* theory for a particle of spin $\frac{1}{2}\hbar$, namely, that in which the particle is described by the Dirac equation. It is not possible to say yet whether Heitler's idea allows of formulation in a way consistent with the theory of relativity. Thus, whereas the treatment of this paper is complete within the limits of classical theory, the approach of Heitler falls short of the completeness of present quantum mechanics by not being in relativistic form. Finally, the assumption of allowing the heavy particles to exist in states of higher spin merely puts the scattering by the spin on the same footing as the scattering by a point charge. In this it does not seem to me to go far enough. For it is well known that the formulation of the interaction of a point charge with a field in quantum theory neglects the effect of radiation reaction and leads to divergence difficulties in higher approximations. These have been discussed at length elsewhere (Bhabha 1940*b*). Dirac (1938) has shown how these difficulties may be removed for a point electron within the limits of classical theory, and the work of this paper is but an extension of the work of Dirac to a spinning particle moving in a meson field, and shows that in classical theory the complete treatment of the spin presents no more difficulty than the treatment of a point charge. Thus it would be reasonable to suppose that the difficulties concerning the spin in the quantum theory are to be removed in the same way as the difficulties concerning a point charge, namely, by introducing into quantum theory the effects of radiation reaction both for a point charge and for a point dipole. A quantum translation of the classical theory of Dirac and of this paper, if possible, would do this, though it may require a far-reaching extension of the basis of quantum mechanics.

An interesting feature appears in the scattering of meson waves. As shown in A, the scattering of longitudinal waves, due to the motion of a point charge is less than the scattering of transverse waves of the same frequency ω_0 by a factor χ^2/ω_0^2. This difference is very much accentuated in the scattering by a dipole. As far as concerns scattering by a magnetomesic dipole which is free to rotate but not to translate, it is only the magnetomesic force of the wave which acts on the dipole and causes it to oscillate. Now the magnetomesic force of a longitudinal wave is zero, so that no scattering of longitudinal waves by the dipole will occur at all. The scattering of longitudinal waves can therefore only be due to the translational motion of the point charge or dipole. With the actual values of the constants as they occur in nature the scattering of transverse waves due to the rotation of the dipole is more than twenty thousand times greater than the scattering of longitudinal waves due to the translational motion of the particle. Hence transverse and longitudinal waves behave quite differently. The difference only disappears for low frequencies, ω_0 very close to χ, when in any case the scattering of both is negligible. However, as shown in A, when a longitudinal wave is scattered it is more probable that the emerging wave will be transversely polarized, and hence the probability of its being scattered again will be very much greater.

The above remarks apply strictly to neutral mesons. The cause of the difference in the scattering of charged and neutral mesons was analysed in a previous paper (Bhabha 1939), and it was shown there (and also by Pauli) that it is due entirely to the fact that whereas in the usual theory a positive meson can only be emitted by a proton and a negative meson by a neutron, a neutral meson can be emitted by both a proton and a neutron. There are thus twice as many intermediate states leading to the scattering of neutral as of charged mesons, and the effects of these largely cancel each other, thus reducing the scattering of neutral mesons. To avoid this difference the present author put forward the idea that the heavy particles could exist in states of all integral charge, positive and negative, having different rest energies, of which only the two of lowest energy (rest mass), namely the proton and neutron, occur normally in nature. This idea was communicated to Dr Heitler and has been adopted by him to calculate the scattering of mesons due to the charge of the heavy particles (Heitler 1940). With this assumption, as was shown in a previous paper (1940*b*), the theory of charged mesons is put on the same footing as the theory of neutral mesons, so that the above-mentioned classical cross-sections which should be valid for energies up to 10^9 eV can at once be applied to charged mesons.†

This idea by itself is, however, not sufficient to remove the difficulties in the scattering of mesons due to the *spin* of the heavy particles. Heitler (1940) has attempted to overcome these difficulties by allowing the heavy particles to exist in states of higher spin, whereas the approach of this paper has been to show that the difficulties disappear if proper account is taken of the effects of radiation reaction. The scattering of mesons due to the *spin* of the heavy particles is therefore different in his theory from that given here. The two cross-sections have a different dependence on energy, and experiment could easily distinguish between them. Thus the cross-section given in this paper is small at low and high energies but becomes very large at energies near $3{\cdot}5\mu$. The effect of this would be that the energy spectrum of transversely polarized mesons would fall off rapidly for energies below about 3μ. Further, one would expect the mesons in cosmic radiation to fall into two groups depending on their polarization, longitudinally polarized mesons being scattered twenty thousand times less than transversely polarized mesons of the same energy and hence being far more penetrating. On the rare occasion, however, that a longitudinally polarized meson is scattered, the probability that the scattered meson is transversely polarized is of the order

† Cf. Bhabha and Madhava Rao, 1941, *Proc. Indian Acad. Sci.* A, **13**, 9–24.

$2\omega_0^2/(2\omega_0^2+\chi^2)$, so that the chance of a subsequent scattering is very much greater. For mesons of cosmic-ray energies, the difference in the scattering of longitudinally and transversely polarized mesons is so great that it might almost lead them to appear as different particles, whereas the difference is only in their polarization and consequently in their penetrating power. This difference vanishes for mesons of extremely small velocities, but then the scattering is in any case very small and negligible compared with a number of other processes.

It must be recognized, however, that the application of the cross-sections of this paper to *charged* mesons definitely implies an assumption that among other things protons of charge $2e$ and $-e$ may be expected to occur in nature. The circumstances under which these particles might be observed and the probability of processes leading to their creation have already been fully investigated in the paper mentioned (Bhabha 1940*b*). If the cross-section for the scattering of mesons given in this paper could be shown experimentally to be in agreement with the scattering of charged mesons, then this would be evidence, though not proof, that the assumption made about the nature of the heavy particles and the possible appearance of protons of charge $2e$ and $-e$ was correct. The theory of the motion of spinning particles in meson fields developed in this paper is, however, complete in itself, and does not depend on the correctness or otherwise of its application to *charged* mesons through the *independent assumption* about the nature of the heavy particles.

Finally, the work of this paper shows that the spin itself does *not* lead to multiple processes and Heisenberg explosions as hitherto supposed. In a previous paper (A) I showed that the mass of the meson cannot lead to explosions either. The only remaining possibility of the appearance of Heisenberg explosions then lies in the difference in the behaviour of charged and neutral mesons, and, as shown in the paper mentioned (Bhabha 1940*b*) this difference is removed by the assumption that the heavy particles can exist in states of all integral charge, so that even this possibility of the appearance of explosions disappears. It may then be concluded that large Heisenberg explosions are not possible theoretically. This does not, of course, mean that occasionally two and on rare occasions more mesons may not be simultaneously produced.

The meson field of a dipole

I keep to the notation of the preceding paper. The velocity of light is put equal to unity, and I take the metric tensor $g_{\mu\nu}$ in the form $g_{00}=1$,

$g_{11} = g_{22} = g_{33} = -1$ with all the other components vanishing. The co-ordinate of the particle is denoted by z_μ which can be considered as a function of the proper time τ, measured from some arbitrary point on the classical world line of the particle. A dot over a symbol denotes differentiation with respect to τ. The velocity $\dot{z}_\mu$ is denoted by v_μ. The spin of the particle is described by an antisymmetric tensor $S_{\mu\nu}$ which may likewise be considered as a function of τ. The mesic charge and current density P_μ at any point x_ρ may then be written

$$\mathrm{P}_\mu(x_\rho) = g_1 \int_{-\infty}^{\infty} d\tau\, v_\mu\, \delta(x_0 - z_0)\, \delta(x_1 - z_1)\, \delta(x_2 - z_2)\, \delta(x_3 - z_3), \tag{1}$$

and the dipole density $\Sigma_{\mu\nu}$ at the point x_ρ by

$$\Sigma_{\mu\nu}(x_\rho) = g_2 \int_{-\infty}^{\infty} d\tau\, S_{\mu\nu}(\tau)\, \delta(x_0 - z_0)\, \delta(x_1 - z_1)\, \delta(x_2 - z_2)\, \delta(x_3 - z_3). \tag{2}$$

As in the preceding paper, the invariant formed from any two tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ is written

$$(XY) \equiv X_{\mu\nu} Y^{\mu\nu}, \quad X^2 \equiv (XX) = X_{\mu\nu} X^{\mu\nu}.$$

The invariant formed from any combination of tensors and two vectors is written in the usual matrix notation; thus

$$(\dot{v}\ddot{S}\dot{S}v) = \dot{v}^\mu \ddot{S}_{\mu\sigma} \dot{S}^{\sigma\nu} v_\nu, \quad \dot{v}^2 = \dot{v}_\rho \dot{v}^\rho.$$

The antisymmetric tensor formed from two other antisymmetric tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ will sometimes be written in the vector product notation

$$[X\,.\,Y]_{\lambda\mu} = X_{\lambda\sigma} Y^\sigma{}_\mu - X_{\mu\sigma} Y^\sigma{}_\lambda.$$

The vector v_μ by definition satisfies

$$v^2 = 1. \tag{3}$$

The equations derived from this by successive differentiation are†

$$(v\dot{v}) = 0, \tag{4a}$$

$$(v\ddot{v}) + \dot{v}^2 = 0, \tag{4b}$$

$$(vv^{\text{iii}}) + 3(\dot{v}\ddot{v}) = 0, \tag{4c}$$

$$(vv^{\text{iv}}) + 4(\dot{v}v^{\text{iii}}) + 3\ddot{v}^2 = 0. \tag{4d}$$

Further

$$S^2 \equiv S_{\mu\nu} S^{\mu\nu} = \text{constant}, \tag{5}$$

† I write v^{iii} and v^{iv} for the third and fourth derivatives with respect to τ.

and the equations derived from this by successive differentiation are

$$(S\dot{S}) = 0, \tag{6a}$$

$$(S\ddot{S})+\dot{S}^2 = 0, \tag{6b}$$

$$(SS^{\text{iii}})+3(\dot{S}\ddot{S}) = 0. \tag{6c}$$

If the potentials of the meson field are denoted by U_ν and the field strengths by $G_{\mu\nu}$, the meson equations may be written in the form†

$$\left(\frac{\partial}{\partial x^\mu}U_\nu - \frac{\partial}{\partial x^\nu}U_\mu\right) = G_{\mu\nu}, \tag{7a}$$

$$\frac{\partial}{\partial x_\mu}G_{\mu\nu}+\chi^2 U_\nu = 4\pi \mathrm{P}_\nu + 4\pi\frac{\partial}{\partial x_\mu}\Sigma_{\mu\nu}. \tag{7b}$$

From (7a) it follows that

$$\frac{\partial}{\partial x^\lambda}G_{\mu\nu}+\frac{\partial}{\partial x^\mu}G_{\nu\lambda}+\frac{\partial}{\partial x^\nu}G_{\lambda\mu} = 0, \tag{8}$$

and from (7b)
$$\chi^2\frac{\partial}{\partial x_\nu}U_\nu = 4\pi\frac{\partial}{\partial x_\nu}\mathrm{P}_\nu = 0, \tag{9}$$

for a neutral meson field,‡ since then the divergence of the charge and current density P_μ vanishes. Inserting (7a) into (7b) and using (9) I get

$$\frac{\partial}{\partial x_\rho}\frac{\partial}{\partial x^\rho}U_\nu+\chi^2 U_\nu = 4\pi\mathrm{P}_\nu+4\pi\frac{\partial}{\partial x_\mu}\Sigma_{\mu\nu}. \tag{10}$$

The usual energy-momentum tensor of the meson field $T_{\mu\nu}$ is given by

$$4\pi T_{\mu\nu} = G_{\mu\rho}G^\rho{}_\nu+\tfrac{1}{4}g_{\mu\nu}G_{\rho\sigma}G^{\rho\sigma}+\chi^2(U_\mu U_\nu-\tfrac{1}{2}g_{\mu\nu}U_\rho U^\rho). \tag{11}$$

The angular momentum density tensor $M_{\lambda\mu\nu}$ of the field is defined as in C by

$$M_{\lambda\mu\nu} = x_\lambda T_{\mu\nu}-x_\mu T_{\lambda\nu}. \tag{12}$$

Since I am interested in investigating the action of the meson field on the dipole, I shall henceforth put $g_1 = 0$ for simplicity. If $g_1 \neq 0$, I have

† These differ from the form given in A in that the field strengths of this paper are equal to $-(G_{\mu\nu}-4\pi\Sigma_{\mu\nu})$ of A. This is only a matter of definition, and is more convenient for our purpose, although superficially it makes the equations less symmetrical.

‡ On the old theory of charged mesons and their interaction with the heavy particles, the charge current density leading to the creation of positive mesons was different from that leading to the creation of negative mesons, and neither satisfied a conservation equation, so that the right-hand side of (9) did not vanish. With the new assumption of allowing the heavy particles to exist in states of all integral charge this is altered, and the right-hand side of (9) vanishes again.

only to add to the equations of motion the terms calculated in A for the case $g_1 \neq 0$, $g_2 = 0$, and further calculate the cross-terms in $g_1 g_2$ exactly on the lines of the preceding paper. These are much simpler than the reaction terms in g_2^2 which will be treated below.

As shown in A, the solution of (10) when $P_\nu = 0$ can be written as an integral with the help of a Green's function G:

$$U_\nu(x_\rho) = \iiint\int_{-\infty}^{\infty} dx'_\rho G(x_\rho, x'_\rho) \left\{ \frac{\partial}{\partial x'_\mu} \Sigma_{\mu\nu}(x'_\rho) \right\}, \tag{13}$$

where x'_ρ stands for the four variables x'_0, x'_1, x'_2 and x'_3. G is a function of $x_\rho - x'_\rho$ only. There are, as usual, two independent solutions corresponding to the retarded and advanced potentials. I shall only deal with the retarded potentials since the result does not depend on which are taken.

If I write, for the moment, $u_\rho \equiv x_\rho - x'_\rho$, $u \equiv (u_\rho u^\rho)^{\frac{1}{2}}$ and $u_r = \left(\sum_1^3 u_k^2\right)^{\frac{1}{2}}$, the Green's function G can be written in the form

$$G(x_\rho, x'_\rho) = \frac{\partial}{u\, \partial u} F(u), \tag{14}$$

with the function F defined, as shown in A (48*a*), for the retarded potentials by

$$F(u) = \begin{cases} J_0(\chi u) & u_0 > u_r, \\ 0 & u_0 < u_r. \end{cases} \tag{15}$$

Here J_0 is the Bessel function of order zero. The function F therefore has a plain discontinuity on the light cone $u = 0$. I shall frequently have to use the following properties of Bessel functions.

$$s^{-n} J_{n+1}(s) = -\frac{d}{ds}\{s^{-n} J_n(s)\}, \tag{16}$$

$$\operatorname*{Lt}_{s\to 0} \frac{1}{s^n} J_n(s) = \frac{1}{2^n\, n!}, \tag{17}$$

$$J_{n-1}(s) + J_{n+1}(s) = \frac{2n}{s} J_n(s). \tag{18}$$

For $\Sigma_{\mu\nu}$ given by (2), the differentiation with respect to x'_μ in (13) can be made to operate on G by partial integration, and, since G is a function of $x_\rho - x'_\rho$ only, it can be written as a differentiation with respect to x_μ, which can then be taken outside the integral. Introducing (2) into (13) I thus find

$$U_\nu^{\text{ret.}}(x_\rho) = g_2 \frac{\partial}{\partial x_\mu} \int_{-\infty}^{\infty} d\tau\, S_{\mu\nu}(\tau) \frac{\partial}{u\, \partial u} F(u), \tag{19}$$

where, in the notation that will be used henceforth,

$$u_\rho = x_\rho - z_\rho(\tau), \tag{20a}$$

$$u = (u_\rho u^\rho)^{\frac{1}{2}}, \tag{20b}$$

$$u_r = \left(\sum_1^3 u_k^2\right)^{\frac{1}{2}}. \tag{20c}$$

I introduce the following symbols as abbreviations:

$$\kappa = u_\mu v^\mu, \quad \kappa' = u_\mu \dot{v}^\mu, \quad \kappa'' = u_\mu \ddot{v}^\mu, \tag{21}$$

and

$$S_\mu = S_{\mu\nu} v^\nu, \quad S'_\mu = \dot{S}_{\mu\nu} v^\nu, \quad S''_\mu = \ddot{S}_{\mu\nu} v^\nu, \quad S'''_\mu = S^{\text{iii}}_{\mu\nu} v^\nu, \quad S''''_\mu = S^{\text{iv}}_{\mu\nu} v^\nu. \tag{22}$$

Now, for any function of u,

$$\frac{\partial}{\partial x_\mu} F(u) = \frac{\partial u}{\partial x_\mu} \frac{\partial F(u)}{\partial u} = \frac{u^\mu}{u} \frac{\partial F(u)}{\partial u}, \tag{23}$$

and, by (20a) for a fixed point x_ρ,

$$\frac{\partial u^2}{\partial \tau} = -2\kappa. \tag{24}$$

By (23), (19) reduces to

$$U_\nu^{\text{ret.}} = g_2 \int_{-\infty}^{\infty} d\tau \frac{u^\rho S_{\rho\nu}}{u} \frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial F}{\partial u}\right),$$

or, by (24),

$$U_\nu^{\text{ret.}} = g_2 \int_{-\infty}^{\infty} d\tau \frac{u^\rho S_{\rho\nu}}{\kappa} \frac{\partial}{\partial \tau}\left(\frac{1}{\kappa}\frac{\partial F}{\partial \tau}\right).$$

After repeated integration by parts we get

$$U_\nu^{\text{ret.}} = g_2 \int_{-\infty}^{\infty} d\tau F(u) \frac{\partial}{\partial \tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial \tau}\left(\frac{u^\rho S_{\rho\nu}}{\kappa}\right)\right\}.$$

Hence, by (15),

$$U_\nu^{\text{ret.}} = g_2 \int_{-\infty}^{\tau_0} d\tau\, J_0(\chi u) \frac{d}{d\tau}\left\{\frac{1}{\kappa}\frac{d}{d\tau}\left(\frac{u^\rho S_{\rho\nu}}{\kappa}\right)\right\} \tag{25a}$$

$$= g_2 \int_0^{\infty} du\, J_0(\chi u) \frac{d}{du}\left\{\frac{d}{u\,du}\left(\frac{u^\rho S_{\rho\nu}}{\kappa}\right)\right\}. \tag{25b}$$

Here τ_0 is the retarded time, determined by

$$\{x_\rho - z_\rho(\tau_0)\}\{x^\rho - z^\rho(\tau_0)\} = 0. \tag{26a}$$

I shall write

$$s_\rho = x_\rho - z_\rho(\tau_0), \tag{26b}$$

so that (26a) may be written $s^2 = 0$.

Now, if $\chi = 0$, the equations (7) become the Maxwell equations, and we should expect (25) to become the expression for the Maxwell potential $\phi_\nu^{(2}$ given in the preceding paper. This is obvious from (25), for when $\chi = 0$, $J_0(\chi u) = 1$ by (17), and (25*a*) does in fact become exactly the expression for $\phi_\nu^{(2)}$ given in (23) of C. Integrating (25*a*) by parts, I have

$$U_\nu^{\text{ret.}} = g_2\left[J_0(\chi u)\frac{1}{\kappa}\frac{d}{d\tau}\left(\frac{u^\rho S_{\rho\nu}}{\kappa}\right)\right]_{-\infty}^{\tau_0} - g_2\int_{-\infty}^{\tau_0} d\tau\frac{1}{\kappa}\frac{d}{d\tau}\left(\frac{u^\rho S_{\rho\nu}}{\kappa}\right)\frac{d}{d\tau}J_0(\chi u). \quad (27)$$

By (17) and (21), the first part just becomes $\phi_\nu^{(2)}$ given by†

$$\phi_\nu^{(2)} = g_2\left\{\frac{\mathcal{S}_\nu}{\kappa^2} + \frac{s^\rho S_{\rho\nu}}{\kappa^3}, \quad -\frac{s^\rho S_{\rho\nu}}{\kappa^3}\kappa' + \frac{s^\rho \dot{S}_{\rho\nu}}{\kappa^2}\right\}, \quad (28)$$

which is just the Maxwell potential given by (23) of C. I make the convention that all quantities are to be taken at the retarded time τ_0 except when they enter in the integrand of an integral along the world line. Thus the first term of (28) is $\mathcal{S}_\nu(\tau_0)\{s_\mu v^\mu(\tau_0)\}^{-2}$. By (24) and (16), the second integral becomes

$$-g_2\chi\int_{-\infty}^{\tau_0}\frac{d}{d\tau}\left(\frac{u^\rho S_{\rho\nu}}{\kappa}\right)\frac{1}{u}J_1(\chi u),$$

which, on again integrating by parts, becomes

$$-g_2\chi\left[\frac{u^\rho S_{\rho\nu}}{\kappa}\frac{J_1(\chi u)}{u}\right]_{-\infty}^{\tau_0} + g_2\chi\int_{-\infty}^{\tau_0}\frac{u^\rho S_{\rho\nu}}{\kappa}\frac{d}{d\tau}\left(\frac{J_1(\chi u)}{u}\right).$$

Hence using (16) and (17) again we may write finally,‡

$$U_\nu^{\text{ret.}} = \phi_\nu^{(2)} + U_\nu^{(\chi)}, \quad (29)$$

where

$$U_\nu^{(\chi)} = -\tfrac{1}{2}g_2\chi^2\frac{s^\rho S_{\rho\nu}}{\kappa} + \tilde{U}_\nu, \quad (30)$$

with

$$\tilde{U}_\nu = g_2\chi^2\int_{-\infty}^{\tau_0} d\tau\frac{u^\rho S_{\rho\nu}}{u^2}J_2(\chi u). \quad (31)$$

The integral in (31) remains finite even when the point χ_ρ approaches the world line. The first term of (30) is finite but not one-valued on the world line, its value depending on the direction from which the point on the world line is approached. $U_\nu^{(\chi)} \to 0$ as $\chi \to 0$.

† Following Dirac, I separate terms which correspond (as $\kappa \to 0$) to singularities of different orders by a comma.

‡ For a consistent notation I should write the second part as $U_\nu^{(2,\chi)}$. I have, however, written it as $U_\nu^{(\chi)}$ for simplicity, since g_1 has been put equal to zero in this paper.

By using ($7a$) we can now find the retarded field $G^{\text{ret.}}_{\mu\nu}$. Corresponding to (29) we can write

$$G^{\text{ret.}}_{\mu\nu} = F^{(2)}_{\mu\nu} + G^{(\chi)}_{\mu\nu}, \tag{32}$$

where $F^{(2)}_{\mu\nu}$ is the expression for the retarded field in the Maxwell theory, and is defined by

$$F^{(2)}_{\mu\nu} = \frac{\partial}{\partial\chi^{\mu}}\phi^{(2)}_{\nu} - \frac{\partial}{\partial\chi^{\nu}}\phi^{(2)}_{\mu}. \tag{33}$$

It is given explicitly in C (114). $G^{(\chi)}_{\mu\nu}$ is derived from $U^{(\chi)}_{\nu}$. In differentiating (30) it must be remembered that a change in the point x_{ρ} also changes the retarded point τ_0. The method has been given in A. I find†

$$G^{(\chi)}_{\mu\nu} = -\tfrac{1}{2}g_2\chi^2\left[\frac{S_{\mu\nu}}{\kappa} + \frac{s_{\mu}\dot{S}_{\nu}}{\kappa^2} - \frac{v_{\mu}s^{\rho}S_{\rho\nu}}{\kappa^2} + \frac{s_{\mu}s^{\rho}S_{\rho\nu}}{\kappa^3}(1-\kappa') + \frac{s_{\mu}s^{\rho}\dot{S}_{\rho\nu}}{\kappa^2}\right]_{-}$$
$$+\tfrac{1}{8}g_2\chi^4\left[\frac{s_{\mu}s^{\rho}S_{\rho\nu}}{\kappa}\right]_{-} + \tilde{G}_{\mu\nu}, \tag{34}$$

with

$$\tilde{G}_{\mu\nu} = 2g_2\chi^2\int_{-\infty}^{\tau_0} d\tau S_{\mu\nu}\frac{J_2(\chi u)}{u^2} - g_2\chi^3\int_{-\infty}^{\tau_0} d\tau\frac{u_{\mu}u^{\rho}S_{\rho\nu} - u_{\nu}u^{\rho}S_{\rho\mu}}{u^3}J_3(\chi u). \tag{35}$$

The part proportional to χ^2 in (34) has a singularity of order κ^{-1}, while the part proportional to χ^4 vanishes on the world line. The expressions (29) and (32) show quite clearly that the retarded meson potential and field is the same as the corresponding Maxwell potential and field plus a part containing χ which is singular to a lesser degree and vanishes when $\chi \to 0$. Both $\tilde{U}_{\nu}$ given by (31) and $\tilde{G}_{\mu\nu}$ given by (35) are finite and one-valued on the world line.

Now, as in the previous papers, I write the actual potentials U_{ν} and field strengths $G_{\mu\nu}$ at a point as the sum of the retarded and ingoing potentials and field strengths respectively at that point; thus

$$U_{\nu} = U^{\text{ret.}}_{\nu} + U^{\text{in.}}_{\nu}, \tag{36a}$$

$$G_{\mu\nu} = G^{\text{ret.}}_{\mu\nu} + G^{\text{in.}}_{\mu\nu}. \tag{36b}$$

The ingoing potentials and field strengths satisfy the equations (7) and (10) with P_{ν} and $\Sigma_{\mu\nu}$ put equal to zero. Using (29) and (32) we may write (36) as

$$U_{\nu} = \phi^{(2)}_{\nu} + U^{(\chi)}_{\nu} + U^{\text{in.}}_{\nu}, \tag{37a}$$

$$G_{\mu\nu} = F^{(2)}_{\mu\nu} + G^{(\chi)}_{\mu\nu} + G^{\text{in.}}_{\mu\nu}. \tag{37b}$$

† The minus sign at the end of a bracket means that the same terms with μ and ν interchanged are to be subtracted.

The ingoing field can always be written as a superposition of plane waves, a typical solution being

$$U_\nu = \gamma_\nu \cos(\omega_\nu x^\nu), \tag{38}$$

where, by (9) and (10),

$$\omega_\nu \gamma^\nu = 0, \tag{39a}$$

$$\omega^2 \equiv \omega_\nu \omega^\nu = \chi^2. \tag{39b}$$

The field strengths for this wave are

$$G_{\mu\nu} = -(\omega_\mu \gamma_\nu - \omega_\nu \gamma_\mu) \sin(\omega_\sigma x^\sigma). \tag{40}$$

A transversely polarized wave is defined as one in which the amplitude† γ_k of the vector potential is perpendicular to the direction of propagation ω_k. Then, by (39*a*), $\gamma_0 = 0$, and $U_0 = 0$. The field strengths $G_{\mu\nu}$ are perpendicular to the direction of propagation. A longitudinally polarized wave is defined as one in which the amplitude γ_k is parallel to the direction of propagation ω_k. Then, by (40), G_{kl} vanishes exactly, so that a longitudinally polarized wave has no magnetomesic force. The 'electromesic force' G_{0k} is now along the direction of propagation. Inserting (38) and (40) into (11), the energy-momentum tensor for this plane wave averaged over a period is

$$-\frac{1}{8\pi}\omega_\mu \omega_\nu \gamma^2. \tag{41}$$

The vector γ_μ is perpendicular to the time-like vector ω_μ, so that γ^2 is negative.

The equations of motion

To find the equations of motion of the particle in a given meson field I use the method of the previous papers. Assume the world line of the particle to be given and the direction of the spin at every point of it. Now surround the world line between the points τ_1 and τ_2 by a thin world tube, and using the tensors (11) and (12) calculate the flow of energy, momentum and angular momentum into the tube. For the conservation laws to hold, the flow of each of these quantities into the tube should equal the increase in the amounts stored at the two ends of the tube. As before, I take the world tube to be defined by

$$\kappa \equiv s_\mu v^\mu = \epsilon, \tag{42}$$

where ultimately ϵ will be made to tend to zero.

† A Latin suffix only takes on the values 1, 2 and 3. A repeated Latin suffix therefore implies a summation from 1 to 3 only.

If we denote an element of the three-dimensional surface of the tube by dS^ν taken as positive when the normal is directed outwards, the flow of energy and momentum into the tube is

$$\int T_{\mu\nu}\, dS^\nu,$$

which, as shown in C, can always be written in the form

$$\int_{\tau_1}^{\tau_2} T_\mu\, d\tau.$$

For conservation of energy and momentum, this must only depend on the conditions at the two ends of the tube, so that the integrand must be a perfect differential. We must then have

$$T_\mu = \dot{A}_\mu, \tag{43}$$

and if A_μ is known, this becomes the equation determining the translational motion of the dipole.

Similarly, the flow of angular momentum into the tube is given by

$$\int M_{\lambda\mu\nu}\, dS^\nu.$$

Defining $M_{\lambda\mu}$ by
$$\int_{\tau_1}^{\tau_2} M_{\lambda\mu}\, d\tau = \int (s_\lambda T_{\mu\nu} - s_\mu T_{\lambda\nu})\, dS^\nu, \tag{44}$$

we can show, as in C, that the conservation of angular momentum requires that $M_{\lambda\mu} - (v_\lambda A_\mu - v_\mu A_\lambda)$ shall be a perfect differential, so that I have to put

$$M_{\lambda\mu} - (v_\lambda A_\mu - v_\mu A_\lambda) = \dot{B}_{\lambda\mu}. \tag{45}$$

If $B_{\lambda\mu}$ is fixed, this becomes the equation determining the rotational motion of the dipole.

The calculation follows the work of C step by step. The only difference is that I get additional terms involving χ in T_μ and $M_{\lambda\mu}$ besides the terms already given in the previous paper. These arise from two causes. First, the retarded field strengths differ from the Maxwell field strengths $F^{(2)}_{\mu\nu}$ by having the extra terms $G^{(\chi)}_{\mu\nu}$. Secondly, the energy tensor (11) itself has additional terms proportional to χ^2 and involving the potentials U_ν explicitly. I can therefore write

$$T_\mu = T_\mu^{\text{max.}} + T_\mu^{\text{mes.}}, \tag{46}$$

where $T_\mu^{\text{max.}}$ is just the expression calculated in C for the Maxwell field, with the only difference that the ingoing meson field strengths $G^{\text{in.}}_{\mu\nu}$ appear in place of the ingoing Maxwell field strengths $F^{\text{in.}}_{\mu\nu}$. Then $T_\mu^{\text{mes.}}$ is the additional part that appears for a meson field due to the two causes mentioned above. It contains χ explicitly and vanishes as $\chi \to 0$. Corresponding to this, I write A_μ in (43) as the sum of two parts,

$$A_\mu = A_\mu^{\text{max.}} + A_\mu^{\text{mes.}}, \tag{47}$$

where $A_\mu^{\text{max.}}$ is the expression for A_μ found in C for a Maxwell field. Similarly, write

$$M_{\lambda\mu} = M_{\lambda\mu}^{\text{max.}} + M_{\lambda\mu}^{\text{mes.}}, \tag{48}$$

$$B_{\lambda\mu} = B_{\lambda\mu}^{\text{max.}} + B_{\lambda\mu}^{\text{mes.}}. \tag{49}$$

By substituting these into (43) and (45), and using the expressions for $T_\mu^{\text{max.}}$, $A_\mu^{\text{max.}}$, $M_{\lambda\mu}^{\text{max.}}$ and $B_{\lambda\mu}^{\text{max.}}$ found in C, the translational equation (43) becomes

$$M\dot{v}_\mu + \frac{d}{d\tau} v_\mu\{\tfrac{1}{4}K\dot{S}^2 + \tfrac{1}{4}K'(\dot{S}\dot{S}^*) - \tfrac{1}{2}g_2(SG^{\text{in.}})\}$$
$$= -\tfrac{1}{2}g_2 S^{\rho\sigma}\frac{\partial}{\partial x^\mu}G_{\rho\sigma}^{\text{in.}} + T_\mu^{\text{react.}} + (T_\mu^{\text{mes.}} - \dot{A}_\mu^{\text{mes.}}), \tag{50}$$

and the rotational equation (45) becomes

$$I\dot{S}_{\lambda\mu} + I'\dot{S}^*_{\lambda\mu} + K[S.\ddot{S}]_{\lambda\mu} + K'[S.\ddot{S}^*]_{\lambda\mu}$$
$$= g_2[S.G^{\text{in.}}]_{\lambda\mu} + [S.C^{\text{react.}}]_{\lambda\mu} + \{M_{\lambda\mu}^{\text{mes.}} - (v_\lambda A_\mu^{\text{mes.}} - v_\mu A_\lambda^{\text{mes.}}) - \dot{B}_{\lambda\mu}^{\text{mes.}}\}. \tag{51}$$

These are just equations (72) and (73) of C which give the motion of a spinning particle in a Maxwell field, with additional terms. M, I, I', K and K' are arbitrary constants, and have the dimensions of mass, angular momentum and moment of inertia respectively. $S^*_{\lambda\mu}$ is the six-vector adjunct† (dual) to $S_{\lambda\mu}$. Finally, $T_\mu^{\text{react.}}$ and $C_{\lambda\mu}^{\text{react.}}$ are the terms proportional to g_2^2 giving the reaction of radiation for a Maxwell field. Now, by the arguments given in C, we must have

$$[M_{\lambda\mu}^{\text{mes.}} - (v_\lambda A_\mu^{\text{mes.}} - v_\mu A_\lambda^{\text{mes.}}) - \dot{B}_{\lambda\mu}^{\text{mes.}}] \equiv S_\lambda{}^\rho C_{\rho\mu}^{\text{mes.}} - S_\mu{}^\rho C_{\rho\lambda}^{\text{mes.}}, \tag{52}$$

that is, $A_\mu^{\text{mes.}}$ and $B_{\lambda\mu}^{\text{mes.}}$ must be so chosen that the left-hand side of (52) has the form of the right-hand side of this equation. Further, in order that the invariant equation got by contracting (50) with v^μ should be consistent with an invariant equation which can be derived from (51) as shown in C, one must demand that

$$(\dot{S}.C^{\text{mes.}}) = 2v^\mu(T_\mu^{\text{mes.}} - \dot{A}_\mu^{\text{mes.}}). \tag{53}$$

† Its components are connected with those of $S_{\mu\nu}$ as in C (26) by $S^{*01} = S_{23}$, $S^*_{01} = -S^{23}$, etc., the other relations being derived by a cyclical permutation of any three suffixes.

Now consider the contribution to $T_\mu^{\text{mes.}}$ that comes from the fact that the retarded field strengths in (37 *b*) contain the extra term $G_{\mu\nu}^{(\chi)}$. The highest singularities in the three terms on the right-hand side of (37 *b*) are respectively of order ϵ^{-3}, ϵ^{-1}, and 1, while the surface element dS^ν is of order ϵ^2. Hence the addition to $T_\mu^{\text{mes.}}$ due to the extra term $G_{\mu\nu}^{(\chi)}$ in (37 *b*) will come from product terms in $T_{\mu\nu}$ involving $F_{\mu\nu}^{(2)}$ and $G_{\mu\nu}^{(\chi)}$, and from terms quadratic in $G_{\mu\nu}^{(\chi)}$. There will be no extra terms containing $G_{\mu\nu}^{\text{in.}}$. Now $G_{\mu\nu}^{(\chi)}$ contains the term $\tilde{G}_{\mu\nu}$ given by (35), which, as has been mentioned before, has no singularities even on the world line. This term will therefore appear in the equations of motion in terms of the same form as those involving the ingoing field $G_{\mu\nu}^{\text{in.}}$. Moreover, since the highest singularity in $F_{\mu\nu}^{(2)}$ is of order ϵ^{-3}, the derivative of $\tilde{G}_{\mu\nu}$ will also appear in the translational equations, just like the derivative of $G_{\mu\nu}^{\text{in.}}$. The derivative appears as before because the value of $\tilde{G}_{\mu\nu}$ at a point x_ρ on the world tube has to be expressed in terms of its value at the retarded point $z_\rho(\tau_0)$ on the world line by a Taylor's series; thus

$$(\tilde{G}_{\mu\nu})_{x_\rho} = (\tilde{G}_{\mu\nu})_{z_\rho(\tau_0)} + s^\rho\left(\frac{\partial}{\partial x^\rho}\tilde{G}_{\mu\nu}\right)_{z_\rho(\tau_0)}. \tag{54}$$

Now, using (35), I find

$$\begin{aligned}\frac{\partial}{\partial x^\rho}\tilde{G}_{\mu\nu} &= \tfrac{1}{4}g_2\chi^4 S_{\mu\nu}\frac{s_\rho}{\kappa} - 2g_2\chi^3\int_{-\infty}^{\tau_0} S_{\mu\nu}u_\rho\frac{J_3(\chi u)}{u^3} \\ &\quad - \tfrac{1}{48}g_2\chi^6(s_\mu s^\lambda S_{\lambda\nu} - s_\nu s^\lambda S_{\lambda\mu})\frac{s_\rho}{\kappa} - g_2\chi^3\int_{-\infty}^{\tau_0} d\tau[g_{\mu\rho}u^\lambda S_{\lambda\nu} + u_\mu S_{\rho\nu}]_-\frac{J_3(\chi u)}{u^3} \\ &\quad + g_2\chi^4\int_{-\infty}^{\tau_0} d\tau(u_\mu u^\lambda S_{\lambda\nu} - u_\nu u^\lambda S_{\lambda\mu})\,u_\rho\frac{J_4(\chi u)}{u^4} \\ &= \tfrac{1}{4}g_2\chi^4 S_{\mu\nu}\frac{s_\rho}{\kappa} - \tfrac{1}{48}g_2\chi^6(s_\mu s^\lambda S_{\lambda\nu} - s_\nu s^\lambda S_{\lambda\mu})\frac{s_\rho}{\kappa} + \tilde{G}_{\mu\nu,\rho},\end{aligned} \tag{55}$$

where

$$\begin{aligned}\tilde{G}_{\mu\nu,\rho} &= -g_2\chi^3\int_{-\infty}^{\tau_0} d\tau[u_\rho S_{\mu\nu} + g_{\mu\rho}u^\lambda S_{\lambda\nu} + u_\mu S_{\rho\nu}]_-\frac{J_3(\chi u)}{u^3} \\ &\quad + g_2\chi^4\int_{-\infty}^{\tau_0} d\tau[u_\mu u^\lambda S_{\lambda\nu}]_-\frac{J_4(\chi u)}{u^4}.\end{aligned} \tag{56}$$

The minus sign means that terms with μ and ν interchanged have to be subtracted. The first two terms on the right-hand side of (55) come from the fact that a change in the point x_ρ changes the retarded point τ_0 according to (26 *a*). The second term is of order ϵ^2 and hence contributes nothing to T_μ and $M_{\lambda\mu}$, as will be seen in the appendix. It vanishes on the world line. The expression (55) is, moreover, contracted with s^ρ in (54), so that the first term on the right-hand side of (55) also vanishes by (26). Hence $\tilde{G}_{\mu\nu,\rho}$ will play the part of $\frac{\partial}{\partial x^\rho}\tilde{G}_{\mu\nu}$ in the equations of motion. Contracting (55) with v^ρ and remembering (21), I get at a point on the world line

$$\frac{d}{d\tau}\tilde{G}_{\mu\nu} \equiv v^\rho \frac{\partial}{\partial x^\rho}\tilde{G}_{\mu\nu} = \tfrac{1}{4}g_2\chi^4 S_{\mu\nu} + \tilde{G}_{\mu\nu,\rho}v^\rho. \tag{57}$$

$T_\mu^{\text{mes.}}$ and $M_{\lambda\mu}^{\text{mes.}}$ are given by (97) and (99) in the appendix, and $A_\mu^{\text{mes.}}$ and $B_{\lambda\mu}^{\text{mes.}}$ are chosen to satisfy (52) and (53). If we introduce the expressions for $T_\mu^{\text{mes.}} - \dot{A}_\mu^{\text{mes.}}$ given in (102) of the appendix, the translational equation (50) becomes

$$\begin{aligned} M\dot{v}_\mu + \frac{d}{d\tau}v_\mu\{\tfrac{1}{4}K\dot{S}^2 + \tfrac{1}{4}K'(\dot{S}\dot{S}^*) - \tfrac{1}{2}g_2(S[G^{\text{in.}} + \tilde{G}])\} \\ = -\tfrac{1}{2}g_2 S^{\rho\sigma}\left(\frac{\partial}{\partial x^\mu}G_{\rho\sigma}^{\text{in.}} + \tilde{G}_{\rho\sigma,\mu}\right) + g_2^2\chi^4(-\tfrac{1}{8}v_\mu S^2 + \tfrac{1}{4}S_{\mu\rho}\mathcal{S}^\rho + \tfrac{1}{4}v_\mu \mathcal{S}^2) \\ + T_\mu^{\text{react.}} + (T_\mu'^{\text{mes.}} - \dot{A}_\mu'^{\text{mes.}}), \end{aligned} \tag{58}$$

where $T_\mu'^{\text{mes.}} - \dot{A}_\mu'^{\text{mes.}}$ is an expression proportional to $g_2^2\chi^2$ and is given by (109) in the appendix. The terms in χ^4 are important. On contracting (58) with v^μ, the last two terms proportional to χ^4 cancel each other, while the first combines according to (57) with $\tilde{G}_{\rho\sigma,\mu}$ to give

$$\tfrac{1}{2}g_2\{S^{\rho\sigma}\tilde{G}_{\rho\sigma,\mu} + \tfrac{1}{4}g_2\chi^4 v_\mu S^2\}v^\mu = \tfrac{1}{2}g_2 S^{\rho\sigma}\frac{d}{d\tau}\tilde{G}_{\rho\sigma},$$

which is analogous to

$$\tfrac{1}{2}g_2\left(S^{\rho\sigma}\frac{\partial}{\partial x^\mu}G_{\rho\sigma}^{\text{in.}}\right)v^\mu = \tfrac{1}{2}g_2 S^{\rho\sigma}\frac{d}{d\tau}G_{\rho\sigma}^{\text{in.}}.$$

By using (104) of the appendix, the rotational equation (51) becomes

$$\begin{aligned} I\dot{S}_{\lambda\mu} + I'\dot{S}_{\lambda\mu}^* + K[S.\ddot{S}]_{\lambda\mu} + K'[S.\ddot{S}^*]_{\lambda\mu} \\ = g_2[S.(G^{\text{in.}} + \tilde{G})]_{\lambda\mu} + [S.C^{\text{react.}}]_{\lambda\mu} + [S.C'^{\text{mes.}}]_{\lambda\mu}, \end{aligned} \tag{59}$$

where $C_{\rho\mu}'^{\text{mes.}}$ is given in the appendix by (110) and is proportional to $g_2^2\chi^2$. The reaction terms $T_\mu'^{\text{mes.}} - \dot{A}_\mu'^{\text{mes.}}$ and $C_{\rho\mu}'^{\text{mes.}}$ contain one arbitrary dimensionless constant k. This completely determines the exact equations of motion of a spinning particle in a meson field. As $\chi \to 0$, the equations go over continuously into those derived in *C* for the motion of a spinning particle in a Maxwell field.

Specialized equations can be derived from (58) and (59) which are consistent with the condition that the mesic dipole moment shall always be a pure 'magnetic' moment in the system in which the particle is at rest. This condition is expressed by

$$S_{\mu\nu} v^{\nu} = 0. \tag{60}$$

To derive equations consistent with (60) I follow the procedure given in C. Take all the terms in (59) on to the left-hand side and contract this expression with v^{μ}. I then have to add the resulting expression to A_{μ} in (43). Any addition to A_{μ} naturally also alters the rotational equation (51) or (59), since A_{μ} occurs on the left-hand side of (45). The alteration in the terms not involving χ has already been given in the previous paper. I have only to consider the alteration in the additional terms involving χ. By (60), the effect of the above operation is to replace $T_{\mu}^{\text{mes.}} - \dot{A}_{\mu}^{\text{mes.}}$ in (50) by

$$T_{\mu}^{\text{mes.}} - \dot{A}_{\mu}^{\text{mes.}} + \frac{d}{d\tau}(S_{\mu}{}^{\rho} C_{\rho\nu}^{\text{mes.}} v^{\nu}). \tag{61}$$

The corresponding change in terms involving χ on the right-hand side of (59) is to replace $[S_{\lambda}{}^{\rho} C_{\rho\mu}^{\text{mes.}}]_{-}$ by $[S_{\lambda}{}^{\rho}(C_{\rho\mu}^{\text{mes.}} - C_{\rho\nu}^{\text{mes.}} v^{\nu} v_{\mu})]_{-}$. Using (102) and (104) of the appendix, and the results of C for the modification of the other terms not involving χ, I find that the translational equation becomes

$$M\dot{v}_{\mu} + \frac{d}{d\tau}\{IS'_{\mu} + \tfrac{1}{4}Kv_{\mu}\dot{S}^2 + KS_{\mu\rho}S''^{\rho} - \tfrac{1}{2}g_2 v_{\mu}(S[G^{\text{in.}} + \tilde{G}]) - g_2 S_{\mu}{}^{\rho}(G_{\rho\nu}^{\text{in.}} + \tilde{G}_{\rho\nu})v^{\nu}\}$$
$$= -\tfrac{1}{2}g_2 S^{\rho\sigma}\left(\frac{\partial}{\partial x^{\mu}} G_{\rho\sigma}^{\text{in.}} + \tilde{G}_{\rho\sigma,\mu}\right) - \tfrac{1}{8}g_2^2 \chi^4 v_{\mu} S^2 + T_{\mu}^{\text{self}} + T_{\mu}^{\text{self},\chi}. \tag{62}$$

Here, as in C, I have put $K' = 0$ for simplicity. I' must necessarily be put equal to zero. T_{μ}^{self} is the reaction term in g_2^2 given explicitly in an earlier paper (Bhabha 1940*a*, referred to here as B). $T_{\mu}^{\text{self},\chi}$ is the reaction term in g_2^2 which involves χ, and is given by (111) in the appendix.

Similarly, the rotational equation (59) is now to be modified to

$$I\{\dot{S}_{\lambda\mu} + v_{\lambda} S'_{\mu} - v_{\mu} S'_{\lambda}\} + K[S_{\lambda}{}^{\rho}(\ddot{S}_{\rho\mu} - S''_{\rho} v_{\mu})]_{-}$$
$$= g_2[S_{\lambda}{}^{\rho}\{(G_{\rho\mu}^{\text{in.}} + \tilde{G}_{\rho\mu}) - (G_{\rho\nu}^{\text{in.}} + \tilde{G}_{\rho\nu}) v^{\nu} v_{\mu}\}]_{-}$$
$$+ [S_{\lambda\rho} D^{\rho}{}_{\mu}]_{-} + g_2^2 \chi^2 [S_{\lambda}{}^{\rho}(\dot{S}_{\rho\mu} - S'_{\rho} v_{\mu})]_{-}. \tag{63}$$

$D_{\rho\mu}$ is the reaction term proportional to g_2^2 given explicitly in B (46) and C (142). It is such that $D_{\rho\mu} v^{\mu} = 0$. We see that $\tilde{G}_{\rho\mu}$ appears in the equations (58) and (59) in the same way as the ingoing field $G_{\mu\nu}^{\text{in.}}$. It embodies the effect of the previous motion of the dipole on the motion of the particle at the instant under consideration and may be interpreted as the resultant of the fields which the particle has itself radiated in moving along the preceding part of the world line. It should be noticed that the arbitrary constant k

in $C'^{\text{mes.}}_{\rho\mu}$ has dropped out of the specialized equations (62) and (63) for the case $\mathcal{S}_\mu = 0$.

Contracting (62) with v^μ and using (53) I get the invariant equation

$$K(\dot{S}\ddot{S}) - 2K(S'S'') - g_2(\dot{S}.\{G^{\text{in.}} + \tilde{G}\}) \\ + 2g_2 S'^\rho(G^{\text{in.}}_{\rho\nu} + \tilde{G}_{\rho\nu})\, v^\nu - (\dot{S}D) - g_2^2\chi^2(\dot{S}^2 - S'^2) = 0. \tag{64}$$

On contracting (63) with v^μ it vanishes identically. Thus (63) vanishes identically in the rest system when either λ or μ takes on the value zero. It therefore only consists of three equations determining $\dot{S}_{lm}$, $\dot{S}_{mk}$ and $\dot{S}_{kl}$, while $\dot{S}_{0k}$, $\dot{S}_{0l}$ and $\dot{S}_{0m}$ are determined explicitly by the equation derived by differentiating (60). Corresponding to this, (64) in the rest system takes the form

$$K(\dot{S}_{kl}\ddot{S}^{kl}) - g_2\dot{S}^{kl}(G^{\text{in.}}_{kl} + \tilde{G}_{kl}) - \dot{S}^{kl}D_{kl} - g_2^2\chi^2\dot{S}^{kl}\dot{S}_{kl} = 0. \tag{65}$$

The scattering of mesons by a dipole

In considering the scattering of meson radiation by a dipole, we may simplify the problem by putting the mass of the dipole $M = \infty$. All derivatives of the velocity will then vanish, and (63) can be considered in the rest system. Put the dipole at the origin of coordinates. Henceforth write t, x, y, z in place of x_0, x_1, x_2 and x_3. If we introduce a space vector $\mathbf{M}$ defined by

$$M_k = S_{lm}, \quad M_l = S_{mk}, \quad M_m = S_{kl},$$

and denoting the magnetomesic force $G^{\text{in.}}_{lm}$, $G^{\text{in.}}_{mk}$, $G^{\text{in.}}_{kl}$ of the ingoing field by $\mathbf{H}$ the equation (63) takes the form†

$$I\mathbf{M} + K[\mathbf{M}.\ddot{\mathbf{M}}] = g_2[\mathbf{M}.\mathbf{H}] + \tfrac{2}{3}g_2^2[\mathbf{M}.\mathbf{M}^{\text{iii}}] + g_2[\mathbf{M}.\tilde{\mathbf{G}}] + g_2^2\chi^2[\mathbf{M}.\dot{\mathbf{M}}], \tag{66}$$

where $\tilde{\mathbf{G}}$ stands for the vector $\tilde{G}_{lm}$, $\tilde{G}_{mk}$, $\tilde{G}_{kl}$. Its components are defined by (35). The quantities u_ρ in (35) written out explicitly are

$$u_k = z_k(\tau) - z_k(\tau') = 0, \quad u_0 = z_0(\tau) - z_0(\tau') = t - t', \quad u = u_0 = t - t', \tag{67}$$

t being the time at the proper time τ. Hence, on account of (60) and (67), the second integral in (35) vanishes, and I get

$$\tilde{\mathbf{G}}(t) = 2g_2\chi^2\int_{-\infty}^{t} dt'\,\mathbf{M}(t')\frac{J_2(\chi u)}{u^2} \\ = 2g_2\chi^2\int_0^\infty du\,\mathbf{M}(t-u)\frac{J_2(\chi u)}{u^2}. \tag{68}$$

† Henceforth the square brackets have their usual meaning in three-dimensional vector theory.

Let us suppose that a periodic force **H** with a frequency ω_0 acts on the dipole:

$$\mathbf{H} = \mathbf{H}_0 \cos \omega_0 t. \tag{69}$$

Solve (66) for small oscillations of the dipole. I write

$$\mathbf{M}(t) = \mathbf{M}_0 + \mathbf{M}_1 \sin \omega_0 t + \mathbf{M}_2 \sin(\omega_0 t + \delta). \tag{70}$$

$\mathbf{M}_0$ is the initial direction of the dipole, and I assume that $\mathbf{M}_0^2 = 1$. The vectors $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{M}_2$ are perpendicular to each other and such that $[\mathbf{M}_0 . \mathbf{M}_1]$ is in the direction $\mathbf{M}_2$. I neglect quantities quadratic in $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{H}_0$.

Introducing (70) into (68), I get

$$\tilde{\mathbf{G}} = 2g_2\chi^2\mathbf{M}_0\int_0^\infty du\,\frac{J_2(\chi u)}{u^2}$$

$$+2g_2\chi^2\int_0^\infty du\,[\mathbf{M}_1 \sin \omega_0(t-u) + \mathbf{M}_2 \sin\{\omega_0(t-u)+\delta\}]\,\frac{J_2(\chi u)}{u^2}.$$

The integrals which occur in this expression can be easily evaluated and are

$$\int_0^\infty du\,\frac{J_2(u)}{u^2} = \tfrac{1}{3},$$

$$\int_0^\infty du\,\frac{J_2(u)}{u^2}\,e^{-i\nu u} = \begin{cases} \tfrac{1}{3}[(1-\nu^2)^{\frac{3}{2}} + i\nu^3 - \tfrac{3}{2}i\nu] & 0<\nu<1, \\ \tfrac{1}{3}[-i(\nu^2-1)^{\frac{3}{2}} + i\nu^3 - \tfrac{3}{2}i\nu] & \nu>1. \end{cases}$$

With the help of these I get at once

$$\tilde{\mathbf{G}} = \tfrac{2}{3}g_2\chi^3\mathbf{M}_0 + \tfrac{2}{3}g_2\chi^3\mathbf{M}_1\{P\sin\omega_0 t + Q\cos\omega_0 t\}$$
$$+\tfrac{2}{3}g_2\chi^3\mathbf{M}_2\{P\sin(\omega_0 t+\delta) + Q\cos(\omega_0 t+\delta)\}, \tag{71}$$

where

$$P = \begin{cases} \left(1-\dfrac{\omega_0^2}{\chi^2}\right)^{\frac{3}{2}} & \omega_0<\chi, \\ 0 & \omega_0>\chi, \end{cases}$$

$$Q = \begin{cases} \dfrac{\omega_0^3}{\chi^3} - \dfrac{3}{2}\dfrac{\omega_0}{\chi} & \omega_0<\chi, \\ \dfrac{\omega_0^3}{\chi^3} - \dfrac{3}{2}\dfrac{\omega_0}{\chi} - \left(\dfrac{\omega_0^2}{\chi^2}-1\right)^{\frac{3}{2}} & \omega_0>\chi. \end{cases} \tag{72}$$

Now write as abbreviations

$$\alpha = \frac{3}{2}\frac{I}{g_2^2}, \quad \beta = \frac{3}{2}\frac{K}{g_2^2}. \tag{73}$$

Introducing (70) and (71) into (66), using (72) and remembering the relation between the three directions $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{M}_2$, I find, neglecting terms quadratic in $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{H}_0$, that (66) becomes

$$\omega_0 \mathbf{M}_1\left[\alpha\cos\omega_0 t + \left|\frac{M_2}{M_1}\right|\{\xi\sin(\omega_0 t+\delta) - \zeta\cos(\omega_0 t+\delta)\}\right]$$
$$+\omega_0\mathbf{M}_2\left[\alpha\cos(\omega_0 t+\delta) - \left|\frac{M_1}{M_2}\right|\{\xi\sin\omega_0 t - \zeta\cos\omega_0 t\}\right] = \frac{3}{2g_2}[\mathbf{M}_0\cdot\mathbf{H}_0]\cos\omega_0 t, \tag{74}$$

where
$$\xi \equiv \beta\omega_0 - \frac{\chi^3}{\omega_0}(1-P) = \begin{cases} \beta\omega_0 - \dfrac{\chi^3}{\omega_0} + \dfrac{(\chi^2-\omega_0^2)^{\frac{3}{2}}}{\omega_0} & \omega_0 < \chi, \\ \beta\omega_0 - \dfrac{\chi^3}{\omega_0} & \omega_0 > \chi, \end{cases} \tag{75a}$$

$$\zeta \equiv -\frac{\chi^3}{\omega_0}\left\{Q - \frac{\omega_0^3}{\chi^3} + \frac{3}{2}\frac{\omega_0}{\chi}\right\} = \begin{cases} 0 & \omega_0 < \chi, \\ \dfrac{(\omega_0^2-\chi^2)^{\frac{3}{2}}}{\omega_0} & \omega_0 > \chi. \end{cases} \tag{75b}$$

Now, each square bracket on the left-hand side of (74) can always be written as the sum of a term proportional to $\cos\omega_0 t$ and a term proportional to $\sin\omega_0 t$, and, since $\mathbf{M}_1$ and $\mathbf{M}_2$ are perpendicular, the coefficients of the terms proportional to $\sin\omega_0 t$ in each bracket must vanish. The coefficient of the term proportional to $\sin\omega_0 t$ in the first bracket vanishes if

$$\xi\cos\delta + \zeta\sin\delta = 0,$$

that is, if
$$\tan\delta = -\frac{\xi}{\zeta}, \tag{76}$$

while the coefficient of the term proportional to $\sin\omega_0 t$ in the second bracket vanishes if

$$\alpha\sin\delta + \left|\frac{M_1}{M_2}\right|\xi = 0.$$

Hence, with the help of (76),

$$\left|\frac{M_1}{M_2}\right| = -\frac{\alpha\sin\delta}{\xi} = \frac{\alpha}{\sqrt{(\xi^2+\zeta^2)}}. \tag{77}$$

By using (76) and (77) and dividing through by $\cos\omega_0 t$, equation (74) becomes

$$\mathbf{M}_1\left(\frac{\alpha^2-\xi^2-\zeta^2}{\alpha}\right) + \mathbf{M}_2\left|\frac{M_1}{M_2}\right|(2\zeta) = \frac{3}{2g_2\omega_0}[\mathbf{M}_0\cdot\mathbf{H}_0].$$

Let θ be the angle between $\mathbf{M}_0$ and $\mathbf{H}_0$, and η the angle between $\mathbf{M}_1$ and $[\mathbf{M}_0 . \mathbf{H}_0]$, $\mathbf{M}_1$ lying in such a direction that the angle between the $\mathbf{M}_1\mathbf{M}_0$ plane and the $\mathbf{M}_0\mathbf{H}_0$ plane is $\frac{1}{2}\pi-\eta$. Then it follows from the above equation that

$$\tan\eta = \frac{2\alpha\zeta}{\alpha^2-\xi^2-\zeta^2}, \tag{78}$$

and

$$|M_1| = \frac{3\alpha H_0 \sin\theta}{2g_2\omega_0\{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\zeta^2\}^{\frac{1}{2}}}. \tag{79}$$

The work done by the external force on the dipole is on the average

$$\begin{aligned} -g_2\overline{(\mathbf{H}\dot{\mathbf{M}})} &= -\tfrac{1}{2}g_2\omega_0\{(\mathbf{H}_0 . \mathbf{M}_1)+(\mathbf{H}_0\mathbf{M}_2)\cos\delta\} \\ &= -\tfrac{1}{2}g_2\omega_0 H_0\{|M_1|\sin\theta\sin\eta-|M_2|\sin\theta\cos\eta\cos\delta\}. \end{aligned}$$

By (76), (77), (78) and (79), this becomes

$$\tfrac{3}{4}H_0^2\sin^2\theta\frac{\zeta(\alpha^2+\xi^2+\zeta^2)}{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\zeta^2}. \tag{80}$$

When $\omega_0<\chi$, this expression vanishes since, by (75*b*), ζ is then zero. This is what one would expect, for no meson waves exist for $\omega_0<\chi$, and hence no radiation by the dipole is possible. The amplitude of the oscillation $\mathbf{M}_1$ as given by (79) is now proportional to $(\alpha^2-\xi^2)^{-1}$. For very low frequencies $\omega_0 \ll \chi$,

$$\xi \approx \beta\omega_0-\tfrac{3}{2}\chi\omega_0 = \frac{3\omega_0}{2g_2^2}(K-g_2^2\chi). \tag{81}$$

This shows quite clearly that as far as slow oscillations are concerned the constant K, which, as was shown in C, is to be interpreted as the moment of inertia of the particle perpendicular to the axis of the spin, is diminished by a quantity $g_2^2\chi$. We must interpret $-g_2^2\chi$ as an added moment of inertia due to the meson field. For the Maxwell field where $\chi=0$, there is no such added moment of inertia due to the field. Now the energy density as given by (11) is a positive definite form, so that if the expression (11) were used for calculating the addition to the moment of inertia due to the field, the addition would always be positive. For a point dipole it would be infinite. The arguments of the preceding paper show that this infinity is spurious and (11) is not the correct expression for the energy density when point charges or point dipoles are present. (11) has to be modified in the presence of point dipoles as Pryce (1938) has modified the energy tensor for a point charge. The result that the addition to the moment of inertia given above is negative then shows that the modified energy tensor will no longer be a positive definite expression.

Exactly as in the cas f a point charge investigated in A, the mass of the particle is changed due to the mass of the meson field. We could calculate it by working out the amplitude of the translatory motion of a dipole of finite mass M due to an external force of low frequency in a way exactly analogous to the one used above for calculating the change in the moment of inertia. Its order of magnitude can be estimated by dimensional considerations. The mass of the field must be proportional to g_2^2, and the only other constant which the expression for it can contain is χ. The only combination of these two constants which has the dimensions of a mass is $g_2^2\chi^3$. With the actual values of the constants for neutrons and mesons as they occur in nature, $g_2^2\chi^3 = \mu(g_2^2\chi^2/\hbar) \sim M/150$, M being the neutron mass, so that this is extremely small compared with the neutron mass, and of the same order as the alteration in the mass calculated in A due to the meson field of a point charge g_1. It is seen that the dipole field makes a very small negative contribution to the mass of a neutron. This result is contrary to a non-relativistic classical theory by Heisenberg (1939) in which the field was made to account for the whole mass of the neutron.

I now return to the more interesting case $\omega_0 > \chi$. To get the scattering cross-section we have to suppose that the force H is the magnetomesic force of a plane wave of the type (38). Now, as mentioned before, the magnetic force of a longitudinal wave is zero, so that a dipole free to rotate only will not scatter longitudinal meson waves at all. Any scattering of these waves must be due to the translation of the dipole, and the scattering will be of the same order as the scattering due to a point charge calculated in A. For transverse waves the rate of energy transfer per unit area of a wave with a magnetic force $\mathbf{H}$ is, by (41), $H_0^2\omega_0(\omega_0^2-\chi^2)^{-\frac{1}{2}}/8\pi$. Dividing (80) by this, it is found that the total effective cross-section for the scattering of a transverse meson wave is

$$6\pi \sin^2\theta \frac{(\omega_0^2-\chi^2)^2}{\omega_0^2} \frac{\alpha^2+\xi^2+\zeta^2}{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\zeta^2}. \tag{82}$$

If χ is put equal to zero, this at once reduces to the cross-section for the scattering of light by a dipole given in C. ζ may be regarded as expressing the effects of radiation reaction for a meson field, for it reduces when $\chi \to 0$ to ω_0^2, the reaction term for a Maxwell field. ξ may then be taken to express the effects of the sum of the mechanical moment of inertia K of the particle perpendicular to the axis of the spin plus that due to the field. When $\omega_0 \gg \chi$,

$$\xi = \beta\omega_0 - \frac{\chi^3}{\omega_0} = \frac{3}{2}\frac{\omega_0}{g_2^2}\left(K - \tfrac{2}{3}g_2^2\frac{\chi^3}{\omega_0^2}\right).$$

This clearly shows that the moment of inertia of the field is $-\frac{2}{3}g_2^2\chi^3/\omega_0^2$, and tends to zero as ω_0 increases. This is what one would expect, since, as the frequency increases, less and less of the static field swings with the dipole. In the absence of the radiation reaction term ζ, the cross-section (82) would become infinite when $\xi = \alpha$, i.e. at frequencies ω_0 given by

$$\omega_0 = \frac{\alpha}{\beta - \chi^3/\omega_0^2} = \frac{I}{K - \frac{2}{3}g_2^2\chi^3/\omega_0^2}. \tag{83}$$

This is a resonance phenomenon. The expression on the right of (83) is the normal gyration frequency of the spin in a meson field for small amplitudes. The effect of the radiation reaction term ζ is to make the scattering finite even for this frequency, but the cross-section still has a more or less strong maximum at this point, as shown by the curves in the figure, which are drawn for different values of β and hence of K. For $I < (K - \frac{2}{3}g_2^2\chi)\chi$, the gyration frequency is less than χ and hence no resonance occurs in the scattering of meson waves. (This is shown below by the curve marked $\beta = 15$ in the figure.)

To get the angular distribution of the scattered radiation one must calculate the potentials at a large distance $\mathbf{r}$ due to the vibration of the dipole given by (70). It is shown in the appendix that, if quantities of the order $1/r^2$ are neglected, the potentials at a large distance r are given by

$$U_0^{\text{ret.}} = 0,$$

$$U_k^{\text{ret.}} = \begin{cases} g_2\sqrt{(\chi^2-\omega_0^2)}\, e^{-r\sqrt{(\chi^2-\omega_0^2)}}\{[\mathbf{r}\,.\,\mathbf{M}_1]\sin\omega_0 t + [\mathbf{r}\,.\,\mathbf{M}_2]\sin(\omega_0 t+\delta)\}/r^2 & \omega_0<\chi, \\ g_2\sqrt{(\omega_0^2-\chi^2)}\{[\mathbf{r}\,.\,\mathbf{M}_1]\cos(\omega_0 t - r\sqrt{(\omega_0^2-\chi^2)}) + [\mathbf{r}\,.\,\mathbf{M}_2]\cos(\omega_0 t - r\sqrt{(\omega_0^2-\chi^2)}+\delta)\}/r^2 & \\ & \omega_0>\chi. \end{cases} \tag{84}$$

For $\omega_0 < \chi$ the field falls off exponentially, and no radiation takes place. For $\omega_0 > \chi$ the radiated wave is purely transverse. The average rate of radiation of energy per unit area in the direction $\mathbf{r}$ is, according to (11),

$$\frac{1}{8\pi}g_2^2\frac{\omega_0(\omega_0^2-\chi^2)^{\frac{3}{2}}}{r^2}\left\{\frac{[\mathbf{r}\,.\,\mathbf{M}_1]^2}{r^2} + \frac{[\mathbf{r}\,.\,\mathbf{M}_2]^2}{r^2} + 2\frac{([\mathbf{r}\,.\,\mathbf{M}_1]\,.\,[\mathbf{r}\,.\,\mathbf{M}_2])}{r^2}\cos\delta\right\}. \tag{85}$$

$\mathbf{M}_1$, $\mathbf{M}_2$ and δ are given by (79), (77) and (76) respectively. Thus the angular distribution has no very simple relation to the direction of the incident field. Moreover, as (78) and (77) show, it varies with the frequency. If we integrate (85) over all directions and use (75*b*), (77) and (79), the total rate of radiation just becomes equal to (80).

Discussion

The scattering cross-section (82) is plotted in the figure for $I = \frac{1}{2}\hbar$, and $\chi = 4{\cdot}42 \times 10^{12}$ cm.$^{-1}$ corresponding to a meson mass μ of 85×10^6 eV. I have taken $\alpha = 10\chi^2$ corresponding to $g_2^2\chi^2/\hbar = 1/13{\cdot}3$, as is required by the theory of nuclear forces. The different curves correspond to different values of K given by $\beta = 0$, χ, 5χ, 10χ and 15χ.

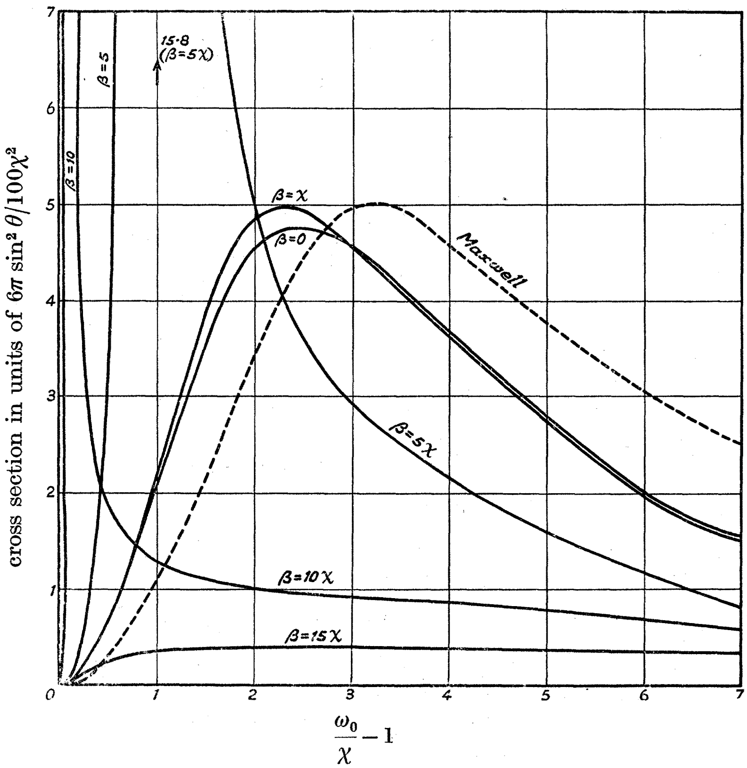

Figure 1

The dipole moment g_2 only appears in (82) in the denominator of α, and of β in ξ. For small g_2 and frequencies which are not too high, these are large compared with ω_0, and (82) can be expanded in a series in ascending powers of g_2. The first term of this series is

$$\frac{8\pi}{3}\sin^2\theta\,\frac{g_2^4}{I^2}\,\frac{(\omega_0^2-\chi^2)^2}{\omega_0^2}\,\frac{1+K^2\omega_0^2/I^2}{(1-K^2\omega_0^2/I^2)^2}. \tag{86}$$

This differs completely from the quantum cross-section for the scattering of mesons by neutrons (or protons) due to the appearance of the constant K, since for frequencies near I/K, (86) has a strong resonance peak and becomes infinite. There is no such constant in the quantum theory. If $K = 0$, then (86) does in fact become the quantum cross-section. To apply it to the scattering of neutral mesons by neutrons or protons I must put $I = \frac{1}{2}\hbar$, and replace $\sin^2\theta$ by 2/3, its value averaged over all initial orientations of the dipole.† If I write for greater similarity with the quantum formula $E = \hbar\omega_0$ for the energy of a meson, and

$$p = \hbar(\omega_0^2 - \chi^2)^{\frac{1}{2}}$$

for its momentum, and further, if I put $g_2 - g_2'/\chi$, I find that (86) becomes, in the case $K = 0$,

$$\frac{64}{9}\pi\left(\frac{g_2'^2}{\hbar}\right)^2 \frac{1}{\chi^2}\frac{p^4}{\mu^2 E^2}, \tag{87}$$

which is exactly the cross-section for the scattering of neutral mesons due to the spin of the neutron as given by quantum theory. Now, as is well known, quantum theory neglects the reaction of the radiated field, so that for high frequencies important deviations from formulae (86) and (87) should be expected, as indeed the exact formula (82) shows. There is, however, no reason to doubt that the Dirac equation adequately describes the mechanical properties of a particle of spin $\frac{1}{2}\hbar$. There is therefore no reason for doubting that the formula (87) is correct for low frequencies. The above argument then shows that for the elementary particles as they occur in nature, K is zero, and the elementary particles have the simplest possible mechanical properties for a spinning particle. Thus the exact classical equations of motion for a neutron or proton (with $g_1 = 0$) in a meson field are got by putting $K = 0$ in (62) and (63). A glance at these equations shows that the case $I = 0$, $K \neq 0$ is formally not as simple as the case $I \neq 0$, $K = 0$ which seems to occur in nature and is described by the quantum theory of elementary particles.

Thus a formula which goes beyond the quantum formula (87) and which may be used for describing the scattering of mesons by neutrons or protons is obtained by putting $K = 0$ in (82). The resulting formula is

$$6\pi\sin^2\theta(\omega_0^2 - \chi^2)^2 \frac{\alpha^2\omega_0^2 + (\omega_0^2 - \chi^2)^3 + \chi^6}{\{\alpha^2\omega_0^2 + (\omega_0^2 - \chi^2)^3 + \chi^6\}^2 - 4\alpha^2\omega_0^2\chi^6}. \tag{88}$$

† $\frac{2}{3}$ is the classical average. Detailed calculations (see second footnote on p. 341) have shown that in the quantum theory of a particle of spin $\hbar/2$, $\sin^2\theta$ is to be replaced by 2. The quantum cross-section is therefore exactly three times (87).

As mentioned at the beginning of this section, $\alpha = 10\chi^2$, so that the explicit χ^6 terms in (88) may be omitted with an error of less than 2 %. To this degree of accuracy the scattering cross-section may be written

$$6\pi \sin^2\theta \frac{\hbar^2 p^4}{\alpha^2\hbar^4 E^2 + p^6}. \tag{88a}$$

The formula (88) has a very wide range of validity. In deriving it the following approximations have been made. First, the translational motion of the particle was neglected. Due to the largeness of the neutron mass this approximation is fully justified. The scattering due to the translational motion of the dipole would be smaller by a factor of the order $(\mu/M)^2$. Secondly, the meson field has been treated classically. A comparison of the Klein-Nishina formula with the Thomson formula, as well as the corresponding quantum formula of Booth and Wilson (1940)† for the scattering of neutral mesons due to the g_1 interaction with the classical formula given in A, shows that this is a good approximation so long as the wave-length of the quanta or mesons is large compared with the Compton wave-length of the electron or neutron respectively. Thus no appreciable error is introduced by the classical treatment provided that $\omega_0 < M/\hbar$. This frequency $M/\hbar$ corresponds to mesons with an energy of the order of 10^9 eV, and thus formula (88) should correctly give the scattering of mesons up to these energies. Quantum deviations from formula (88) would only make their appearance near and above this energy. Lastly, the spin of the neutron (or proton) has been treated classically. Since the neutron spin is $\frac{1}{2}\hbar$, so that its angular momentum in any direction can only take on the two values $\pm\frac{1}{2}\hbar$, it may be doubted whether a classical treatment of the spin would give correct results. The complete agreement‡ of the classical formula (88) with the quantum formula (87) for low frequencies shows

† I am indebted to Dr A. H. Wilson for communicating this result to me before publication.

‡ Subsequent calculation has shown (Bhabha and Madhava Rao, 1941, *Proc. Indian Acad. Sci.* A, **13**, 9–24) that the quantum cross-section is greater than the classical by a constant factor 3, though *the two agree completely in their dependence on frequency, scattering angle, and polarization of the incident and scattered meson.* This difference of a constant factor 3 results from differences in the averaging over the initial orientations of the spin of the heavy particle. In the quantum theory the average value of the square of the cosine of the angle which the spin makes with some fixed direction is 1, if the spin is $\hbar/2$, whereas in the classical theory it is always $\frac{1}{3}$. As a consequence it can be shown that $\sin^2\theta$ in formulae (88) and (88a) must be replaced by 2 and not $\frac{2}{3}$.

that a classical treatment also gives the right result. There is no reason to suppose that it will go wrong at high frequencies on this account alone.

We therefore meet with a very unusual state of affairs. The above arguments show that the quantum theory of the interaction of the heavy particles with mesons goes wrong owing to its neglect of the effects of radiation reaction on the rotation of the spin at the point where the quantum cross-section (87) diverges from the classical cross-section (88a), i.e. at momenta satisfying $p^3 \lesssim E\alpha\hbar^2$. With $\alpha = 10\chi^2$, this gives $E \sim 3\mu$. On the other hand, quantum effects would not invalidate the classical formulae (88) and (88a) until we reach energies comparable with the rest energy of the heavy particles. Thus, there is a region of energies E, defined by $3\mu < E < M$, for which the classical formulae (88) and (88a) still hold, but the quantum formula (87) is quite wrong.

The cross-section (88) is given by the curve marked $\beta = 0$ in the figure. For comparison the corresponding cross-section for the scattering of light is given in the figure by the broken curve marked 'Maxwell'. This is got from (88) by putting $\chi = 0$, in which case (88) becomes the cross-section for the scattering of light given in the previous papers (B (56) and C (111)). The abscissa for the broken curve is ω_0/χ and not $\omega_0\chi^{-1} - 1$. The difference between the two curves is considerable, contrary to what is found for the scattering by a point charge. This is what one would expect from the argument given in the introduction.

With the assumption that the heavy particles can exist in states of all integral charge positive or negative (Bhabha 1940b), the above results can at once be applied to charged mesons as they appear in cosmic radiation. To sum up, then, the scattering of *transversely* polarized mesons is given correctly by formula (88) (curve marked $\beta = 0$ in the figure) up to energies of 10^9 eV. The scattering is greatest at energies $E \sim 3{\cdot}5\mu$, the cross-section at this point being 3×10^{-26} cm.^{2}. For higher and lower energies the scattering is much less, so that it is *not* possible to exclude the above cross-section as being too large to be reconciled with experiment. That would only be the case if the cross-section at higher energies were also of the same order, whereas (88) decreases roughly as E^{-2} at high energies. The largeness of the cross-section for $E \sim 3{\cdot}5\mu$, however, will have the effect that relatively very few transversely polarized mesons will fall below these energies. Thus the energy spectrum of mesons will fall away rapidly for energies below about 3μ, if they are transversely polarized.

Longitudinally polarized mesons are not scattered by the rotation of the spin. They can only be scattered by the translational motion of the dipole; the cross-section for this is of the same order as the cross-section for the scattering of longitudinal waves by the g_1 term as calculated in A (43), namely†

$$6\pi\frac{\chi^2}{\omega_0^2}\left(1+\frac{1}{2}\frac{\chi^2}{\omega_0^2}\right)\frac{1}{\left\{\left(\frac{3M}{2g_1^2}\right)^2+\frac{1}{2}\frac{\chi^3}{\omega_0^2}\right\}^2+\left\{\omega_0^2-\frac{3}{4}\frac{\chi^4}{\omega_0^2}-\frac{1}{4}\frac{\chi^6}{\omega_0^4}\right\}},$$

The ratio of this cross-section to the cross-section given above for scattering by the spin at a frequency $\omega_0 \sim 3{\cdot}5\mu$ where the latter is a maximum, is of the order $(g_1^2/\hbar)^2\,(\mu/M)^2$. With $(\mu/M)^2 \sim 1/137$ and $g_1^2/\hbar \sim 1/14$, this ratio is of the order 4×10^{-5}. For higher energies the cross-section due to the spin falls away very rapidly but so also does the cross-section given above for scattering of *longitudinally polarized* mesons by a point charge, due mainly to factor χ^2/ω_0^2. The ratio is therefore not much altered at high energies. Thus the scattering of longitudinally polarized mesons is always more than twenty thousand times less than the scattering of transversely polarized mesons except for very small velocities, when in any case the scattering is negligibly small for both. Scattering will therefore play an important part in the absorption of transversely polarized mesons, but a quite unimportant part in the absorption of longitudinally polarized mesons.

In comparing the theory of this paper with experiment it should be remembered that if the observed mesons do not show the above large scattering at energies near $3{\cdot}5\mu$, as given by formulae (88) and (88*a*) with $\sin^2\theta$ replaced by 2, in the absence of other evidence this could be interpreted to mean that the observed mesons were mostly longitudinally polarized at the place of observation. On the other hand, if the observed mesons showed a large scattering for energies near $3{\cdot}5\mu$, it would mean that a large number of the actual mesons at the place of observation were transversely polarized. It would also be evidence in favour of the correctness of the theory of this paper, for no other theory predicts a behaviour of this type for mesons.

I wish to express my thanks to Professor Sir C. V. Raman and the Director, Dr J. C. Ghosh, for having afforded me every facility for doing this work at the Institute.

Appendix

One must calculate $T_\mu^{\text{mes.}}$ and $M_{\lambda\mu}^{\text{mes.}}$ as defined by (46) and (48). First consider the contribution to these due to the extra terms containing the potentials in (11). These are proportional to χ^2. The potential U_μ at a point

† Obvious misprints in A (43) have been corrected.

on the world tube is the sum of three parts as given by (37 a), the first being independent of χ and of order ϵ^{-2} and the second proportional to χ^2 and of order 1. Thus, remembering that, for the world tube defined by (42),

$$dS^\nu = \{s^\nu(1-\kappa') - \epsilon v^\nu\}\,\epsilon\, d\Omega\, d\tau, \tag{89}$$

as shown in A, where $d\Omega$ is a solid angle about the retarded point in the rest system of that point, it is seen that the potentials will contribute to $T_\mu^{\text{mes.}}$ an amount

$$\frac{\chi^2}{4\pi}\int\{\phi_\mu^{(2)}\phi_\nu^{(2)} - \tfrac{1}{2}g_{\mu\nu}\phi_\rho^{(2)}\phi^{(2)\rho}\}\{s^\nu(1-\kappa') - \epsilon v^\nu\}\,\epsilon\, d\Omega$$
$$+\frac{\chi^2}{4\pi}\int\{\phi_\mu^{(2)}(U_\nu^{(\chi)} + U_\nu^{\text{in.}}) + (U_\mu^{(\chi)} + U_\mu^{\text{in.}})\,\phi_\nu^{(2)} - g_{\mu\nu}\phi^{(2)\rho}(U_\rho^{(\chi)} + U_\rho^{\text{in.}})\}\,(s^\nu - \epsilon v^\nu)\,\epsilon\, d\Omega. \tag{90}$$

By using (28), the first integral can be evaluated as shown in C. All its terms are proportional to χ^2, and are of orders ϵ^{-1} and 1. As far as the second integral is concerned, the two parts of $U_\nu^{(\chi)}$ given in (30) can be treated separately. The first part will give a contribution proportional to χ^4 and calculation shows that it is

$$g_2^2\chi^4(\tfrac{1}{3}S_{\mu\rho}\,\mathcal{S}^\rho + \tfrac{1}{6}v_\mu\,\mathcal{S}^2). \tag{91}$$

The second part $\tilde{U}_\nu$ is non-singular and can be treated along with the ingoing field. This gives

$$g_2\chi^2\{\tfrac{2}{3}S_{\mu\sigma} + \tfrac{1}{3}v_\mu\,\mathcal{S}_\sigma - \tfrac{1}{3}v_\sigma\,\mathcal{S}_\mu\}\,(\tilde{U}^\sigma + U^{\text{in.}\,\sigma}). \tag{92}$$

The terms containing the ingoing field, $T_\mu^{\text{mix.}}$, as calculated in C (119), are

$$T_\mu^{\text{mix.}} = g_2\Big[\frac{d}{d\tau}\{\tfrac{2}{3}S_\mu{}^\sigma\, G_{\sigma\nu}^{\text{in.}}\, v^\nu - \tfrac{2}{3}G_{\mu\sigma}^{\text{in.}}\,\mathcal{S}^\sigma + \tfrac{1}{3}v_\mu\, S^{\rho\sigma}G_{\rho\sigma}^{\text{in.}}\}$$
$$- \tfrac{1}{2}S^{\rho\sigma}\frac{\partial}{\partial x^\mu}G_{\rho\sigma}^{\text{in.}} - \{\tfrac{2}{3}S_{\mu\sigma} + \tfrac{1}{3}v_\mu\,\mathcal{S}_\sigma - \tfrac{1}{3}v_\sigma\,\mathcal{S}_\mu\}\Big]\frac{\partial}{\partial x^\nu}G^{\text{in.}\,\sigma\nu}, \tag{93}$$

where the ingoing meson field strengths $G_{\mu\nu}^{\text{in.}}$ have been written in place of the ingoing Maxwell field strengths $F_{\mu\nu}^{\text{in.}}$. Now the terms containing $U_\nu^{\text{in.}}$ in (92) combine with the last three terms in (93) to form

$$(\tfrac{2}{3}S_{\mu\sigma} + \tfrac{1}{3}v_\mu\,\mathcal{S}_\sigma - \tfrac{1}{3}v_\sigma\,\mathcal{S}_\mu)\left(\frac{\partial}{\partial x^\nu}G^{\text{in.}\,\nu\sigma} + \chi^2 U^{\text{in.}\,\sigma}\right),$$

which vanishes on account of (7 b). $T_\mu^{\text{mix.}}$ therefore remains the same as in the Maxwell case, except that $G_{\mu\nu}^{\text{in.}}$ now replaces $F_{\mu\nu}^{\text{in.}}$.

Now consider the contribution to $T_{\mu}^{\text{mes.}}$ which comes from the first two terms in (11). It is

$$\frac{1}{4\pi}\int\{F^{(2)}_{\mu\rho}G^{(\chi)\rho}{}_{\nu}+G^{(\chi)}_{\mu\rho}F^{(2)\rho}{}_{\nu}+\tfrac{1}{2}g_{\mu\nu}F^{(2)\rho\sigma}G^{(\chi)}_{\rho\sigma}\}\{s^{\nu}(1-\kappa')-\epsilon v^{\nu}\}\epsilon\,d\Omega$$
$$+\frac{1}{4\pi}\int\{G^{(\chi)}_{\mu\rho}G^{(\chi)\rho}{}_{\nu}+\tfrac{1}{4}g_{\mu\nu}G^{(\chi)}_{\rho\sigma}G^{(\chi)\rho\sigma}\}(s^{\nu}-\epsilon v^{\nu})\,\epsilon\,d\Omega. \tag{94}$$

$G^{(\chi)}_{\mu\nu}$ consists of three parts as shown by (34). The first is of order ϵ^{-1}, the second of order ϵ and the last of order 1. Therefore, as far as the second integral of (94) is concerned, only the first will give a finite contribution and it will be proportional to χ^4. Calculation shows that it vanishes. In the first integral, the first part of $G^{(\chi)}_{\mu\nu}$ will give a contribution proportional to χ^2, the second to χ^4. The latter is

$$\tfrac{1}{8}g_2^2\chi^4\{-\tfrac{2}{3}S_{\mu\nu}\mathcal{S}^{\nu}-\tfrac{1}{3}v_{\mu}S^2+\tfrac{2}{3}v_{\mu}\mathcal{S}^2\}. \tag{95}$$

The third term in $G^{(\chi)}_{\mu\nu}$ is non-singular, and therefore it can be treated with $G^{\text{in.}}_{\mu\nu}$. Its value on the world tube must be expressed in terms of its value at the retarded point by a Taylor series as given by (54). $\tilde{G}_{\mu\nu,\rho}$ will play the part of the derivative $\dfrac{\partial}{\partial x^{\rho}}\tilde{G}_{\mu\nu}$. Hence there will be terms containing $\tilde{G}_{\mu\nu}$ and $\tilde{G}_{\mu\nu,\rho}$ exactly like the terms containing $G^{\text{in.}}_{\mu\nu}$ and $\dfrac{\partial}{\partial x^{\rho}}G^{\text{in.}}_{\mu\nu}$. Thus, corresponding to the last three terms of (93) the first integral of (94) gives

$$-g_2\{\tfrac{2}{3}S_{\mu\sigma}+\tfrac{1}{3}v_{\mu}\mathcal{S}_{\sigma}-\tfrac{1}{3}v_{\sigma}\mathcal{S}_{\mu}\}\tilde{G}^{\sigma\nu}{}_{,\nu}.$$

This combines with the terms containing $\tilde{U}^{\sigma}$ in (92) to give an expression containing $\tilde{G}_{\sigma\nu,}{}^{\nu}-\chi^2\tilde{U}_{\sigma}$ as a factor. Using (31) and (56), and remembering (20*b*) we get

$$\tilde{G}_{\sigma\nu,}{}^{\nu}-\chi^2\tilde{U}_{\sigma}=-6g_2\chi^3\int_{-\infty}^{\tau_0}d\tau\,S_{\sigma\nu}u^{\nu}\frac{J_3(\chi u)}{u^3}+g_2\chi^4\int_{-\infty}^{\tau_0}d\tau\,S_{\sigma\nu}u^{\nu}\frac{J_4(\chi u)}{u^2}$$
$$+g_2\chi^4\int_{-\infty}^{\tau_0}d\tau\,S_{\sigma\nu}u^{\nu}\frac{J_2(\chi u)}{u^2}$$
$$=g_2\chi^4\int_{-\infty}^{\tau_0}d\tau\frac{S_{\sigma\nu}u^{\nu}}{u^2}\left\{J_2(\chi u)+J_4(\chi u)-6\frac{J_3(\chi u)}{\chi u}\right\}=0$$

by (18).

Now consider the second term in (93). It really appears as the sum of two terms

$$-\tfrac{2}{3}g_2\left[v^{\rho}\left(\frac{\partial}{\partial x^{\rho}}G^{\text{in.}}_{\mu\sigma}\right)\mathcal{S}^{\sigma}+G^{\text{in.}}_{\mu\sigma}\dot{\mathcal{S}}^{\sigma}\right]=-\tfrac{2}{3}g_2\frac{d}{d\tau}(G^{\text{in.}}_{\mu\sigma}\mathcal{S}^{\sigma}).$$

Corresponding to the first two terms on the left of this equation, the first integral of (94) will give

$$-\tfrac{2}{3}g_2[v^\rho \tilde{G}_{\mu\sigma,\rho}\mathcal{S}^\sigma + \tilde{G}_{\mu\sigma}\dot{\mathcal{S}}^\sigma],$$

which, on account of (57), may be written

$$-\tfrac{2}{3}g_2\left[\left(\frac{d}{d\tau}\tilde{G}_{\mu\sigma} - \tfrac{1}{4}g_2\chi^4 S_{\mu\sigma}\right)\mathcal{S}^\sigma + \tilde{G}_{\mu\sigma}\dot{\mathcal{S}}^\sigma\right] = \tfrac{1}{6}g_2^2\chi^4 S_{\mu\sigma}\mathcal{S}^\sigma - \tfrac{2}{3}g_2\frac{d}{d\tau}(\tilde{G}_{\mu\sigma}\mathcal{S}^\sigma).$$

Thus, corresponding to the first four terms of (93), it is seen by the same reasoning that (94) will give

$$g_2\left[\frac{d}{d\tau}\{\tfrac{2}{3}S_{\mu\sigma}\tilde{G}^{\sigma\nu}v_\nu - \tfrac{2}{3}\tilde{G}_{\mu\sigma}\mathcal{S}^\sigma + \tfrac{1}{3}v_\mu S^{\rho\sigma}\tilde{G}_{\rho\sigma}\} - \tfrac{1}{12}g_2\chi^4 v_\mu S^2 - \tfrac{1}{2}S^{\rho\sigma}\tilde{G}_{\rho\sigma,\mu}\right]. \tag{96}$$

The calculation of the terms proportional to χ^2 is straightforward. The terms proportional to χ^4 are given by (91), (95) and (96). Adding up all the terms the final result is

$$\begin{aligned} T_\mu^{\text{mes.}} = g_2\Big[&\frac{d}{d\tau}\{\tfrac{2}{3}S_{\mu\sigma}\tilde{G}^{\sigma\nu}v_\nu - \tfrac{2}{3}\tilde{G}_{\mu\sigma}\mathcal{S}^\sigma + \tfrac{1}{3}v_\mu S^{\rho\sigma}\tilde{G}_{\rho\sigma}\} \\ &- \tfrac{1}{2}S^{\rho\sigma}\tilde{G}_{\rho\sigma,\mu} + g_2\chi^4(-\tfrac{1}{8}v_\mu S^2 + \tfrac{1}{4}S_{\mu\sigma}\mathcal{S}^\sigma + \tfrac{1}{4}v_\mu\mathcal{S}^2)\Big] \\ &- g_2^2\frac{\chi^2}{\epsilon}\frac{d}{d\tau}(\tfrac{1}{6}v_\mu\mathcal{S}^2) + T'^{\text{mes.}}_\mu, \end{aligned} \tag{97}$$

where

$$\begin{aligned} T'^{\text{mes.}}_\mu = g_2^2\chi^2[&v_\mu\{\tfrac{1}{2}\dot{S}^2 - \tfrac{5}{3}S'^2 + \tfrac{1}{3}(\mathcal{S}S'') - \tfrac{11}{15}(\mathcal{S}\dot{S}\dot{v}) - \tfrac{11}{15}(S'S\dot{v}) \\ &+ \tfrac{4}{15}(\mathcal{S}S\ddot{v}) - \tfrac{1}{6}S^2\dot{v}^2 + \mathcal{S}^2\dot{v}^2 + \tfrac{1}{15}(\dot{v}SS\dot{v})\} - \dot{v}_\mu\{\tfrac{7}{5}(\mathcal{S}S') + \tfrac{4}{5}(\mathcal{S}S\dot{v})\} \\ &+ \tfrac{1}{30}\ddot{v}_\mu S^2 - \tfrac{1}{5}\ddot{v}_\mu\mathcal{S}^2 + \tfrac{1}{3}\mathcal{S}_\mu(S'\dot{v}) + \tfrac{1}{3}\mathcal{S}_\mu(\mathcal{S}\ddot{v}) + \tfrac{1}{3}S_{\mu\rho}\dot{v}^\rho(\mathcal{S}\dot{v}) \\ &+ \tfrac{1}{3}S'_\mu(\mathcal{S}\dot{v}) + \tfrac{1}{2}S_{\mu\nu}S''^\nu + \tfrac{13}{30}S_{\mu\nu}\dot{S}^{\nu\rho}\dot{v}_\rho + \tfrac{4}{15}S_{\mu\nu}S^{\nu\rho}\ddot{v}_\rho \\ &+ \tfrac{1}{3}S_{\mu\nu}\mathcal{S}^\nu\dot{v}^2 - \tfrac{2}{3}\dot{S}_{\mu\nu}S'^\nu - \tfrac{7}{30}\dot{S}_{\mu\nu}S^{\nu\rho}\dot{v}_\rho - \tfrac{1}{6}\ddot{S}_{\mu\nu}\mathcal{S}^\nu]. \end{aligned} \tag{98}$$

The calculation of $M_{\lambda\mu}^{\text{mes.}}$ is straightforward. I have to calculate the integrals (90) and (94) with an extra s_λ in the integrand, the expressions being then made antisymmetrical in λ and μ. As before I obtain an expression like $M_{\lambda\mu}^{\text{mix.}}$ given in C (59), with $\tilde{G}_{\mu\nu}$ written in place of $F_{\mu\nu}^{\text{in.}}$. The final result is

$$M_{\lambda\mu}^{\text{mes.}} = g_2[S_\lambda{}^\sigma\tilde{G}_{\sigma\mu} + \tfrac{2}{3}v_\lambda S_\mu{}^\sigma\tilde{G}_{\sigma\nu}v^\nu - \tfrac{2}{3}v_\lambda\tilde{G}_{\mu\sigma}\mathcal{S}^\sigma]_- + M'^{\text{mes.}}_{\lambda\mu}, \tag{99}$$

where

$$M'^{\text{mes.}}_{\lambda\mu} = g_2^2\chi^2[S_{\lambda\rho}\,\dot{S}^{\rho}{}_{\mu} + \mathcal{S}_{\lambda}\,S'_{\mu} + \tfrac{1}{2}\dot{v}_{\lambda}\,S_{\mu\sigma}\,\mathcal{S}^{\sigma}$$
$$+ v_{\lambda}\{S_{\mu\sigma}\,S'^{\sigma} - \tfrac{2}{3}\dot{S}_{\mu\sigma}\,\mathcal{S}^{\sigma} + \tfrac{13}{30}S_{\mu\sigma}\,S^{\sigma\nu}\dot{v}_{\nu} + \tfrac{1}{5}\dot{v}_{\mu}\,S^2 - \tfrac{13}{30}\dot{v}_{\mu}\,\mathcal{S}^2 + \tfrac{1}{3}\mathcal{S}_{\mu}(\mathcal{S}\dot{v})\}]_{-}. \quad (100)$$

I may straightaway write

$$A^{\text{mes.}}_{\mu} = g_2\{\tfrac{2}{3}S_{\mu\sigma}\,\tilde{G}^{\sigma\nu}v_{\nu} - \tfrac{2}{3}\tilde{G}_{\mu\sigma}\,\mathcal{S}^{\sigma} + \tfrac{1}{6}v_{\mu}\,S^{\rho\sigma}\,\tilde{G}_{\rho\sigma}\} - g_2^2\frac{\chi^2}{\epsilon}(\tfrac{1}{6}v_{\mu}\mathcal{S}^2) + A'^{\text{mes.}}_{\mu}, \quad (101)$$

where $A'^{\text{mes.}}_{\mu}$ contains only terms proportional to $g_2^2\chi^2$, so that

$$T^{\text{mes.}}_{\mu} - \dot{A}^{\text{mes.}}_{\mu} = -\tfrac{1}{2}g_2\,S^{\rho\sigma}\,\tilde{G}_{\rho\sigma,\mu} + g_2^2\chi^4(-\tfrac{1}{8}v_{\mu}\,S^2 + \tfrac{1}{4}S_{\mu\rho}\,\mathcal{S}^{\rho}$$
$$+ \tfrac{1}{4}v_{\mu}\,\mathcal{S}^2) + \tfrac{1}{2}g_2\frac{d}{d\tau}(v_{\mu}S^{\rho\sigma}\,\tilde{G}_{\rho\sigma}) + (T'^{\text{mes.}}_{\mu} - \dot{A}'^{\text{mes.}}_{\mu}), \quad (102)$$

and
$$M^{\text{mes.}}_{\lambda\mu} - (v_{\lambda}\,A^{\text{mes.}}_{\mu} - v_{\mu}\,A^{\text{mes.}}_{\lambda}) - \dot{B}^{\text{mes.}}_{\lambda\mu} = [S_{\lambda}{}^{\rho}\,C^{\text{mes.}}_{\rho\mu}]_{-}, \quad (103)$$

where
$$C^{\text{mes.}}_{\rho\mu} = g_2\,\tilde{G}_{\rho\mu} + C'^{\text{mes.}}_{\rho\mu} \quad (104)$$

and
$$[S_{\lambda\rho}C'^{\text{mes.}\,\rho}{}_{\mu}]_{-} = M'^{\text{mes.}}_{\lambda\mu} - (v_{\lambda}\,A'^{\text{mes.}}_{\mu} - v_{\mu}\,A'^{\text{mes.}}_{\lambda}) - \dot{B}^{\text{mes.}}_{\lambda\mu}. \quad (105)$$

The expressions (102) and (103) show that $\tilde{G}_{\rho\sigma,\mu}$ occurs in the translational equation (50) exactly like the ingoing field $\frac{\partial}{\partial x^{\mu}}\,G^{\text{in.}}_{\rho\sigma}$, while $\tilde{G}_{\rho\mu}$ occurs in the rotational equation (51) exactly like $G^{\text{in.}}_{\rho\mu}$. The terms of order χ^4 in (102) are most essential. Not having any differentiation with respect to τ, they cannot be altered by any addition to $A^{\text{mes.}}_{\mu}$. They are just such that, on contracting (102) with v^{μ}, and remembering (57), I get

$$v^{\mu}(T^{\text{mes.}}_{\mu} - \dot{A}^{\text{mes.}}_{\mu}) = \tfrac{1}{2}g_2(\dot{S}^{\rho\sigma}\,\tilde{G}_{\rho\sigma}) + v^{\mu}(T'^{\text{mes.}}_{\mu} - \dot{A}'^{\text{mes.}}_{\mu}). \quad (106)$$

This, according to (53), must be equal to $\tfrac{1}{2}(\dot{S}C^{\text{mes.}})$, that is, by (104),

$$\tfrac{1}{2}g_2(\dot{S}\tilde{G}) + \tfrac{1}{2}(\dot{S}\,.\,C'^{\text{mes.}}). \quad (107)$$

If any substitution other than (101) had been made for $A^{\text{mes.}}_{\mu}$, (106) and (107) would not have been equal.

Finally, then, $A'^{\text{mes.}}_{\mu}$ and $B^{\text{mes.}}_{\lambda\mu}$ have to be determined so that

$$M'^{\text{mes.}}_{\lambda\mu} - (v_{\lambda}\,A'^{\text{mes.}}_{\mu} - v_{\mu}\,A'^{\text{mes.}}_{\lambda}) - \dot{B}^{\text{mes.}}_{\lambda\mu} \equiv [S_{\lambda}{}^{\rho}\,.\,C'^{\text{mes.}}_{\rho\mu}]_{-}, \quad (108a)$$

$$(\dot{S}C'^{\text{mes.}}) = 2v^{\mu}(T'^{\text{mes.}}_{\mu} - \dot{A}'^{\text{mes.}}_{\mu}). \quad (108b)$$

Both $A'^{\text{mes.}}_{\mu}$ and $B^{\text{mes.}}_{\lambda\mu}$ must be proportional to $g_2^2\chi^2$, as is seen from (98) and (100). They must be quadratic in $S_{\lambda\mu}$ and contain terms with respectively one and no dots. They may be found by the method given in C. There are eight possible independent terms in $A'^{\text{mes.}}_\mu$ and only one in $B^{\text{mes.}}_{\lambda\mu}$. The conditions (107) determine the coefficients of all these in terms of one arbitrary constant k. The result is that

$$
\begin{aligned}
T'^{\text{mes.}}_\mu - \dot{A}'^{\text{mes.}}_\mu = g_2^2\chi^2[&k\ddot{S}_{\mu\nu}\,\not{S}^\nu + (2k-1)\,\dot{S}_{\mu\nu}\,S'^\nu + k\dot{S}_{\mu\nu}\,S^{\nu\rho}\dot{v}_\rho \\
&+ kS_{\mu\nu}\,S''^\nu + kS_{\mu\nu}\,\dot{S}^{\nu\rho}\dot{v}_\rho + \tfrac{1}{3}S_{\mu\nu}\,S^{\nu\rho}\ddot{v}_\rho + \tfrac{1}{3}S_{\mu\nu}\not{S}^\nu\dot{v}^2 \\
&+ v_\mu\{2(k-1)\,S'^2 + (2k-1)\,(S'S\dot{v}) + 2k(\not{S}S'') + (2k-1)\,(\not{S}\dot{S}\dot{v}) \\
&+ \tfrac{1}{3}(\not{S}S\ddot{v}) + \tfrac{1}{2}\dot{S}^2 - \tfrac{1}{6}S^2\dot{v}^2 + \not{S}^2\dot{v}^2\} + 2k\,\dot{v}_\mu(\not{S}\dot{S}) \\
&+ \dot{v}_\mu(\not{S}S\dot{v}) - \tfrac{1}{6}\ddot{v}_\mu\,S^2 + \tfrac{2}{3}\ddot{v}_\mu\,\not{S}^2],
\end{aligned} \tag{109}
$$

and

$$
\begin{aligned}
C'^{\text{mes.}}_{\rho\mu} = g_2^2\chi^2[&(k-1)\,(S_{\rho\nu}\,\dot{v}^\nu v_\mu - S_{\mu\nu}\,\dot{v}^\nu v_\rho) \\
&+ (k-1)\,(\not{S}_\rho\dot{v}_\mu - \not{S}_\mu\dot{v}_\rho) - (S'_\rho v_\mu - S'_\mu v_\rho) + \dot{S}_{\rho\mu}].
\end{aligned} \tag{110}
$$

According to (61),

$$
\begin{aligned}
T^{\text{self},\,\chi}_\mu &\equiv T'^{\text{mes.}}_\mu - \dot{A}'^{\text{mes.}}_\mu + \frac{d}{d\tau}(S_\mu{}^\rho C'^{\text{mes.}}_{\rho\nu}v^\nu) \\
&= g_2^2\chi^2\{v_\mu(\tfrac{1}{2}\dot{S}^2 - S'^2 - \tfrac{1}{6}S^2\dot{v}^2) - \tfrac{1}{6}\ddot{v}_\mu S^2 + \tfrac{2}{3}S_{\mu\nu}\,S''^\nu + \tfrac{1}{3}S_{\mu\nu}\,\dot{S}^{\nu\rho}\dot{v}_\rho.
\end{aligned} \tag{111}
$$

Further,

$$
C'^{\text{mes.}}_{\rho\mu} - (C'^{\text{mes.}}_{\rho\nu}v^\nu v_\mu - C'^{\text{mes.}}_{\mu\nu}v^\nu v_\rho) = g_2^2\chi^2\{\dot{S}_{\rho\mu} - (S'_\rho v_\mu - S'_\mu v_\rho)\}. \tag{112}
$$

This has already been introduced into (63) and gives the last term of that equation.

I wish to find the potentials at a distant point x_k at time t due to an oscillation of the dipole of the type

$$
\dot{S}_{\mu\nu} = \omega_0 L_{\mu\nu}\cos\omega_0 t,
$$

the $L_{\mu\nu}$ being constants. The dipole is taken to be fixed at the origin. Then, by (20a), $u_k = x_k$ and $u_0 = t - t'$, for a point on the world line corresponding to a time t', and, by (20b), $u_0 = \sqrt{(u^2+r^2)}$, where $r = \sum_1^3 x_k^2$; hence

$$
t' = t - \sqrt{(u^2+r^2)}.
$$

Further, by (21), $\kappa = u_0$. The potentials are given by (25b), and, for any given value of u, the position of the dipole has to be taken at a time t' as defined above. Both u_0 and t' are functions of u as the preceding expressions show, but, since I am only interested in the potentials at a very large distance r, it is sufficient to let the differentiation act only on t' and not on u_0. Differentiation of u_0 merely adds terms to the potentials which are of a higher order in $1/r$. I thus find

$$U_\nu^{\text{ret.}} \approx -g_2 \int_0^\infty u\,du\,J_0(\chi u)\left\{\frac{L_{0\nu}}{u_0^2}+\frac{x^k L_{k\nu}}{u_0^3}\right\}\omega_0^2 \sin\omega_0(t-\sqrt{(u^2+r^2)}). \quad (113)$$

Now, by a well-known theorem of Bessel functions,

$$\int_0^\infty du\,J_0(\chi u)\frac{u}{\sqrt{(u^2+r^2)}}e^{-\omega_0\sqrt{(u^2+r^2)}} = \frac{e^{-r\sqrt{(\chi^2+\omega_0^2)}}}{\sqrt{(\chi^2+\omega_0^2)}};$$

whence

$$\int_0^\infty du\,J_0(\chi u)\frac{u}{u^2+r^2}e^{-i\omega_0\sqrt{(u^2+r^2)}} = \int_{i\omega_0}^{\infty+i\omega_0} d\omega' \int_0^\infty du\,J_0(\chi u)\frac{u}{\sqrt{(u^2+r^2)}}e^{-\omega'\sqrt{(u^2+r^2)}}$$

$$= \begin{cases} \dfrac{e^{-r\sqrt{(\chi^2-\omega_0^2)}}}{ir\omega_0} & \omega_0<\chi, \\[2ex] \dfrac{e^{-ir\sqrt{(\omega_0^2-\chi^2)}}}{ir\omega_0} & \omega_0>\chi, \end{cases}$$

where terms of order $1/r^2$ have been neglected. Similarly, by integrating twice with respect to ω_0, I find, omitting terms of order $1/r^3$, that

$$\int_0^\infty du\,J_0(\chi u)\frac{u}{(u^2+r^2)^{\frac{3}{2}}}e^{-i\omega_0\sqrt{(u^2+r^2)}} = \begin{cases} -\dfrac{\sqrt{(\chi^2-\omega_0^2)}}{r^2\omega_0^2}e^{-r\sqrt{(\chi^2-\omega_0^2)}} & \omega_0<\chi, \\[2ex] -\dfrac{i\sqrt{(\omega_0^2-\chi^2)}}{r^2\omega_0^2}e^{-ir\sqrt{(\omega_0^2-\chi^2)}} & \omega_0>\chi. \end{cases}$$

By using these integrals, (113) gives at once

$$U_0^{\text{ret.}} = \begin{cases} g_2\sqrt{(\chi^2-\omega_0^2)}\dfrac{x^k L_{k0}}{r^2}e^{-r\sqrt{(\chi^2-\omega_0^2)}}\sin\omega_0 t & \omega_0<\chi \\[2ex] g_2\sqrt{(\omega_0^2-\chi^2)}\dfrac{x^k L_{k0}}{r^2}\cos(\omega_0 t-r\sqrt{(\omega_0^2-\chi^2)}) & \omega_0>\chi \end{cases}$$

and

$$U_k^{\text{ret.}} = \begin{cases} g_2\left(\omega_0\dfrac{L_{0k}}{r}\cos\omega_0 t+\sqrt{(\chi^2-\omega_0^2)}\dfrac{x^l L_{lk}}{r^2}\sin\omega_0 t\right)e^{-r\sqrt{(\chi^2-\omega_0^2)}} & \omega_0<\chi, \\[2ex] g_2\left(\omega_0\dfrac{L_{0k}}{r}+\sqrt{(\omega_0^2-\chi^2)}\dfrac{x^l L_{lk}}{r^3}\right)\cos(\omega_0 t-r\sqrt{(\omega_0^2-\chi^2)}) & \omega_0>\chi. \end{cases}$$

Use of these results leads at once to (84) of the text.

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