General classical theory of spinning particles in a Maxwell field

By H. J. Bhabha, Ph.D. and H. C. Corben, Ph.D.

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The purpose of this paper is to give the complete classical theory of a spinning particle moving in a Maxwell field. The particle is assumed to be a point, and its interaction with the field is described by a point charge g_1 and a point dipole g_2. The Maxwell equations are assumed to hold right up to the point representing the particle. Exact equations are then derived for the motion of the particle in a given external field which are strictly consistent with the conservation of energy, momentum and angular momentum, and hence contain the effects of radiation reaction on the motion of the particle. It is shown that in the presence of a point dipole the energy tensor of the field can and must be redefined so as to make the total energy finite. The mass, the angular momentum of the spin, and the moment of inertia perpendicular to the spin axis appear in the equations as arbitrary mechanical constants. Reasons are given for believing that for an elementary particle the last constant is zero, in agreement with relativistic quantum theory.

In the general theory there is no relation between the electric and magnetic dipole moments of the particle and the state of its translational motion. A procedure is given for deriving from the general equations specialized equations consistent with the condition that the dipole is always a purely magnetic or electric one in the system in which the particle is instantaneously at rest. The radiation reaction terms are very much simpler in the former of these specialized cases than in the general case. The effect of radiation reaction is to make the scattering of light by a rotating dipole *decrease* inversely as the square of the frequency for high frequencies, just as for scattering by a point charge.

The quantum treatment of a point charge and its interaction with quantized fields gives rise to a number of difficulties, for example, those connected with self-energies, which become very much greater when the particle has an *explicit* spin interaction with the field, as in the case of protons or neutrons and their interaction with the meson field. These difficulties are due at least in part to a neglect of the effects of radiation reaction. For a point charge the effects of the radiation reaction can be estimated by a comparison with the classical theory of Lorentz, and it is generally assumed that the quantum theory will give correct results in those energy regions and for those processes where the effects of this radiation reaction are negligible. For a point dipole the position is less satisfactory; for in the absence of a classical theory giving the effects of radiation reaction on the motion of the dipole, it has not been possible to delimit the region in which the quantum theory might be expected to give correct results. In particular, it has been impossible to decide whether the multiple processes and Heisenberg explosions predicted by the quantum theory fall, at least partly, in a region where the theory should be valid and are therefore to be regarded as deserving of credence, or whether they take place entirely in energy regions where the theory loses all claim to validity by its vital neglect of radiation reaction, and are hence to be regarded as entirely spurious. An exact classical theory of spinning particles taking into account the effects of radiation reaction is therefore of considerable physical interest. The purpose of this paper is to supply a complete classical theory of spinning particles moving in a Maxwell field. The extension of this theory to a meson field is carried out in the paper which immediately follows this.

It will be shown in this paper that our classical theory is the parallel of a quantum theory in which the particle has an *explicit* spin interaction with the Maxwell field. It is *not* the classical equivalent of a theory in which the interaction of the particle with the field is expressed only through the potentials, as in Dirac's theory of the electron. All comparisons must therefore be between this theory and a quantum theory in which the particle has an explicit spin interaction with the field, such as could always be introduced mathematically.† It appears, however, that the electron as it occurs in nature does not have such an explicit interaction with the Maxwell field, so that the theory of this paper is not applicable, even in the classical limit, to an electron. On the other hand, protons and neutrons have an explicit spin interaction with the meson field. Our classical theory, or rather the extension of it to a meson field carried out in the following paper, is then applicable to this problem. All the remarks we make below will then only refer to a quantum theory containing an added spin interaction with the Maxwell field of the type mentioned.

It must be demanded of any complete classical theory that it shall be consistent with the principle of relativity, and this almost inevitably requires that the dipole be considered as a point with no extension. For if a particle of finite extension be considered, then it is not possible to specify the distribution of charge and dipole moment over the finite volume occupied by the particle in a relativistically consistent way without introducing a field (other than the Maxwell field) which shall be responsible for preserving the shape of the particle. The introduction of another field for this purpose

† A further interaction term of the type $g_2 F_{\mu\nu} \gamma^\mu \gamma^\nu \psi$ could be added to the Dirac equation, where $F_{\mu\nu}$ are the field strengths, and γ_μ the Dirac matrices as defined by Pauli (1933, p. 220).

would be artificial and in any case much more complicated than treating the dipole as a point. We now meet the difficulty that the field energy as usually defined becomes infinite in the immediate neighbourhood of the dipole. This difficulty may be avoided either by modifying Maxwell's equations so that the field remains finite in the immediate neighbourhood of the dipole, or by assuming that the Maxwell equations hold exactly right up to the point singularity representing the dipole, and modifying the definition of field energy when singularities are present in the field. The latter method has been applied successfully by Dirac (1938) and Pryce (1938) to a point electron, and it is the method we shall follow in this paper.

The procedures of Dirac and Pryce are different, but lead to the same result, and both amount in essence to adding terms to the usual Maxwellian expression for the field energy which just cancel some of the singular terms and make the total field energy of a point charge finite. We believe that this procedure is not a mere mathematical device but is physically sensible. One can see this at once by going back to the origin of the idea of field energy. Consider the static case. The potential energy of any non-singular distribution of charge or dipole moment, that is, the work done in bringing this distribution of charge and dipole moment from infinity into the actual configuration, can be transformed by using Maxwell's equations into an integral over the whole of space occupied by the field, and hence may be regarded as energy stored in the field. But this transformation is no longer possible if singularities, for example point charges or point dipoles, are present in the field. The self-energy difficulties in the classical theory therefore arise by using a definition of field energy which is no longer valid in the presence of singularities.

It is of course possible to look upon a point electron as the limiting case of a distribution in which the same charge is initially spread over a finite volume. The Maxwellian definition of field energy can be used now, and in the limit when the charge becomes concentrated in a point the total field energy becomes infinite. This infinite field energy has a physical meaning, for it corresponds to the evident fact that the work done against electrostatic force in compressing a charge originally spread over a finite volume into a point is infinite. There are, however, no purely logical or mathematical reasons why a point charge or point dipole should be regarded as limiting cases in this manner. Moreover, it would be contrary to our present views about elementary particles to look upon them as made up in this way. It therefore seems to us logical and physically reasonable to look upon the elementary particles in nature as point charges or point dipoles, and to reformulate the definition of field energy when singularities are present in the field. It will be shown in this paper that this can be done for a point dipole in a way consistent with the principle of relativity.

We treat a particle as a point having an interaction with the field characterized by a charge g_1 and a dipole moment g_2, and we assume that the Maxwell equations are valid right up to the point. We then derive equations for the motion of the particle which are consistent with the conservation of energy, momentum and angular momentum for the system as a whole consisting of the particle and the field. These equations will automatically include the effects of radiation reaction. The conservation of angular momentum has to be demanded explicitly, for it gives the equation for the rotation of the dipole. The mass of the particle, the angular momentum of the spin and the moment of inertia perpendicular to the axis of the spin appear in the equations as arbitrary and independent constants.

In the general classical theory there is no connexion between the translational motion of the dipole and its rotational motion (by rotation is here understood a rotation in the direction of the time axis as well as a space rotation). It would be quite possible theoretically, for example, for an initially pure magnetic dipole with no translational motion to develop in a suitable external field an electric dipole moment at right angles to its magnetic moment while still continuing to be at rest. Dipoles of this sort do not seem to occur in nature. It is possible, however, to impose the condition that a dipole shall always remain a pure electric or magnetic dipole in the rest system, and a procedure is given for deriving from our general equations specialized equations consistent with either of these conditions. The special equations for a dipole which remains a pure magnetic dipole in the rest system have already been derived by a direct method (Bhabha 1940*a*, referred to in this paper as B) and are the same as those which are derived as a particular case of the present general theory. As has been mentioned before (Bhabha 1940*b*), the effect of radiation reaction is very much more complicated in the general than in either of the two special cases mentioned above, but there seems to be no reason within the limits of the classical theory for excluding the general case. It is therefore to be regarded as an achievement of relativistic quantum theory that it automatically demands that an elementary particle shall have only a pure magnetic moment in the system in which it is at rest.

As has been mentioned earlier, the dipole may also have a moment of inertia perpendicular to the axis of the spin, and this enters in the theory as an entirely independent constant. The equations, however, take their simplest form if the moment of inertia perpendicular to the axis of the spin is put equal to zero, that is, when the mechanical properties of the particle are those of a *pure* gyroscope. A comparison of the classical equations with those of the quantum theory, as well as a comparison of the respective cross-sections for the scattering of light, clearly shows that in the quantum theory the particle automatically has the mechanical properties of a pure gyroscope. The quantum theory therefore again appears to be an advance on classical theory in that it allows only the classically simplest case, in conformity with what is found in nature.

Where the present classical theory goes beyond the quantum theory is in being able to treat the effects of radiation reaction exactly, whereas the quantum theory neglects these completely. Our theory shows quite clearly the frequency in the neighbourhood of which radiation reaction first begins to dominate the scattering of light, and hence allows us to fix the region of validity of the quantum theory. It appears that within this region the classical and quantum theories give the same scattering, showing that *quantum* effects have not yet come in, as is to be expected also from quite general arguments. The present classical theory, therefore, apart from being of general theoretical interest, actually allows us to give a scattering formula which should be valid for frequencies far beyond the region of validity of the quantum formula under certain circumstances.

An important point may be noticed. The classical theory shows that for very high frequencies *the scattering of light does not depend either on the charge or on the dipole moment or on the mechanical constants of the particle, but is a pure function of the frequency, decreasing as the inverse square of the frequency*. This is true for both scattering by a pure charge as calculated by Dirac and for scattering by a dipole as calculated below, and appears to be a fundamental property of radiation. It is shown in the following paper that it is also true of a meson field. It is clear that a series in ascending powers of the charge or dipole moment cannot approximate to a behaviour of this sort. The proper treatment of radiation reaction, therefore, will probably require a far-reaching extension of quantum theory. The classical theory developed in this paper gives an exact treatment of radiation reaction, but its quantization appears to present very great difficulties.

General theory

We use tensor notation throughout, and for convenience put the velocity of light equal to unity. The metric tensor $g_{\mu\nu}$ is assumed to have the form $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$, with all the other components vanishing.

As stated in the introduction, we treat the particle as a point. Its co-ordinates are denoted by z_μ, which may be considered as functions of the proper time τ measured from an arbitrary point on the world line of the particle. A dot over a letter is used to denote differentiation with respect to the proper time τ. The velocity $\dot{z}_\mu$ of the particle is denoted by v_μ. The spin of the particle is described by an antisymmetric tensor $S_{\mu\nu}$ which may also be considered as a function of τ. The particle is assumed to have a charge g_1. The charge and current density P_μ at any point of space x_ρ may then be written with the help of δ-functions in the form

$$\mathrm{P}_\mu = g_1\int_{-\infty}^{\infty} d\tau\, v_\mu \delta(x_0-z_0)\,\delta(x_1-z_1)\,\delta(x_2-z_2)\,\delta(x_3-z_3). \tag{1}$$

The dipole moment of the particle is denoted by g_2. It gives rise to a dipole density $\Sigma_{\mu\nu}$ at a point x_ρ which may be written in the form

$$\Sigma_{\mu\nu} = g_2\int_{-\infty}^{\infty} d\tau\, S_{\mu\nu}(\tau)\,\delta(x_0-z_0)\,\delta(x_1-z_1)\,\delta(x_2-z_2)\,\delta(x_3-z_3). \tag{2}$$

It is found convenient to adopt the following notation. The invariant formed from any two tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ is written in the scalar product notation:

$$(XY) \equiv X_{\mu\nu}Y^{\mu\nu}, \quad X^2 \equiv (XX) = X_{\mu\nu}X^{\mu\nu}.$$

The invariant formed by any combination of tensors and two vectors is written in the usual matrix notation; thus

$$(\dot{v}\ddot{S}\dot{S}v) \equiv \dot{v}^\mu \ddot{S}_{\mu\rho}\dot{S}^{\rho\nu}v_\nu, \quad \dot{v}^2 \equiv \dot{v}_\rho \dot{v}^\rho.$$

The antisymmetric tensor formed from *two other antisymmetric* tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ is sometimes written in the vector product notation

$$[X.Y]_{\lambda\mu} = X_{\lambda\sigma}Y^\sigma{}_\mu - X_{\mu\sigma}Y^\sigma{}_\lambda.$$

Then if $X_{\mu\nu}$, $Y_{\mu\nu}$ and $Z_{\mu\nu}$ are three antisymmetric tensors, the following identity holds:

$$(X[Y.Z]) = (Y[Z.X]). \tag{3}$$

The vector v_μ by definition satisfies the equation

$$v^2 = 1. \tag{4}$$

The equations derived from this by successive differentiation are†

$$(v\dot{v}) = 0, \tag{5a}$$

$$(v\ddot{v}) + \dot{v}^2 = 0, \tag{5b}$$

$$(vv^{\mathrm{iii}}) + 3(\dot{v}\ddot{v}) = 0, \tag{5c}$$

$$(vv^{\mathrm{iv}}) + 4(\dot{v}v^{\mathrm{iii}}) + 3\ddot{v}^2 = 0. \tag{5d}$$

† We write v^{iii} and v^{iv} for the third and fourth derivatives with respect to τ.

Now in order that the constant g_2 should have a meaning, the absolute magnitude of the tensor $S_{\mu\nu}$ must remain constant. Thus we must demand that

$$S^2 \equiv S_{\mu\nu} S^{\mu\nu} = \text{constant}. \tag{6}$$

The equations derived from this by successive differentiation are

$$(S\dot{S}) = 0, \tag{7a}$$

$$(S\ddot{S}) + \dot{S}^2 = 0, \tag{7b}$$

$$(SS^{\text{iii}}) + 3(\dot{S}\ddot{S}) = 0. \tag{7c}$$

The equations of motion of the dipole must be such as to be consistent with the equations (4), (5), (6) and (7).

If we denote the Maxwell potentials by ϕ_ν and the field strengths by $F_{\mu\nu}$, the Maxwell equations may be written in the form

$$\frac{\partial}{\partial x^\mu}\phi_\nu - \frac{\partial}{\partial x^\nu}\phi_\mu = F_{\mu\nu}, \tag{8a}$$

$$\frac{\partial}{\partial x_\mu}F_{\mu\nu} = 4\pi \mathrm{P}_\nu + 4\pi \frac{\partial}{\partial x_\mu}\Sigma_{\mu\nu}. \tag{8b}$$

From (8*a*) it follows at once that

$$\frac{\partial}{\partial x^\lambda}F_{\mu\nu} + \frac{\partial}{\partial x^\mu}F_{\nu\lambda} + \frac{\partial}{\partial x^\nu}F_{\lambda\mu} = 0. \tag{9}$$

The potentials may be taken to satisfy the equation

$$\frac{\partial}{\partial x_\mu}\phi_\mu = 0, \tag{10}$$

and hence it follows from (8) that

$$\frac{\partial}{\partial x_\rho}\frac{\partial}{\partial x^\rho}\phi_\nu = 4\pi \mathrm{P}_\nu + 4\pi \frac{\partial}{\partial x_\mu}\Sigma_{\mu\nu}. \tag{11}$$

The usual energy-momentum density tensor of the field $T_{\mu\nu}$ is given by the expression

$$4\pi T_{\mu\nu} = F_{\mu\rho} F^\rho{}_\nu + \tfrac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{12}$$

It satisfies the equation of conservation

$$\frac{\partial}{\partial x_\nu}T_{\mu\nu} = 0 \tag{13}$$

in free space.

As in a previous paper (Bhabha 1939*b*) we introduce a tensor $M_{\lambda\mu\nu}$, antisymmetric in λ and μ, to represent the angular momentum density of the field:

$$M_{\lambda\mu\nu} \equiv x_\lambda T_{\mu\nu} - x_\mu T_{\lambda\nu}. \tag{14}$$

It likewise satisfies the equation of conservation in free space

$$\frac{\partial}{\partial x_\nu} M_{\lambda\mu\nu} = 0. \tag{15}$$

The solutions of (11) giving the retarded and advanced potentials are well known. Our theory is quite symmetrical in retarded and advanced potentials, and we therefore only deal with retarded potentials for simplicity. Due to the two terms on the right-hand side of (11), the retarded potential $\phi_\nu^{\text{ret.}}$ is a sum of two parts:

$$\phi_\nu^{\text{ret.}} = \phi_\nu^{(1)} + \phi_\nu^{(2)}, \tag{16}$$

the first being proportional to g_1, the second to g_2.

We introduce the vector s_μ to represent the distance from the actual point to the retarded point $z_\mu(\tau_0)$:

$$s_\mu \equiv x_\mu - z_\mu(\tau_0). \tag{17}$$

The retarded time τ_0 is defined by the equation

$$s_\mu s^\mu \equiv \{x_\mu - z_\mu(\tau_0)\}\{x^\mu - z^\mu(\tau_0)\} = 0. \tag{18}$$

We introduce the following symbols:

$$\kappa \equiv s_\mu(\tau_0)\, v^\mu(\tau_0), \quad \kappa' \equiv s_\mu(\tau_0)\, \dot{v}^\mu(\tau_0), \quad \kappa'' \equiv s_\mu(\tau_0)\, \ddot{v}^\mu(\tau_0), \tag{19}$$

and

$$\mathcal{S}_\mu = S_{\mu\nu} v^\nu, \quad \mathcal{S}'_\mu = \dot{S}_{\mu\nu} v^\nu, \quad \mathcal{S}''_\mu = \ddot{S}_{\mu\nu} v^\nu, \quad \mathcal{S}'''_\mu = S^{\text{iii}}_{\mu\nu} v^\nu, \quad \mathcal{S}''''_\mu = S^{\text{iv}}_{\mu\nu} v^\nu. \tag{20}$$

The retarded potentials may then be written in the well-known forms

$$\phi_\nu^{(1)} = g_1 \frac{v_\nu}{\kappa}, \tag{21}$$

$$\phi_\nu^{(2)} = g_2 \frac{\partial}{\partial x_\rho}\left(\frac{S_{\rho\nu}}{\kappa}\right). \tag{22}$$

All the quantities on the right-hand side are understood to be taken at the retarded time τ_0. In carrying out the differentiation in (22) we must remember that a change in the point x_ρ not only changes the distance s_ρ but also changes the retarded time τ_0. The general method of carrying out the differentiation has been given in an earlier paper (Bhabha 1939*a* referred to here as A). We thus find†

$$\phi_\nu^{(2)} = g_2 \left\{ \frac{\dot{S}_\nu}{\kappa^2} + \frac{s^\rho S_{\rho\nu}}{\kappa^3}, \quad -\frac{s^\rho S_{\rho\nu}}{\kappa^3}\kappa' + \frac{s^\rho \dot{S}_{\rho\nu}}{\kappa^2} \right\}. \tag{23}$$

Now s_μ, κ and κ' are of the same order, so that the first two terms are of order κ^{-2} and the next two of order κ^{-1}.

The retarded field strengths can be calculated at once by using (8*a*). Corresponding to (16) we write

$$F_{\mu\nu}^{\text{ret.}} = F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)}. \tag{24}$$

$F_{\mu\nu}^{(1)}$ and $F_{\mu\nu}^{(2)}$ are given in the appendix. The first is the field of a pure point charge and is proportional to g_1. It has terms of order κ^{-2} and κ^{-1}. The second is the field of a pure point dipole and is proportional to g_2. It has terms of order κ^{-3}, κ^{-2} and κ^{-1}. Following Dirac we write the actual field $F_{\mu\nu}$ at any point as the sum of the retarded field $F_{\mu\nu}^{\text{ret.}}$ at that point plus an ingoing field $F_{\mu\nu}^{\text{in.}}$:

$$F_{\mu\nu} = F_{\mu\nu}^{\text{ret.}} + F_{\mu\nu}^{\text{in.}}. \tag{25}$$

The ingoing field $F_{\mu\nu}^{\text{in.}}$ satisfies the Maxwell equations for empty space, i.e. (8) and (9), with the right-hand side of (8*b*) put equal to zero.

We now proceed to find the exact equations of motion for the particle. It is necessary at this stage to introduce the idea of an antisymmetric tensor $X_{\mu\nu}^*$ adjunct (dual) to a given antisymmetric tensor $X_{\mu\nu}$. Its components are connected with those of $X_{\mu\nu}$ by the equations

$$\left.\begin{aligned} X^{*01} &= X_{23}, \quad X^{*02} = X_{31}, \quad X^{*03} = X_{12}, \\ X^{*23} &= X_{01}, \quad X^{*31} = X_{02}, \quad X^{*12} = X_{03}. \end{aligned}\right\} \tag{26a}$$

Any of the relations (26*a*) can be deduced from any other by successively changing any three suffices in rotation. Using the tensor $g_{\mu\nu}$, we can deduce from (26*a*) that

$$\left.\begin{aligned} X_{01}^* &= -X^{23}, \quad X_{02}^* = -X^{31}, \quad X_{03}^* = -X^{12}, \\ X_{23}^* &= -X^{01}, \quad X_{31}^* = -X^{02}, \quad X_{12}^* = -X^{03}. \end{aligned}\right\} \tag{26b}$$

It follows at once from (26) that

$$(X^*)^* = -X. \tag{27}$$

† Following Dirac, we separate terms which correspond (as $\kappa \to 0$) to singularities of different orders by a comma.

If $X_{\mu\nu}$, $Y_{\mu\nu}$ are two antisymmetric tensors, then it can be proved easily that

$$(XY^*) = (X^*Y), \tag{28a}$$

$$[X.Y^*] = [X^*.Y], \tag{28b}$$

and

$$[X.Y]^* = [X.Y^*] = [X^*.Y]. \tag{29}$$

It will appear presently that the six-vector $S^*_{\mu\nu}$ adjunct to $S_{\mu\nu}$ plays an important part in the equations and is on the same footing as $S_{\mu\nu}$. Using (26) we have

$$(SS^*) \equiv S_{\mu\nu} S^{*\mu\nu} = 4(S_{01}S_{23} + S_{02}S_{31} + S_{03}S_{12}), \tag{30}$$

and, by (29),

$$[S.S^*] = [S.S]^* = 0. \tag{31}$$

Now the change $\delta S_{\lambda\mu}$ in the components of the spin due to the most general type of rotation possible (by a rotation we always mean a spatial rotation as well as a rotation involving the time component) can always be written in the form

$$\delta S_{\lambda\mu} = S_{\lambda\rho}\phi^{\rho}{}_{\mu} - S_{\mu\rho}\phi^{\rho}{}_{\lambda},$$

where $\phi_{\rho\mu}$ is an infinitesimal antisymmetric tensor. Thus the most general form for the equation giving the rotation of the spin is

$$\dot{S}_{\lambda\mu} = S_{\lambda\rho}\Theta^{\rho}{}_{\mu} - S_{\mu\rho}\Theta^{\rho}{}_{\lambda}, \tag{32a}$$

$\Theta_{\rho\mu}$ being some six-vector. Θ may itself involve derivatives of S. In our abbreviated notation this equation reads

$$\dot{S} = [S.\Theta]. \tag{32b}$$

This expression for $\dot{S}$ automatically satisfies (7a), for, by (3),

$$(S\dot{S}) = (S[S.\Theta]) = (\Theta[S.S]) = 0; \tag{33}$$

Similarly it follows from (32) and (31) that

$$(S^*\dot{S}) = (S^*[S.\Theta]) = (\Theta[S^*.S]) = 0; \tag{34}$$

whence

$$(S^*S) = \text{constant}. \tag{34a}$$

The equation adjunct to (32) is

$$\dot{S}^* = [S.\Theta]^* = [S.\Theta^*] = [S^*.\Theta]. \tag{35}$$

Multiplying (32) and (35) by arbitrary constants I and I' and adding, we find

$$I\dot{S} + I'\dot{S}^* = [S.\Theta']. \tag{36}$$

We have written Θ' in place of $I\Theta + I'\Theta^*$. This is the most general form that can be taken by the equations of motion for the rotation of the dipole.

It will be seen below that the condition that the rotational equations must have the form (36) takes us a long way towards determining the radiation reaction terms.

Conversely, we can go back from equation (36) to the equations (32) and (35). For the equation adjunct to (36) is

$$I\dot{S}^* - I'\dot{S} = [S.\Theta'^*]; \tag{37}$$

whence, by (36), we deduce

$$(I^2+I'^2)\dot{S} = [S.(I\Theta' - I'\Theta'^*)] = (I^2+I'^2)[S.\Theta],$$

which is just equation (32). It follows from this, as can also be proved directly, that it is a necessary consequence of an equation of the type (36) that

$$(S\dot{S}) = 0, \tag{33}$$

$$(S^*\dot{S}) = 0. \tag{34}$$

It further follows from the symmetry of the right-hand side of (36) in S and Θ', as also from (32) and (35), that

$$(\dot{S}\Theta) = 0, \tag{38}$$

$$(\dot{S}^*\Theta) = 0. \tag{39}$$

Thus two invariant equations are a consequence of the rotational equation (36). As will be seen below, another invariant equation can be deduced from the equations of motion giving the translation of the particle, and the requirement that this equation shall be consistent with (38) gives us further conditions determining the equations of motion.

To proceed further we follow a method first used by Dirac (1938). Consider the world line of the particle to be given, and also the direction of the spin at every point. Now surround this world line by a thin world tube, the radius of which will ultimately be made to tend to zero, and calculate by using the tensors $T_{\mu\nu}$ and $M_{\lambda\mu\nu}$ the flow of field energy, momentum and angular momentum through the three-dimensional surface of a finite length of this tube bounded by the proper times τ_1 and τ_2. For conservation, the flow of energy, momentum and angular momentum through the surface of this length of tube must equal the difference in these quantities at the two ends of the tube. The rate of flow of these quantities must therefore be a perfect differential. This will only be the case if, depending on the ingoing field, the world line has a certain shape and the dipole a certain orientation at each point. In other words, this condition will give the equations of motion of the particle in the given ingoing field.

It is easy to show that our results do not depend on the shape of the tube. For convenience we take the tube defined by

$$\kappa \equiv s_\mu(\tau_0)\, v^\mu(\tau_0) = \epsilon, \tag{40}$$

where ϵ will be treated as a small quantity which will ultimately be made to tend to zero. We fix our attention on some point τ_0 of the world line and consider it in the particular Lorentz system in which the velocity at that point has the special form $v_0 = 1$, $v_1 = v_2 = v_3 = 0$. This will be called the rest system of the point. Then from (40) it follows that in this system $s_0 = \epsilon$, and since $s_\mu s^\mu = 0$, a two-dimensional sphere of radius ϵ with the point τ_0 as centre taken at a time ϵ later forms a section of the tube. All points on the surface of this sphere correspond to the same retarded point τ_0. Thus any integral over the surface of the tube can always be regarded as a two-dimensional integral over the surface of a sphere centred round a point with the proper time τ, and then an integral along the world line with respect to the proper time τ.

If we denote by dS^ν an element of the surface of the tube taken as positive when the normal is directed outwards, then the flow of energy and momentum into a length of the tube whose ends correspond to the proper times τ_1 and τ_2 is

$$\int T_{\mu\nu}\, dS^\nu, \tag{41}$$

which, in view of what has been said above about integration over the surface of the tube, can always be written in the form

$$\int_{\tau_1}^{\tau_2} T_\mu\, d\tau. \tag{42}$$

For conservation of energy and momentum this must only depend on the conditions at the two ends of the tube, so that the integral must be a perfect differential. Therefore we must have

$$T_\mu = \dot{A}_\mu, \tag{43}$$

where A_μ is some tensor which has to be found. Equation (43) determines the translational motion of the particle.

The flow of angular momentum into the tube is given by

$$\int M_{\lambda\mu\nu}\, dS^\nu. \tag{44}$$

By (14) and the definition (17) of s_μ, (44) can be written

$$\int (s_\lambda T_{\mu\nu} - s_\mu T_{\lambda\nu})\, dS^\nu + \int (z_\lambda T_{\mu\nu} - z_\mu T_{\lambda\nu})\, dS^\nu. \tag{44a}$$

The first integral can again be written in the form

$$\int_{\tau_1}^{\tau_2} M_{\lambda\mu}\, d\tau, \tag{44b}$$

while the second becomes

$$\int_{\tau_1}^{\tau_2} (z_\lambda T_\mu - z_\mu T_\lambda)\, d\tau.$$

Thus, by (43),

$$\int M_{\lambda\mu\nu}\, dS^\nu = \int_{\tau_1}^{\tau_2} d\tau \{M_{\lambda\mu} + (z_\lambda \dot{A}_\mu + z_\mu \dot{A}_\lambda)\}$$

$$= \int_{\tau_1}^{\tau_2} d\tau \left\{ M_{\lambda\mu} - (v_\lambda A_\mu - v_\mu A_\lambda) + \frac{d}{d\tau}(z_\lambda A_\mu - z_\mu A_\lambda) \right\}.$$

For conservation of angular momentum, the integrand of this must be a perfect differential. Put it equal to $\dot{B}_{\lambda\mu} + \dfrac{d}{d\tau}(z_\lambda A_\mu - z_\mu A_\lambda)$. This leads to

$$M_{\lambda\mu} - (v_\lambda A_\mu - v_\mu A_\lambda) = \dot{B}_{\lambda\mu}, \tag{45}$$

where $B_{\lambda\mu}$ has to be found. Equation (45) then determines the rotational motion of the dipole.

We now return to T_μ. Owing to the quadratic form (12) of $T_{\mu\nu}$ in the field strengths and the splitting of $F_{\mu\nu}$ into two parts by (25), $T_{\mu\nu}$ can be written as the sum of three parts:

$$T_{\mu\nu} = T_{\mu\nu}^{\text{ret.}} + T_{\mu\nu}^{\text{mix.}} + T_{\mu\nu}^{\text{in.}}.$$

The first contains the retarded field only, the second the product of the retarded and ingoing fields, and the last only the ingoing field. Since $F_{\mu\nu}^{\text{in.}}$ is not singular on the world line, $T_{\mu\nu}^{\text{in.}}$ is also not singular, and hence in the limit $\epsilon \to 0$ will make no contribution to (41). Thus we may write

$$T_\mu = T_\mu^{\text{ret.}} + T_\mu^{\text{mix.}}. \tag{46}$$

Since, according to (24), $F_{\mu\nu}^{\text{ret.}}$ consists of two parts, $T_\mu^{\text{ret.}}$ will contain terms proportional to g_1^2, $g_1 g_2$ and g_2^2, while $T_\mu^{\text{mix.}}$ will contain the ingoing field and terms proportional to g_1 and g_2. Corresponding to this we write A_μ as the sum of three parts:

$$A_\mu = A_\mu^{\text{ret.}} + A_\mu^{\text{mix.}} + A_\mu^{\text{mech.}}. \tag{47}$$

The first only contains terms proportional to g_1^2, $g_1 g_2$ and g_2^2, the second contains the ingoing field and terms proportional to g_1 or g_2, while the third does not contain g_1 or g_2. We call $A_\mu^{\text{mech.}}$ the mechanical part of the energy.

If we write
$$T_\mu^{\text{react.}} \equiv T_\mu^{\text{ret.}} - \dot{A}_\mu^{\text{ret.}}, \tag{48}$$
the translational equation (43) becomes
$$T_\mu^{\text{react.}} + T_\mu^{\text{mix.}} - \dot{A}_\mu^{\text{mix.}} = \dot{A}_\mu^{\text{mech.}}. \tag{49}$$

For the same reasons as lead to (46), $M_{\lambda\mu}$ can also be written as the sum of two parts:
$$M_{\lambda\mu} = M_{\lambda\mu}^{\text{ret.}} + M_{\lambda\mu}^{\text{mix.}}, \tag{50}$$
the first of which contains terms proportional to g_1^2, $g_1 g_2$ and g_2^2, while the second has terms proportional to g_1 and g_2 containing the ingoing field. Corresponding to (50) we write $B_{\lambda\mu}$ as the sum of two parts:
$$B_{\lambda\mu} = B_{\lambda\mu}^{\text{ret.}} + B_{\lambda\mu}^{\text{mech.}}. \tag{51}$$
Here $B_{\lambda\mu}^{\text{ret.}}$ contains all the terms proportional to g_1^2, $g_1 g_2$ and g_2^2, while $B_{\lambda\mu}^{\text{mech}}$ only contains terms independent of g_1 and g_2, and represents the mechanical properties of the spin, as will be seen below. $B_{\lambda\mu}^{\text{mix.}}$ cannot exist for dimensional reasons. It would have to be a product of g_2 times the ingoing field times a quantity of the dimensions of a length, and no such quantity can be made up of $S_{\mu\nu}$ or v_μ and their derivatives (provided, naturally, that the reciprocals of these quantities are excluded). If we write
$$M_{\lambda\mu}^{\text{react.}} \equiv M_{\lambda\mu}^{\text{ret.}} - (v_\lambda A_\mu^{\text{ret.}} - v_\mu A_\lambda^{\text{ret.}}) - \dot{B}_{\lambda\mu}^{\text{ret.}}, \tag{52}$$
the rotational equation (45) can be written
$$M_{\lambda\mu}^{\text{react.}} + \{M_{\lambda\mu}^{\text{mix.}} - (v_\lambda A_\mu^{\text{mix.}} - v_\mu A_\lambda^{\text{mix.}})\} = \dot{B}_{\lambda\mu}^{\text{mech.}} + (v_\lambda A_\mu^{\text{mech.}} - v_\mu A_\lambda^{\text{mech.}}). \tag{53}$$
The terms $T_\mu^{\text{react.}}$ in (49) and $M_{\lambda\mu}^{\text{react.}}$ in (53), being quadratic in g_1 and g_2, represent the effects of radiation reaction.

The derivation of (53) shows that as far as the conservation laws are concerned $B_{\lambda\mu}^{\text{mech.}}$ is arbitrary. The general arguments given at the beginning show that the rotational equation must be of the form (36). We may therefore put
$$B_{\lambda\mu}^{\text{mech.}} = I S_{\lambda\mu} + I' S_{\lambda\mu}^* + B_{\lambda\mu}'^{\text{mech.}}, \tag{54}$$
where I and I' are arbitrary constants. (53) then becomes
$$I\dot{S}_{\lambda\mu} + I'\dot{S}_{\lambda\mu}^* + \{v_\lambda A_\mu^{\text{mech.}} - v_\mu A_\lambda^{\text{mech.}} + \dot{B}_{\lambda\mu}'^{\text{mech.}}\}$$
$$= \{M_{\lambda\mu}^{\text{mix.}} - (v_\lambda A_\mu^{\text{mix.}} - v_\mu A_\lambda^{\text{mix.}})\} + M_{\lambda\mu}^{\text{react.}}. \tag{55}$$

Now the reaction and the mixed terms in (55) contain different powers of the fundamental constants g_1 and g_2, while the mechanical terms are independent of them. Each group by itself must therefore have the form of the right-hand side of (36). Hence we must have

$$M_{\lambda\mu}^{\text{react.}} \equiv S_\lambda{}^\rho C_{\rho\mu}^{\text{react.}} - S_\mu{}^\rho C_{\rho\lambda}^{\text{react.}}. \tag{56}$$

In other words, $A_\mu^{\text{ret.}}$ and $B_{\lambda\mu}^{\text{ret.}}$ must be so chosen that the left-hand side of (56) has the form of the right-hand side of this equation. $C_{\rho\mu}^{\text{react.}}$ is an antisymmetric tensor which is not yet known. Similarly, $A_\mu^{\text{mech.}}$ and $B_{\lambda\mu}'^{\text{mech.}}$ must be so chosen that

$$v_\lambda A_\mu^{\text{mech.}} - v_\mu A_\lambda^{\text{mech.}} + \dot{B}_{\lambda\mu}'^{\text{mech.}} \equiv S_\lambda{}^\rho C_{\rho\mu}^{\text{mech.}} - S_\mu{}^\rho C_{\rho\lambda}^{\text{mech.}}. \tag{57}$$

Consider now the mixed terms. It is shown in the appendix that

$$T_\mu^{\text{mix.}} = g_1 F_{\mu\sigma}^{\text{in.}} v^\sigma + g_2\left[-\tfrac{1}{2}S^{\rho\sigma}\frac{\partial}{\partial x^\mu}F_{\rho\sigma}^{\text{in.}} + \frac{d}{d\tau}\{\tfrac{2}{3}S_{\mu\sigma}F^{\text{in.}\sigma\nu}v_\nu - \tfrac{2}{3}F_{\mu\sigma}^{\text{in.}}\not{S}^\sigma + \tfrac{1}{3}v_\mu S^{\rho\sigma}F_{\rho\sigma}^{\text{in.}}\}\right], \tag{58}$$

$$M_{\lambda\mu}^{\text{mix.}} = g_2[S_\lambda{}^\sigma F_{\sigma\mu}^{\text{in.}} - S_\mu{}^\sigma F_{\sigma\lambda}^{\text{in.}} + \tfrac{2}{3}v_\lambda S_\mu{}^\sigma F_{\sigma\nu}^{\text{in.}} v^\nu - \tfrac{2}{3}v_\mu S_\lambda{}^\sigma F_{\sigma\nu}^{\text{in.}} v^\nu - \tfrac{2}{3}v_\lambda F_{\mu\sigma}^{\text{in.}}\not{S}^\sigma + \tfrac{2}{3}v_\mu F_{\lambda\sigma}^{\text{in.}}\not{S}^\sigma]. \tag{59}$$

Put

$$A_\mu^{\text{mix.}} = g_2\{\tfrac{2}{3}S_\mu{}^\sigma F_{\sigma\nu}^{\text{in.}} v^\nu - \tfrac{2}{3}F_{\mu\sigma}^{\text{in.}}\not{S}^\sigma - \tfrac{1}{6}v_\mu S^{\rho\sigma}F_{\rho\sigma}^{\text{in.}}\}. \tag{60}$$

This at once gives

$$M_{\lambda\mu}^{\text{mix.}} - (v_\lambda A_\mu^{\text{mix.}} - v_\mu A_\lambda^{\text{mix.}}) = g_2\{S_\lambda{}^\sigma F_{\sigma\mu}^{\text{in.}} - S_\mu{}^\sigma F_{\sigma\lambda}^{\text{in.}}\}, \tag{61}$$

which is just of the required form. In fact, the condition that the mixed terms in (55) must have the form of the right-hand side of (36) uniquely determines the first two terms of $A_\mu^{\text{mix.}}$. A term consisting of v_μ multiplied by an invariant could always be added to $A_\mu^{\text{mix.}}$ without altering the left-hand side of (61). But this term would alter the mixed terms in the translational equation (49). It will appear almost immediately that even a term of this type is uniquely determined, and has to be taken to be the third term of (60). By (58) and (60), the mixed terms in (49) then reduce to

$$T_\mu^{\text{mix.}} - \dot{A}_\mu^{\text{mix.}} = g_1 F_{\mu\sigma}^{\text{in.}} v^\sigma - \tfrac{1}{2}g_2 S^{\rho\sigma}\frac{\partial}{\partial x^\mu}F_{\rho\sigma}^{\text{in.}} + \tfrac{1}{2}g_2\frac{d}{d\tau}(v_\mu S^{\rho\sigma}F_{\rho\sigma}^{\text{in.}}). \tag{62}$$

By (62), the translational equation (49) becomes

$$\dot{A}_\mu^{\text{mech.}} = g_1 F_{\mu\sigma}^{\text{in.}} v^\sigma - \tfrac{1}{2}g_2 S^{\rho\sigma}\frac{\partial}{\partial x^\mu}F_{\rho\sigma}^{\text{in.}} + \tfrac{1}{2}g_2\frac{d}{d\tau}(v_\mu S^{\rho\sigma}F_{\rho\sigma}^{\text{in.}}) + T_\mu^{\text{react.}}. \tag{63}$$

By substituting (56), (57) and (61) in (55), the rotational equation may be written

$$I\dot{S}_{\lambda\mu}+I'\dot{S}^*_{\lambda\mu}+[S\,.\,C^{\text{mech.}}]_{\lambda\mu} = g_2[S\,.\,F^{\text{in.}}]_{\lambda\mu}+[S\,.\,C^{\text{react.}}]_{\lambda\mu}. \tag{64}$$

The simplest invariant equation which can be deduced from (63) is obtained by contracting it with v^μ. Remembering (4) and (5a), and using the obvious relation

$$v^\mu S^{\rho\sigma}\frac{\partial}{\partial x^\mu}F^{\text{in.}}_{\rho\sigma} = S^{\rho\sigma}\frac{d}{d\tau}F^{\text{in.}}_{\rho\sigma},$$

we find

$$v^\mu \dot{A}^{\text{mech.}}_\mu - \tfrac{1}{2}g_2(\dot{S}F^{\text{in.}}) = v^\mu T^{\text{react}}_\mu. \tag{65}$$

By the reasoning which leads from (36) to (38) and (39), we see that the rotational equation (64) leads to two invariant equations

$$(\dot{S}C^{\text{mech.}})-g_2(\dot{S}F^{\text{in.}}) = (\dot{S}C^{\text{react.}}) \tag{66}$$

and

$$(\dot{S}^*C^{\text{mech.}})-g_2(\dot{S}^*F^{\text{in.}}) = (\dot{S}^*C^{\text{react.}}). \tag{67}$$

The expressions containing the ingoing fields in (65) and (66) are identical, so that consistency requires that

$$(\dot{S}C^{\text{mech.}}) = 2v^\mu \dot{A}^{\text{mech.}}_\mu \tag{68}$$

and

$$(\dot{S}C^{\text{react.}}) = 2v^\mu T^{\text{react.}}_\mu. \tag{69}$$

We now see that the last term of $A^{\text{mix.}}_\mu$ in (60) was uniquely determined. For the addition to $A^{\text{mix.}}_\mu$ of a term consisting of v_μ multiplied by an invariant containing the ingoing field would have left (61) and hence (66) unchanged, while changing (62) and hence the terms in (65) containing the ingoing field. This would have made (65) and (66) inconsistent.

The mechanical constants

We now come to $A^{\text{mech.}}_\mu$ and $B^{\text{mech.}}_{\lambda\mu}$. These may be any expressions which satisfy the condition that the left-hand side of (57) shall be identically of the form of the right-hand side of that equation:

$$v_\lambda A^{\text{mech.}}_\mu - v_\mu A^{\text{mech.}}_\lambda + \dot{B}'^{\text{mech.}}_{\lambda\mu} \equiv S_\lambda{}^\rho C^{\text{mech.}}_{\rho\mu} - S_\mu{}^\rho C^{\text{mech.}}_{\rho\lambda}. \tag{57}$$

They must further satisfy (68).

Expressions which satisfy these conditions are

$$A^{\text{mech.}}_\mu = v_\mu\{M+\tfrac{1}{4}K\dot{S}^2+\tfrac{1}{4}K'(\dot{S}\dot{S}^*)\} \tag{70}$$

and

$$B'^{\text{mech.}}_{\lambda\mu} = K[S\,.\dot{S}]_{\lambda\mu}+K'[S\,.\dot{S}^*]_{\lambda\mu}, \tag{71}$$

where M, K and K' are arbitrary constants. M has the dimensions of a mass, and K and K' those of a moment of inertia.

If we introduce (70) into (63), the translational equation can be written in its final form

$$M\dot{v}_\mu + \frac{d}{d\tau} v_\mu\{\tfrac{1}{4}K\dot{S}^2 - \tfrac{1}{4}K'(\dot{S}\dot{S}^*) - \tfrac{1}{2}g_2(SF^{\text{in.}})\}$$
$$= g_1 F^{\text{in.}}_{\mu\sigma} v^\sigma - \tfrac{1}{2}g_2 S^{\rho\sigma}\frac{\partial}{\partial x^\mu} F^{\text{in.}}_{\rho\sigma} + T^{\text{react.}}_\mu. \quad (72)$$

By substituting (71) into (64), the rotational equation becomes

$$I\dot{S}_{\lambda\mu} + I'\dot{S}^*_{\lambda\mu} + K[S.\ddot{S}]_{\lambda\mu} + K'[S.\ddot{S}^*]_{\lambda\mu} = g_2[S.F^{\text{in.}}]_{\lambda\mu} + [S.C^{\text{react.}}]_{\lambda\mu}, \quad (73)$$

and the invariant equation (66) becomes

$$K(\dot{S}\ddot{S}) + K'(\dot{S}\ddot{S}^*) - g_2(\dot{S}F^{\text{in.}}) = (\dot{S}C^{\text{react.}}). \quad (74)$$

The substitutions (70) and (71) are not the most general which are possible and they have been chosen for their simplicity. Moreover, the five constants M, I, I', K and K' are capable of simple physical interpretations, and represent the mechanical properties of a spinning particle which are well known from ordinary mechanics. Reasons will be given in the last section for believing that even (70) and (71) lead to equations which are too general, and that for the elementary particles which occur in nature, I', K and K' are zero.

I and I' represent the gyroscopic properties of the particle, and their physical meaning becomes particularly clear in certain special cases which will be investigated in a later section. It will be shown there that when the equations are such that the dipole is always a pure magnetic dipole in the rest system, I is the angular momentum of the spin about the axis of the dipole and K is the moment of inertia about an axis perpendicular to the direction of the dipole. In this case I' is zero. When the equations are such that the dipole is always a pure electric dipole in the rest system, I' represents the angular momentum of the spin about the axis of the *electric* dipole, and K' is the moment of inertia about a perpendicular axis, while I must now be zero.

It should be emphasized that all the five constants M, I, I', K and K' are completely independent and arbitrary, and may be given any positive or negative values. This may seem to be in contradiction with the properties of bodies in ordinary mechanics, where, for example, a non-vanishing value of I, the angular momentum about one axis, necessarily requires that the moment of inertia about a perpendicular axis shall be positive. This contradiction is only apparent, and results from the fact that in ordinary mechanics the mass density is always assumed to be positive at all points of the particle. In general theory, however, the mass density need not be positive at every point any more than the total mass of the particle need be positive, and once the mass density is allowed to take on both positive and negative values, it can easily be seen that a mass distribution can be given in a body of finite size for which the mechanical constants have arbitrarily assigned values, positive or negative. It is therefore not surprising that in our theory, where the particle is treated as a point, the five constants are entirely independent and arbitrary.

It is now easy to see the physical meaning of all the terms on the left-hand side of the translational equation (72). The first term just expresses the ordinary mechanical properties of a particle whose rest mass is M. Next, $[S\,.\,\dot{S}]$ represents the rate of rotation of the direction of the spin, and the kinetic energy associated with this motion is $\frac{1}{4}K\dot{S}^2$. In conformity with relativistic ideas this appears as an addition to the mass in (70) and (72). The meaning of the next term is similar. Lastly, $-\frac{1}{2}g_2(SF^{\text{in.}})$ is the potential energy of the dipole in the given ingoing field, and this also appears as an addition to the mass of the particle in (72).

The meaning of equation (74) is also clear. It states that $\frac{1}{2}g_2(\dot{S}F^{\text{in.}})$, the rate of decrease of the potential energy of the dipole in the ingoing field *due to a rotation of the dipole*, is equal to the rate of increase of the rotational kinetic energy plus the rate at which energy is radiated away, $-\frac{1}{2}(\dot{S}C^{\text{react.}})$.

To find other possible additions to (70) and (71), we have to find further solutions of (57) and (68). The easiest way to do this is to proceed methodically by taking terms which can combine with each other in groups. We illustrate the method of finding solutions by two examples which will be of use later when we come to discuss the radiation reaction terms. Possible additions to $A^{\text{mech.}}_{\mu}$ which do not contain the spin $S_{\mu\nu}$ at all have already been investigated in detail in a previous paper (Bhabha 1939*b*). It is shown there that besides the first term of (70) there is only one other solution possible which is reasonably simple, and this would lead to a motion of the particle quite unlike anything that is known in nature.

Now consider all possible additions to $A^{\text{mech.}}_{\mu}$ which are quadratic in $S_{\lambda\mu}$ and do not contain any differentiations with respect to τ. The most general combination of this type is

$$A''^{\text{mech.}}_{\mu} = e_1 v_{\mu} S^2 + e_2 v_{\mu} \mathcal{S}^2 + e_3 S_{\mu\rho} \mathcal{S}^{\rho}. \tag{75}$$

The e's are arbitrary constants having the dimensions of a mass. By (57), terms of this sort must combine with a certain $\dot{B}''^{\text{mech.}}_{\lambda\mu}$, so that $B''^{\text{mech.}}_{\lambda\mu}$ must contain one differentiation less than $A''^{\text{mech.}}_{\mu}$. This is not possible with the substitution (75), so that in this case $B''^{\text{mech.}}_{\lambda\mu}$ is zero. We find

$$\begin{aligned} v_\lambda A''^{\text{mech.}}_{\mu} - v_\mu A''^{\text{mech.}}_{\lambda} &= e_3(v_\lambda S_{\mu\rho}\mathcal{S}^\rho - v_\mu S_{\lambda\rho}\mathcal{S}^\rho) \\ &= -e_3\{S_{\lambda\sigma}(\mathcal{S}^\sigma v_\mu - v^\sigma \mathcal{S}_\mu) - S_{\mu\sigma}(\mathcal{S}^\sigma v_\lambda - v^\sigma \mathcal{S}_\lambda)\}, \qquad (76) \end{aligned}$$

which satisfies the condition (57). By (4), (5*a*) and (6), the equation (68) then requires that

$$-e_3(\mathcal{S}S') = 2e_2(\mathcal{S}S') + 2e_2(\mathcal{S}S\dot{v}) - 2e_3(\mathcal{S}S') - e_3(\mathcal{S}S\dot{v}).$$

This gives
$$e_3 = 2e_2.$$

Writing M for e_1 and M' for e_2, we see that possible additions to (70) are given by

$$A''^{\text{mech.}}_{\mu} = Mv_\mu S^2 + M'(v_\mu \mathcal{S}^2 + 2S_{\mu\sigma}\mathcal{S}^\sigma), \qquad (77)$$

where M and M' are independent and arbitrary constants. Since S^2 is a constant, the first term has in fact already been included in (70). The second term gives a possible addition to the translational equation. We should expect the elementary particles in nature to obey the simplest possible equation, and therefore we should expect M' to be zero.

Next consider additions to $A^{\text{mech.}}_{\mu}$ which are quadratic in $S_{\lambda\mu}$ or its derivatives and contain one differentiation with respect to τ. It can easily be shown by an analysis similar to that given above that there is only one possible solution of (57) and (68) of this type, namely,

$$\left.\begin{aligned} A'''^{\text{mech.}}_{\mu} &= I''\{2v_\mu(\mathcal{S}S') + \dot{S}_{\mu\sigma}\mathcal{S}^\sigma + S_{\mu\sigma}S'^\sigma\}, \\ B'''^{\text{mech.}}_{\lambda\mu} &= -I''\{v_\lambda S_{\mu\sigma}\mathcal{S}^\sigma - v_\mu S_{\lambda\sigma}\mathcal{S}^\sigma\}. \end{aligned}\right\} \qquad (78)$$

I'' is an arbitrary constant having the dimensions of mass times length, i.e. of an angular momentum (since the velocity of light is put equal to unity). The addition (78) is obviously far more complicated than anything in (70) and (71), and we should expect I'' to be zero for an elementary particle.

There appears to be no limit to the number of solutions which can be found satisfying (57) and (68), but they are all very much more complicated than (70) and (71), and we should be justified in believing that they do not occur in the description of an elementary particle.

The reaction of radiation

The radiation reaction terms still remain to be determined. $T^{\text{ret.}}_{\mu}$ and $M^{\text{ret.}}_{\lambda\mu}$ are given in the appendix. We have to fix $A^{\text{ret.}}_{\mu}$ and $B^{\text{ret.}}_{\lambda\mu}$. As mentioned before, they have to be so chosen that $M^{\text{react.}}_{\lambda\mu}$ is identically of the form given by (56), that is, by (52),

$$\{M^{\text{ret.}}_{\lambda\mu} - (v_\lambda A^{\text{ret.}}_\mu - v_\mu A^{\text{ret.}}_\lambda) - \dot{B}^{\text{ret.}}_{\lambda\mu}\} \equiv S_\lambda{}^\rho\, C^{\text{react.}}_{\rho\mu} - S_\mu{}^\rho\, C^{\text{react.}}_{\rho\lambda}, \tag{79}$$

with $C^{\text{react.}}_{\rho\mu}$ and $T^{\text{react.}}_\mu$ satisfying (69), which, by the definition (48) of $T^{\text{react.}}_\mu$, may be written

$$(\dot{S}C^{\text{react.}}) = 2v^\mu(T^{\text{ret.}}_\mu - \dot{A}^{\text{ret.}}_\mu). \tag{69a}$$

We separate the terms in (79) and (69a) which are proportional respectively to g_1^2, $g_1 g_2$ and g_2^2 into three groups. It is obvious that the terms proportional to g_1^2 will not contain $S_{\lambda\mu}$ or its derivatives, each term proportional to $g_1 g_2$ will contain $S_{\lambda\mu}$ or one of its derivatives once, while $S_{\lambda\mu}$ and its derivatives will appear twice in each term proportional to g_2^2. Each group of terms by itself has to satisfy the equations (79) and (69a). It is convenient at this stage to introduce a notation which will be of use later. We distinguish the symbols containing only terms of the first group by writing a (0) after the symbol, those of the second by a (1), and those of the third by a (2), thus expressing the fact that the groups are respectively independent of, linear and quadratic in $S_{\lambda\mu}$ and its derivatives. Thus, for example,

$$M^{\text{react.}}_{\lambda\mu} \equiv g_1^2 M^{\text{react.}}_{\lambda\mu}(0) + g_1 g_2 M^{\text{react.}}_{\lambda\mu}(1) + g_2^2 M^{\text{react.}}_{\lambda\mu}(2). \tag{80}$$

The general method has already been given in B. Since the retarded field of a point charge or dipole tends to infinity as we approach the dipole, both $T^{\text{ret.}}_\mu$ and $M^{\text{ret.}}_{\lambda\mu}$ will contain terms which tend to infinity as $\epsilon \to 0$. We therefore write

$$T^{\text{ret.}}_\mu = T^{(\epsilon)}_\mu + T^{(0)}_\mu, \tag{81}$$

where $T^{(\epsilon)}_\mu$ contains all the terms which are singular. Similarly, we write

$$M^{\text{ret.}}_{\lambda\mu} = M^{(\epsilon)}_{\lambda\mu} + M^{(0)}_{\lambda\mu}. \tag{82}$$

$T^{(\epsilon)}_\mu$ is given in the appendix by (123), (125) and (127). It appears that it is a perfect differential. It is therefore possible to split $A^{\text{ret.}}_\mu$ into two parts:

$$A^{\text{ret.}}_\mu = A^{(\epsilon)}_\mu + A^{(0)}_\mu, \tag{83}$$

of which $A^{(\epsilon)}_\mu$ alone contains the singular terms and is so chosen that

$$T^{(\epsilon)}_\mu - \dot{A}^{(\epsilon)}_\mu = 0. \tag{84}$$

Thus, remembering (48), we find that

$$T^{\text{react.}}_\mu = T^{(0)}_\mu - \dot{A}^{(0)}_\mu, \tag{85}$$

and is now entirely free from singularities. It remains finite as $\epsilon \to 0$.

$M^{(\epsilon)}_{\lambda\mu}$ is also given in the appendix by (130) and (132) and is not a perfect differential. However, it only appears in the rotational equation (55) in the combination (52):

$$M^{\text{ret.}}_{\lambda\mu} - (v_\lambda A^{\text{ret.}}_\mu - v_\mu A^{\text{ret.}}_\lambda) - \dot{B}^{\text{ret.}}_{\lambda\mu}.$$

(83) shows that $A^{\text{ret.}}_\mu$ also contains singular terms. It now appears that, with $A^{(\epsilon)}_\mu$ determined by (84),

$$M^{(\epsilon)}_{\lambda\mu} - (v_\lambda A^{(\epsilon)}_\mu - v_\mu A^{(\epsilon)}_\lambda)$$

is a perfect differential. It is therefore possible to split $B^{\text{ret.}}_{\lambda\mu}$ into two parts:

$$B^{\text{ret.}}_{\lambda\mu} = B^{(\epsilon)}_{\lambda\mu} + B^{(0)}_{\lambda\mu}, \tag{86}$$

where $B^{(\epsilon)}_{\lambda\mu}$ alone contains the singular terms and is so chosen that

$$M^{(\epsilon)}_{\lambda\mu} - (v_\lambda A^{(\epsilon)}_\mu - v_\mu A^{(\epsilon)}_\lambda) - \dot{B}^{(\epsilon)}_{\lambda\mu} = 0. \tag{87}$$

Hence, by (52),

$$M^{\text{react.}}_{\lambda\mu} = M^{(0)}_{\lambda\mu} - (v_\lambda A^{(0)}_\mu - v_\mu A^{(0)}_\lambda) - \dot{B}^{(0)}_{\lambda\mu}. \tag{88}$$

This is now entirely free from singularities and remains finite as $\epsilon \to 0$. Thus the rotational equation (55) also has no singular terms.

The above analysis shows that it is *possible* to eliminate the singularities which result from taking the energy tensor of the field to have the form (12). The question now arises as to how far this elimination is mathematically *necessary* in order that the conservation laws should hold. Consider the terms of order ϵ^{-3} in $T'^{(\epsilon)}_\mu$. These are quadratic in $S_{\lambda\mu}$ and contain one differentiation with respect to τ. The terms of order ϵ^{-3} in $A^{(\epsilon)}_\mu$ must therefore be quadratic in $S_{\lambda\mu}$ and contain no differentiation with respect to τ. The equations (79) and (69a) therefore cannot determine such terms uniquely, because we could always add an expression like (77) with arbitrary M and M', since this satisfies (57) and (68). The same applies to terms of order ϵ^{-2} in $A^{(\epsilon)}_\mu$, since these must be quadratic in $S_{\lambda\mu}$ and contain one differentiation with respect to τ. These and the corresponding terms in $B^{(\epsilon)}_\mu$ must therefore be arbitrary to the extent of a possible addition of the expressions (78) with arbitrary I''. Since there are seven terms of order ϵ^{-2} in $T'^{(\epsilon)}_\mu$ and only three in $A'''^{\text{mech.}}_\mu$ of (78), it is clear that the elimination of four of the singular terms of this order is *necessary*, while the elimination of the other three is a matter of choice. We may then sum up the position as follows. While the elimination of some of the singular terms is necessary and is achieved automatically by our method, the elimination of others is at our choice. But the non-elimination of such singular terms is merely equivalent to putting one of the arbitrary mechanical constants in our equations, for example, M, I or I'', equal to infinity. To put one of these constants, say M, equal to infinity is equivalent to making a *physical* assumption as definite as giving it some particular finite value, and this can only be decided by comparison with experiment. Reasons will be given in the last section for the belief that in order to describe the elementary particles *as they appear in nature* within the limits of classical theory, all the mechanical constants except M and I have to be put equal to zero. In this sense the conservation laws demand that all the singular terms *must* be eliminated.

It should be noticed that the elimination of the singularities is not trivial. It is easy to see that the coefficients of some of the terms in $T_\mu^{(\epsilon)}$ and $M_{\lambda\mu}^{(\epsilon)}$ might have had such values as to make the simultaneous elimination of all the singular terms impossible. The singularities would then have been inherent in the problem. That this is not the case shows that the singularities introduced by taking the energy tensor to have the form (12) even in the presence of point charges or point dipoles are entirely spurious, for they do not enter into the equations of motion. It would therefore be logical to seek to alter the expression for the energy tensor (12) when point charges and point dipoles are present so as to make the total energy of the field finite. This has already been done by Pryce for a point charge, and the results of this paper show that it must be possible for a point dipole also.

It only remains to determine $A_\mu^{(0)}$ and $B_{\lambda\mu}^{(0)}$. This is done in the appendix. The method is similar to the one we have used in deducing expressions (77) and (78), but is very much more complicated. $T_\mu^{(0)}(2)$ is given in the appendix and has four differentiations with respect to τ. $A_\mu^{(0)}(2)$ must therefore have three differentiations with respect to τ. The number of possible independent terms in $A_\mu^{(0)}(2)$ is therefore very large, and in fact there are fifty-seven terms in $A_\mu^{(0)}(2)$, and twenty-nine terms in $B_{\lambda\mu}^{(0)}(2)$. It is remarkable, however, that the conditions (79) and (69a) are so stringent as to determine the coefficients of all these terms either uniquely or in terms of six arbitrary constants. Thus $T_\mu^{\text{react.}}(2)$ and $M_{\lambda\mu}^{\text{react.}}(2)$ are determined in terms of six arbitrary constants. A considerable simplification can be introduced by giving some of these constants particular numerical values, but we do not need to go deeper into this point. $T_\mu^{\text{react.}}(2)$ contains all derivatives of v_μ and $S_{\lambda\mu}$ up to the fourth, and $M_{\lambda\mu}^{\text{react.}}(2)$ contains all derivatives of these quantities up to the third. Similarly, it is shown in the appendix that $T_\mu^{\text{react.}}(1)$ and $M_{\lambda\mu}^{\text{react.}}(1)$ can be determined entirely in terms of one arbitrary constant. $T_\mu^{\text{react.}}(1)$ and $M_{\lambda\mu}^{\text{react.}}(1)$ respectively contain all derivatives of v_μ and $S_{\lambda\mu}$ up to the third and second. $T_\mu^{\text{react.}}(0)$ is just the usual radiation reaction term for a point charge, while $M_{\lambda\mu}^{\text{react.}}(0)$ is zero. The complete expression for $M_{\lambda\mu}^{\text{react.}}$ is given in the appendix.

This completely determines the translational equation (72) and the rotational equation (73). Since, however, all derivatives of v_μ and $S_{\lambda\mu}$ up to the fourth appear in $T_\mu^{\text{react.}}$ and all derivatives up to the third in $M_{\lambda\mu}^{\text{react.}}$, the conditions under which the solutions of these equations are definite have to be investigated. If $\dot{v}_\mu$, $\ddot{v}_\mu$, v_μ^{iii}, $\dot{S}_{\lambda\mu}$ and $\ddot{S}_{\lambda\mu}$ vanish, then in the absence of an ingoing field all the terms in equation (73) vanish except $M_{\lambda\mu}^{\text{react.}}$, and from the expression for this given in the appendix we see that (73) reduces to

$$[S_{\lambda\rho}\{\tfrac{2}{3}S^{\text{iii}\,\rho}{}_{\mu}+\tfrac{4}{3}(v_\rho S'''_\mu - v_\mu S'''_\rho)\}] = 0. \tag{89}$$

Moreover, the second term of (7c) now also vanishes, and hence this equation demands that

$$(SS^{\text{iii}}) = 0. \tag{90}$$

From equations (89) and (90) it can be deduced that $S_{\lambda\mu}^{\text{iii}}$ must vanish. (The easiest way to do this is to introduce two space vectors† for S_{0k}, S_{0l}, S_{0m} and S_{lm}, S_{mk}, S_{kl}, and to consider equation (89) in the rest system.) Thus, as a consequence of the rotational equation (73), $\dot{S}_{\lambda\mu}$ and $\ddot{S}_{\lambda\mu}$ will continue to be zero, and hence all higher derivatives will also vanish, in particular $S_{\lambda\mu}^{\text{iv}}$. Now consider the translational equation (72). When $\dot{v}_\mu$, $\ddot{v}_\mu$, v_μ^{iii}, $\dot{S}_{\lambda\mu}$ and $\ddot{S}_{\lambda\mu}$ vanish, all the terms in (72) vanish in the absence of an ingoing field except $T_\mu^{\text{react.}}$, and hence this equation demands that $T_\mu^{\text{react.}}$ shall vanish. As shown above, it follows from the rotational equation that $S_{\lambda\mu}^{\text{iii}}$ and $S_{\lambda\mu}^{\text{iv}}$ must also vanish, and in these circumstances it follows from (138) in the appendix that $T_\mu^{\text{react.}} = 0$ reduces to

$$v_\mu^{\text{iv}}(\tfrac{2}{5}\not{S}^2 - \tfrac{1}{15}S^2) + \tfrac{2}{15}v_\mu(\not{S}Sv^{\text{iv}}) + \tfrac{2}{15}S_{\mu\nu}S^{\nu\rho}v_\rho^{\text{iv}} = 0. \tag{91}$$

Further, (5d) now becomes $(vv^{\text{iv}}) = 0$, so that in the rest system $v_0^{\text{iv}} = 0$. Thus in the rest system (91) just reduces to the three equations

$$v_k^{\text{iv}}(\tfrac{2}{5}\not{S}^2 - \tfrac{1}{15}S^2) + \tfrac{2}{15}S_{k\nu}S^{\nu l}v_l^{\text{iv}} = 0,$$

from which it can be deduced that $v_k^{\text{iv}} = 0$. Hence the particle will continue in a state of uniform motion with its spin pointing in a fixed direction. Thus the solutions of the equations of motion (72) and (73) will be perfectly definite if the initial velocity and direction of the spin are given, and only those solutions are allowed for which $\dot{v}_\mu$, $\ddot{v}_\mu$, v_μ^{iii}, $\dot{S}_{\lambda\mu}$ and $\ddot{S}_{\lambda\mu}$ all vanish after the ingoing field has died down. We may take (72) and (73) to be the exact equations of motion taking radiation reaction into account for a spinning particle moving in a Maxwell field.

† We make the convention that Latin suffices only take on the values 1, 2 and 3.

Specialized equations

Although the equations (72) and (73) give a consistent mathematical scheme for the motion of a spinning particle they are in some ways too general to be used for a description of the elementary particles which occur in nature. There is no connexion between the velocity of the particle and the state of its dipole moment. If we introduce a space vector **M** to denote the magnetic dipole moment S_{lm}, S_{mk}, S_{kl} and a space vector **D** to denote the electric dipole moment S_{0k}, S_{0l}, S_{0m}, the equations are such that, according to (6), (30) and (34*a*),

$$\mathbf{M}^2-\mathbf{D}^2 = \text{constant}, \tag{92a}$$

$$(\mathbf{MD}) = \text{constant}. \tag{92b}$$

Except for these two constants of the motion, the magnitudes of **M** and **D** may change without any relation to the velocity. For example, if we had a dipole of the special type which initially had only a magnetic moment but no electric moment in its rest system, an ingoing field could easily be found which after a time brought the dipole to rest again, but this time, without violating (92*a*) and (92*b*), with an electric moment perpendicular to the magnetic moment and a larger value of the magnetic moment. (This is particularly evident when $g_1 = 0$, for then a constant electric field would do this.) The elementary particles in nature do not behave in this way, and the question now arises as to whether it is possible to give a procedure for deriving specialized equations from (72) and (73) which leave a pure magnetic dipole always a pure magnetic dipole in the rest system and a pure electric dipole always a pure electric dipole.

Consider first the case where the dipole is a pure magnetic dipole in the rest system. This is expressed in mathematical form by the equation

$$S_{\mu\nu}v^\nu = 0. \tag{93}$$

The equations obtained by differentiating this are

$$S'_\mu + S_{\mu\nu}\dot{v}^\nu = 0, \tag{94a}$$

$$S''_\mu + 2\dot{S}_{\mu\nu}\dot{v}^\nu + S_{\mu\nu}\ddot{v}^\nu = 0, \tag{94b}$$

$$S'''_\mu + 3\ddot{S}_{\mu\nu}\dot{v}^\nu + 3\dot{S}_{\mu\nu}\ddot{v}^\nu + S_{\mu\nu}v^{\text{iii}\,\nu} = 0, \tag{94c}$$

$$S''''_\mu + 4S^{\text{iii}}_{\mu\nu}\dot{v}^\nu + 6\ddot{S}_{\mu\nu}\ddot{v}^\nu + 4\dot{S}_{\mu\nu}v^{\text{iii}\,\nu} + S_{\mu\nu}v^{\text{iv}\,\nu} = 0. \tag{94d}$$

Suppose the equations of motion as derived previously without the condition (93) are (43) and (64), which we write in the more general form (36):

$$\dot{A}_\mu = T_\mu, \tag{43}$$

$$I\dot{S}_{\lambda\mu} + I'\dot{S}^*_{\lambda\mu} = [S\,.\,\Theta]_{\lambda\mu}. \tag{36}$$

We have to find whether, when the constraint (93) is imposed, a procedure can be found within the method used for deducing (43) and (36) which allows (43) and (36) to be so altered as to make them consistent with the equations (93) and (94). The only condition that A_μ has to satisfy is that the two invariant equations (65) and (66) shall be consistent. In our present notation this requires that

$$v^\mu(T_\mu - \dot{A}_\mu) = \tfrac{1}{2}(\dot{S}_{\lambda\mu}\Theta^{\lambda\mu}). \tag{95}$$

Contract (36) with v^μ and replace A_μ by $A_\mu + A'_\mu$, where

$$A'_\lambda = \{I\dot{S}_{\lambda\mu} + I'\dot{S}^*_{\lambda\mu} - [S.\Theta]_{\lambda\mu}\}v^\mu. \tag{96}$$

An addition to A_μ of A'_μ adds according to (45) a term $v_\lambda A'_\mu - v_\mu A'_\lambda$ to the left-hand side of (36). Thus (36) is changed to†

$$I\{\dot{S}_{\lambda\mu} + v_\lambda S'_\mu - v_\mu S'_\lambda\} + I'\{\dot{S}^*_{\lambda\mu} + v_\lambda \dot{S}^*_{\mu\nu}v^\nu - v_\mu \dot{S}^*_{\lambda\nu}v^\nu\}$$
$$= [S_\lambda{}^\rho + \{\Theta_{\rho\mu} - (\Theta_{\rho\nu}v^\nu v_\mu - \Theta_{\mu\nu}v^\nu v_\rho)\}]_-. \tag{97}$$

In general (97) would not be permissible as the rotational equation because it has not the form (36). But in the special case when (93) and (94) are valid it can be written in the form

$$I\dot{S}_{\lambda\mu} + I[S_{\lambda\rho}(\dot{v}^\rho v_\mu - v^\rho \dot{v}_\mu)]_- + I'\{\dot{S}^*_{\lambda\mu} + v_\lambda \dot{S}^*_{\mu\nu}v^\nu - v_\mu \dot{S}^*_{\lambda\nu}v^\nu\}$$
$$= [S_\lambda{}^\rho\{\Theta_{\rho\mu} - (\Theta_{\rho\nu}v^\nu v_\mu - \Theta_{\mu\nu}v^\nu v_\rho)\}]_-. \tag{98}$$

This is of the required form (36) *provided* $I' = 0$. Thus (43) and (36) must be replaced by

$$\dot{A}_\mu + \frac{d}{d\tau}\{IS'_\mu - S_{\mu\rho}\Theta^{\rho\nu}v_\nu\} = T_\mu \tag{99}$$

and
$$I\{\dot{S}_{\lambda\mu} + v_\lambda S'_\mu - v_\mu S'_\lambda\} = [S_\lambda{}^\rho\{\Theta_{\rho\mu} - \Theta_{\rho\nu}v^\nu v_\mu\}]_-. \tag{100}$$

(98) with $I' = 0$ leads to the invariant equation

$$-2I\dot{v}^\rho \dot{S}_{\rho\mu}v^\mu + (\dot{S}\Theta) - 2S'^\rho\Theta_{\rho\nu}v^\nu = 0,$$

while (99) leads to

$$v^\mu(T_\mu - \dot{A}_\mu) - S'^\rho\Theta_{\rho\nu}v^\nu - Iv^\mu \dot{S}_{\mu\rho}\dot{v}^\rho = 0.$$

Since, by (94*a*),
$$\dot{v}^\rho \dot{S}_{\rho\mu}v^\mu = -\dot{v}^\rho S_{\rho\mu}\dot{v}^\mu = 0,$$

it follows in view of (95) that these two equations are consistent. Contracting (100) with v^μ, we see that it vanishes identically, so that *in the rest system* this equation determines the rotational motion only for $\lambda, \mu \neq 0$. In other

† The minus sign behind a bracket indicates that the same terms with ν and μ interchanged are to be subtracted.

words, it only determines $\dot{S}_{kl}$. In this system $\dot{S}_{0k}$ are then determined explicitly by (94 *a*). We have thus given a procedure for deriving from equations (43) and (36) others of the correct form which are consistent with the constraint (93) and satisfy the conservation laws.

Let us apply this method to the equations (72) and (73), with $K' = 0$ for simplicity. According to (99) the translational equation (72) is to be replaced by

$$M\dot{v}_\mu + \frac{d}{d\tau}\{IS'_\mu + \tfrac{1}{4}Kv_\mu \dot{S}^2 + KS_{\mu\rho}S''^\rho - \tfrac{1}{2}g_2 v_\mu (SF^{\text{in.}}) - g_2 S^\mu{}_\sigma F^{\text{in.}}_{\sigma\nu} v^\nu\}$$
$$= g_1 F^{\text{in.}}_{\mu\sigma} v^\sigma - \tfrac{1}{2}g_2 S^{\rho\sigma}\frac{\partial}{\partial x^\mu}F^{\text{in.}}_{\rho\sigma} + T^{\text{self}}_\mu, \tag{101}$$

where we have written

$$T^{\text{self}}_\mu \equiv T^{\text{react.}}_\mu + \frac{d}{d\tau}(S_\mu{}^\rho C^{\text{react.}}_{\rho\nu} v^\nu). \tag{102}$$

According to (100) the rotational equation (73) must be replaced by

$$I\{\dot{S}_{\lambda\mu} + v_\lambda S'_\mu - v_\mu S'_\lambda\} + K[S_\lambda{}^\rho\{\ddot{S}_{\rho\mu} - S''_\rho v^\mu\}]_-$$
$$= g_2[S_\lambda{}^\rho\{F^{\text{in.}}_{\rho\mu} - F^{\text{in.}}_{\rho\nu} v^\nu v_\mu\}]_- + [S.D]_{\lambda\mu}, \tag{103}$$

where
$$D_{\lambda\mu} \equiv C^{\text{react.}}_{\lambda\mu} - (C^{\text{react.}}_{\lambda\rho} v^\rho v_\mu - C^{\text{react.}}_{\mu\rho} v^\rho v_\lambda). \tag{104}$$

(101) and (103) are just the equations for this case derived in a previous paper (B, equations (30) and (31)) by a direct method.† The constants g_1 and K were there put equal to zero. It can easily be seen from the expression for $C^{\text{react.}}_{\lambda\mu}(1)$ given by (137 *b*) of the appendix that the $g_1 g_2$ terms vanish in the expression on the right-hand side of (104), so that $D_{\lambda\mu}$ contains only the pure spin reaction terms proportional to g_2^2. In other words, for a dipole which is a pure magnetic one in the rest system, the equations for the rotation of the dipole are the same whether the particle has a charge g_1 or not. Now, as mentioned before, $C^{\text{react.}}_{\lambda\mu}(2)$ given by (140) has six arbitrary constants. When we build the expression on the right-hand side of (104) and use the equations (93) and (94), three of the arbitrary constants drop out, while the remaining three always appear together in a certain combination, so that the resulting expression for $D_{\lambda\mu}$ has in effect but one arbitrary constant. $D_{\lambda\mu}$ is given in the appendix by (142) and is exactly the radiation reaction term given in the previous paper (B, (46)).

† Owing to a slip the g_2 terms in B appear with the same sign as here, whereas they should appear with the opposite sign since the field strengths as defined by (8 *a*) are equal to minus the field strengths in B. This is of no physical consequence, for it is merely equivalent to reversing the sign of g_2, which is always possible.

The radiation reaction terms in the translational equation (101) are given in the appendix. $T_\mu^{\text{self}}(0)$ is the same as $T_\mu^{\text{react.}}(0)$ by (102), since $C_{\lambda\mu}^{\text{react.}}(0)$ is zero. It is the well-known expression for a point charge. $T_\mu^{\text{self}}(1)$ has no arbitrary constant in it and is given by (143) in the appendix, while $T_\mu^{\text{self}}(2)$ can be expressed in terms of *one* arbitrary constant and is exactly the radiation reaction term given in the previous paper (B, appendix).

We thus see that, if the radiation reaction terms are neglected, the equations (101) and (103) are formally more complicated than (72) and (73) (with I' and K' equal to zero), but *with the radiation reaction* terms the equations (72) and (73) are vastly more complicated than (101) and (103). Moreover, whereas the damping terms in (101) and (103) have but *one* arbitrary dimensionless constant, those in (72) and (73) have no less than seven arbitrary constants. Thus, although the classical theory cannot exclude the general case treated in the previous section, it at least gives us a reason why the elementary particles in nature might be expected to belong to the specialized case treated in this section. Relativistic quantum theory is an advance on classical theory in that in it the elementary particles automatically and necessarily have only a magnetic moment in their rest system, as is found in nature.

Lastly, we consider the case where the dipole is a pure *electric* dipole in the rest system. This is expressed mathematically by the equation

$$S^*_{\mu\nu} v^\nu = 0. \tag{105}$$

To alter (43) and (36) so as to be compatible with (105) we proceed exactly as before and add (96) to A_μ. The rotational equation then becomes (97), and in order that this should be of the form (36) I must now be put equal to zero, while I' remains arbitrary. By using the identities (27) and (28*b*) and the equation (105) it can easily be shown that the right-hand side of (97) can be brought into the form of the right-hand side of (36), if desired. It is, however, convenient to keep it in the form which it has in (97), with $I = 0$.

The scattering of light by a dipole

To get an insight into the rotation of the dipole alone we may simplify the problem by putting M equal to infinity. In this case it follows from the translational equation (101) that all the derivatives of the velocity vanish, and we may conveniently consider the particle in the rest system. If we use the space vector $\mathbf{M}$ introduced at the beginning of the previous section to denote the spin, the rotational equation (103) then takes the particularly simple form

$$I\dot{\mathbf{M}} + K[\mathbf{M}\,.\,\dot{\mathbf{M}}] = g_2[\mathbf{M}\,.\,\mathbf{H}] - \tfrac{2}{3}g_2^2[\mathbf{M}\,.\,\mathbf{M}^{\text{111}}], \tag{106}$$

where the square brackets now denote the usual vector product, and **H** is the magnetic force of the ingoing field. If we put the constant K equal to zero we get just the equation given in the previous paper (B, (51)). As shown there, the last term in (106), which embodies the effects of radiation reaction can be derived quite simply from dimensional arguments and the condition that the work done by an external periodic force on the dipole shall be equal to the energy radiated by the dipole. The general reaction terms in (72) and (73) or (101) and (103) naturally cannot be derived so easily.

We now consider the scattering of light by this dipole. The calculation is but a generalization of the calculation given in the previous paper and is a particular case of the problem of the scattering of meson waves by a dipole dealt with in detail in the paper which immediately follows this. We therefore only give the result. We henceforth write x, y, z for the space coordinates and t for x_0. Let the ingoing light wave of frequency ω be described by

$$\mathbf{H} = \mathbf{H}_0 \cos \omega(z-t), \tag{107}$$

where $\mathbf{H}_0$ lies along the x-axis. We consider the scattering for weak fields, so that the oscillation of the dipole is small and we may write

$$\mathbf{M}(t) = \mathbf{M}_0 + \mathbf{M}_1 \sin \omega t + \mathbf{M}_2 \sin(\omega t + \delta),$$

$\mathbf{M}_0$ being the initial value of **M**, the length of which may be taken to be unity. Denote by θ the angle between $\mathbf{M}_0$ and **H**. The vectors $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{M}_2$ are mutually perpendicular and such that $[\mathbf{M}_0 . \mathbf{M}_1]$ is in the direction $\mathbf{M}_2$. Let η denote the angle between the vectors $[\mathbf{M}_0 . \mathbf{H}]$ and $\mathbf{M}_1$, $\mathbf{M}_1$ lying in such a direction that the angle† between the planes $[\mathbf{M}_0 . \mathbf{M}_1]$ and $[\mathbf{M}_0 . \mathbf{H}]$ is $\frac{1}{2}\pi - \eta$. If we write for brevity

$$\alpha = \frac{3}{2}\frac{I}{g_2^2}, \quad \beta = \frac{3}{2}\frac{K}{g_2^2}, \tag{108}$$

it can be shown that

$$|M_1| = \frac{3}{2} \frac{H_0 \alpha \sin\theta}{g_2^2 \omega\{(\alpha^2 - \beta^2\omega^2 - \omega^4)^2 + 4\alpha^2\omega^2\}^{\frac{1}{2}}}, \tag{109a}$$

$$\left|\frac{M_2}{M_1}\right| = (\beta^2\omega^2 + \omega^4)^{\frac{1}{2}} \alpha^{-1}, \tag{109b}$$

$$\tan \eta = \frac{2\alpha\omega^2}{\alpha^2 - \beta^2\omega^2 - \omega^4}, \tag{109c}$$

$$\tan \delta = -\frac{\beta}{\omega}. \tag{109d}$$

† A slight slip has occurred in B. With the angle ϕ as defined there, the sign of the right-hand side of (55b) should be reversed.

Thus, as we should expect, the dipole does not carry out a pure oscillation as in the case investigated in B, but, owing to the presence of the inertial term K, it executes a small elliptic gyration about its original direction.

The scattering cross-section, i.e. the energy scattered divided by the energy carried by the incident light wave per unit area, is

$$6\pi \sin^2\theta \frac{\omega^2(\alpha^2+\beta^2\omega^2+\omega^4)}{(\alpha^2-\beta^2\omega^2-\omega^4)^2+4\alpha^2\omega^4}. \tag{110}$$

For $\beta = 0$ this just goes over into the cross-section given in B, namely,

$$6\pi \sin^2\theta \frac{\omega^2}{\alpha^2+\omega^4}, \tag{111}$$

while for $\alpha = 0$ it reduces to

$$6\pi \sin^2\theta \frac{1}{\beta^2+\omega^2}. \tag{112}$$

The effect of radiation reaction is contained in the *explicit* ω^4 terms in (110). Their effect is twofold. For very high frequencies they cause the scattering to diminish as ω^{-2} quite irrespective of the values of α or β. Moreover, if we neglect these reaction terms, the denominator of (110) becomes zero for a frequency

$$\omega = \frac{\alpha}{\beta} = \frac{I}{K},$$

which gives an infinite scattering for this frequency. As is well known, I/K is the natural precession frequency of the spin for small oscillations, so that we have to do here with a resonance phenomenon. The effect of radiation reaction is to make the scattering finite even for this resonance frequency, as we should expect. It is interesting to note that while the scattering due to a pure gyroscopic spin ($K = 0$) tends to infinity like ω^2 with increasing frequency, the addition of a finite moment of inertia perpendicular to the spin axis ($K \neq 0$) makes the scattering cross-section tend to a finite value for high frequencies, *even in the absence of radiation reaction.* That the effect of radiation reaction on the *rotation* of the dipole is to make the scattering diminish for very high frequencies as ω^{-2}, bears a striking resemblance to the scattering by a point charge calculated by Dirac (1938), the cross-section for which has exactly the dependence on frequency given by (112), where the effect of radiation reaction on the *translation* of the point charge is also to make the scattering decrease as ω^{-2} for high frequencies. Since a point charge and a point dipole are entirely different things and the mechanism of scattering is also different, this leads us to suspect that it may be a fundamental property of radiation that for high frequencies the scattering should decrease as ω^{-2}.

It is of interest to compare our classical theory with the quantum theory for a particle of spin $\frac{1}{2}\hbar$ as described by the Dirac equation. In this quantum theory it is possible to add in addition to the usual interaction which is described by means of the potentials, an explicit spin interaction of the particle with the Maxwell field equivalent to the g_2 term. The calculation of the scattering of light by a particle with this explicit spin interaction, leads, if the mass of the particle is ultimately allowed to tend to infinity, to a scattering cross-section of the form (111) with the absence of the ω^4 term in the denominator. The absence of the ω^4 term is understandable, for it is the result of radiation reaction, and as is well known, this is neglected in the quantum theory. However, the fact that the quantum theory then leads to a cross-section agreeing in form with (111) and not (110), with the *explicit* ω^4 terms omitted in each case, shows clearly that the Dirac equation automatically describes a particle for which β is zero, that is, for which K vanishes. Thus the Dirac equation automatically describes a spinning particle which has the simplest possible mechanical properties, namely those of a pure gyroscope, characterized by $I = \frac{1}{2}\hbar$, $K = 0$. Moreover, the cross-section (111), which correctly contains the effect of radiation reaction, shows that the validity of the quantum theory of an explicit spin interaction would be restricted to frequencies for which the ω^4 term is unimportant, i.e., for frequencies

$$\omega \ll \sqrt{\alpha} = \sqrt{\frac{3I}{2g_2^2}}.$$

If the g_2 term in the quantum theory is put equal to zero, the scattering reduces to just that given by the Klein-Nishina formula and vanishes as the mass of the particle tends to infinity.

The question now arises as to whether the above theory can be applied in the classical limit to an electron or not, for as is well known, the magnetic moment of the electron is a pure quantum effect. The argument of the previous paragraph shows that the magnetic moment of the electron is not to be described in this way for that *would be equivalent to describing the magnetic moment in the quantum theory by an explicit spin interaction term with* $g_2 = e\hbar/2m$. This would not give agreement with nature, as can be seen by comparing the theoretical scattering for an explicit g_2 term with that found experimentally for free electrons. It is justifiable to calculate the theoretical scattering by using (111), for, as has been mentioned before, this completely agrees for frequencies $\omega \ll \sqrt{\alpha}$ with the quantum formula derived with a g_2 term.† The fact that the electron was considered as fixed at a point in calculating (111) is of no account since the scattering due to the rotation of the spin with an explicit g_2 term is far greater than that due to the translation of the dipole. Putting $I = \frac{1}{2}\hbar$, $K = 0$, $g_2 = e\hbar/2m$, thus making $\alpha = 3(\hbar/e^2)(m/\hbar)^2$, we find that the scattering given by (111) for $\omega = 5m/\hbar$ is already about twenty times larger than the scattering given by the Klein-Nishina formula for the same frequency. Since the Klein-Nishina formula has been checked experimentally up to these frequencies (see for example, Heitler 1936), we must conclude that the above scattering cross-section is not applicable to an electron. This is entirely to be attributed to the fact that *to describe the electron and its interaction with the Maxwell field as it occurs in nature we must put $g_2 = 0$ both in the classical and quantum theories.* The equations of the preceding section are naturally applicable to an electron if we specialize them by putting $g_2 = 0$, $K = 0$, $I = \frac{1}{2}\hbar$, in which case (103) shows that the spin continues to point in the same direction in the rest system, while in the limit $\hbar \to 0$ (101) just becomes the well-known Lorentz-Dirac equation. The quantum theory of the electron might therefore be expected to be valid up to energies of $137mc^2$ as hitherto supposed, and contrary to a tentative suggestion recently made by one of us (Bhabha 1940*b*).

For the meson it is not yet known from experiment whether an explicit spin interaction term is necessary to describe its interaction with the Maxwell field or not. If it is found that such a term is required in the quantum description of the meson's interaction with the Maxwell field, then the above theory would certainly be applicable to this case in the classical limit, and (111) shows that the neglect of the effect of radiation reaction on the rotation of the spin would restrict the validity of quantum theory to frequencies such that‡

$$\omega \ll \sqrt{\frac{3I}{2g_2^2}} = \sqrt{\frac{3\hbar}{2g_2^2}}.$$

It is known that the heavy particles have an explicit spin interaction with the meson field, so that the theory of this paper would certainly describe

† Detailed calculations by Bhabha and Madhava Rao (1941, *Proc. Indian Acad. Sci.* A, **13**, 9–24) have shown that the quantum and classical cross-sections have the same dependence on energy, scattering angle and polarization of the incident and scattered light, but that the former is larger than the latter by a constant factor 3. This factor results from physically understandable differences in the quantum and classical averaging over the initial orientations of the spin of the scattering particle (see footnote on p. 341 of the subsequent paper).

‡ Here g^2 denotes the strength of a possible explicit interaction of the meson with the Maxwell field, and not as usual the strength of the spin interaction of the heavy particles with the meson field.

their behaviour in the classical limit, but we shall not consider this problem here as it is treated in detail in the paper which immediately follows this, where the present theory is extended to cover meson fields.

Appendix

The field strengths can be derived according to ($8a$) by differentiating the potentials (21) and (23). The method of differentiation has been given in a previous paper (A). We give only the result:†

$$F^{(1)}_{\mu\nu} = g_1\left[\frac{s_\mu v_\nu}{\kappa^3}, \quad -\frac{s_\mu v_\nu}{\kappa^3}\kappa' + \frac{s_\mu \dot{v}_\nu}{\kappa^2}\right]_- . \tag{113}$$

This is the usual field of a point charge. Further

$$\begin{aligned} F^{(2)}_{\mu\nu} = g_2\Bigg[& \frac{S_{\mu\nu}}{\kappa^3} + 3\frac{s_\mu \not{S}_\nu}{\kappa^4} - 2\frac{v_\mu \not{S}_\nu}{\kappa^3} + 3\frac{s^\rho S_{\rho\nu}}{\kappa^5}(s_\mu - \kappa v_\mu), \\ & + \frac{\dot{S}_{\mu\nu}}{\kappa^2} - \frac{S_{\mu\nu}}{\kappa^3}\kappa' - 3\frac{s_\mu \not{S}_\nu}{\kappa^4}\kappa' + 2\frac{s_\mu S'_\nu}{\kappa^3} + \frac{s_\mu S_{\nu\rho}\dot{v}^\rho}{\kappa^3} - \dot{v}_\mu \frac{s^\rho S_{\rho\nu}}{\kappa^3} \\ & + 3\frac{s_\mu s^\rho \dot{S}_{\rho\nu}}{\kappa^4} - 2\frac{v_\mu s^\rho S_{\rho\nu}}{\kappa^3} - 6\frac{s_\mu s^\rho S_{\rho\nu}}{\kappa^5}\kappa' + 3\frac{v_\mu s^\rho S_{\rho\nu}}{\kappa^4}\kappa', \\ & + \frac{s_\mu s^\rho \ddot{S}_{\rho\nu}}{\kappa^3} - 3\frac{s_\mu s^\rho \dot{S}_{\rho\nu}}{\kappa^4}\kappa' + 3\frac{s_\mu s^\rho S_{\rho\nu}}{\kappa^5}\kappa'^2 - \frac{s_\mu s^\rho S_{\rho\nu}}{\kappa^4}\kappa''\Bigg]_- . \end{aligned} \tag{114}$$

We proceed to calculate T_μ and $M_{\lambda\mu}$. For the world tube defined by (40) it was shown in A (formula (54)) that the directed surface element dS^ν of the tube is given by

$$dS^\nu = \{s^\nu(1-\kappa') - \epsilon v^\nu\}\,\epsilon\, d\Omega\, d\tau. \tag{115}$$

Here $d\Omega$ represents an element of solid angle subtended at the point τ of the world line in the rest system of this point by a portion of the sphere of radius ϵ with this point as centre. As mentioned in the text, this sphere taken at a time ϵ later than τ is a section of the world tube. For all points on this sphere, $s_0 = \epsilon$, while $\sum_1^3 s_k^2 = \epsilon^2$. Hence, if A_μ is any vector which is not a function of position on the surface of the sphere,

$$\frac{1}{4\pi\epsilon}\int s^\lambda A_\lambda\, d\Omega = A_0 = v^\lambda A_\lambda, \tag{116a}$$

† The minus sign behind a bracket indicates that the same terms with ν and μ interchanged are to be subtracted.

since in the rest system $v_0 = 1$, $v_1 = v_2 = v_3 = 0$. Similarly, if B_μ is another vector which is also not a function of position on the surface of the sphere,

$$\frac{1}{4\pi\epsilon^2}\int s^\lambda A_\lambda s^\mu B_\mu d\Omega = A_0 B_0 + \tfrac{1}{3}\sum_1^3 A_k B_k.$$

We can at once write this in tensor form by combining the terms with v_μ; thus

$$\frac{1}{4\pi\epsilon^2}\int (sA)(sB)\,d\Omega = -\tfrac{1}{3}(A_\lambda B^\lambda) + \tfrac{4}{3}(v^\lambda A_\lambda)(v^\mu B_\mu). \tag{116b}$$

Proceeding in this way we can easily show that

$$\frac{1}{4\pi\epsilon^3}\int (sA)(sB)(sC)\,d\Omega = -\tfrac{1}{3}P(AB)(Cv) + \tfrac{6}{3}(Av)(Bv)(Cv). \tag{116c}$$

The symbol P denotes that we have to sum over all possible combinations of the vectors occurring in the product, each combination being taken only once; thus

$$P(AB)(Cv) = (AB)(Cv) + (BC)(Av) + (CA)(Bv).$$

The generalization of the formulae (116) is easily found. In these calculations only integrals involving products of not more than eight vectors A, B, C, D, E, F, G and H appear. We have

$$\frac{1}{4\pi\epsilon^4}\int (sA)(sB)(sC)(sD)\,d\Omega = \frac{1}{3.5}P(AB)(CD) - \frac{6}{3.5}P(AB)(Cv)(Dv) + \frac{6.8}{3.5}(Av)(Bv)(Cv)(Dv). \tag{116d}$$

$$\frac{1}{4\pi\epsilon^5}\int (sA)\ldots(sE)\,d\Omega = \frac{1}{3.5}P(AB)(CD)(Ev) - \frac{8}{3.5}P(AB)(Cv)(Dv)(Ev) + \frac{8.10}{3.5}(Av)\ldots(Ev), \tag{116e}$$

$$\frac{1}{4\pi\epsilon^6}\int (sA)\ldots(sF)\,d\Omega = -\frac{1}{3.5.7}P(AB)(CD)(EF) + \frac{8}{3.5.7}P(AB)(CD)(Ev)(Fv) - \frac{8.10}{3.5.7}P(AB)(Cv)\ldots(Fv) + \frac{8.10.12}{3.5.7}(Av)\ldots(Fv), \tag{116f}$$

$$\frac{1}{4\pi\epsilon^7}\int (sA)\dots(sG)\,d\Omega = -\frac{1}{3.5.7}P(AB)(CD)(EF)(Gv)$$
$$+\frac{10}{3.5.7}P(AB)(CD)(Ev)\dots(Gv)$$
$$-\frac{10.12}{3.5.7}P(AB)(Cv)\dots(Gv)+\frac{10.12.14}{3.5.7}(Av)\dots(Gv), \quad (116g)$$

$$\frac{1}{4\pi\epsilon^8}\int (sA)\dots(sH)\,d\Omega = \frac{1}{3.5.7.9}P(AB)(CD)(EF)(GH)$$
$$-\frac{10}{3.5.7.9}P(AB)(CD)(EF)(Gv)(Hv)$$
$$+\frac{10.12}{3.5.7.9}P(AB)(CD)(Ev)\dots(Hv)-\frac{10.12.14}{3.5.7.9}P(AB)(Cv)\dots(Hv)$$
$$+\frac{10.12.14.16}{3.5.7.9}(Av)\dots(Hv). \quad (116h)$$

Introducing (12) and (115) into (41) and remembering that this is equal to (42), we find

$$T_\mu = \frac{1}{4\pi}\int \{F_{\mu}{}^{\rho}F_{\rho\nu}+\tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\}\{s^\nu(1-\kappa')-\epsilon v^\nu\}\,\epsilon\, d\Omega. \quad (117)$$

Consider first $T_\mu^{\text{mix.}}$. Remembering (25), we have

$$T_\mu^{\text{mix.}} = \frac{1}{4\pi}\int \{F^{\text{in.}}_{\mu\rho}F^{\text{ret.}\,\rho}{}_{\nu}+F^{\text{ret.}}_{\mu\rho}F^{\text{in.}\,\rho}{}_{\nu}+\tfrac{1}{2}g_{\mu\nu}F^{\text{in.}}_{\rho\sigma}F^{\text{in.}\,\rho\sigma}\}\{s^\nu(1-\kappa')-\epsilon v^\nu\}\,\epsilon\, d\Omega. \quad (118)$$

Now the ingoing field has no singularity on the world line. Hence its value at any point x_ρ on the surface of the sphere can be expressed by the help of Taylor's theorem in terms of its value at the retarded point $z_\rho(\tau_0)$; thus

$$(F^{\text{in.}}_{\mu\nu})_{x_\rho} = (F^{\text{in.}}_{\mu\nu})_{z_\rho(\tau_0)} + s^\rho\left(\frac{\partial}{\partial x^\rho}F^{\text{in.}}_{\mu\nu}\right)_{z_\rho(\tau_0)}$$

Now dS^ν is at least of order ϵ^2, while the highest singularity in $F^{(1)}_{\mu\nu}$ given by (113) is of order ϵ^{-2}. Hence the g_1 terms in $T_\mu^{\text{mix.}}$ will contain the ingoing field strengths but not their derivatives. The highest singularity in $F^{(2)}_{\mu\nu}$ is of order ϵ^{-3}, and hence the g_2 terms in $T_\mu^{\text{mix.}}$ will contain both the ingoing field strengths and their derivatives on the world line. At first sight it would seem possible for singular terms of order ϵ^{-1} to appear in $T_\mu^{\text{mix.}}$, but calculation shows that these vanish, as we should expect. Introducing (114) for $F^{\text{ret.}}_{\mu\nu}$

in (118) and using some of the relations (116), we find after some calculation that the g_2 terms in $T_\mu^{\text{mix.}}$ reduce to

$$g_2\left[\frac{d}{d\tau}\{\tfrac{2}{3}S_\mu{}^\sigma F^{\text{in.}}_{\sigma\nu}v^\nu - \tfrac{2}{3}F^{\text{in.}}_{\mu\sigma}\not{S}^\sigma + \tfrac{1}{3}v_\mu S^{\rho\sigma}F^{\text{in.}}_{\rho\sigma}\}\right.$$
$$\left. - \tfrac{1}{2}S^{\rho\sigma}\frac{\partial}{\partial x^\mu}F^{\text{in.}}_{\rho\sigma} + \{\tfrac{2}{3}S_{\mu\sigma} + \tfrac{1}{3}v_\mu\not{S}_\sigma - \tfrac{1}{3}v_\sigma\not{S}_\mu\}\frac{\partial}{\partial x^\nu}F^{\text{in.}\nu\sigma}\right]. \tag{119}$$

Use has been made of the fact that the ingoing field satisfies (9). The last term vanishes since the ingoing field satisfies (8*b*) with the right-hand side equal to zero. We thus get the result quoted in (58) in the text. The g_1 term there is the usual one which was also calculated in *A*.

Inserting (12) and (115) into the first integral of (44*a*) and remembering that this is equal to (44*b*), we find

$$M_{\lambda\mu} = \frac{1}{4\pi}\int [s_\lambda\{F_{\mu\rho}F^\rho{}_\nu + \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\}\{s^\nu(1-\kappa') - \epsilon v^\nu\}]_{-}\epsilon\, d\Omega. \tag{120}$$

$M^{\text{mix.}}_{\lambda\mu}$ therefore differs from (119) in containing an extra s_λ in the integrand, the same terms with λ and μ interchanged being then subtracted. The result of this extra s_λ is that there will be no g_1 terms in $M^{\text{mix.}}_{\lambda\mu}$, while the g_2 terms will contain the ingoing field but not its derivatives. After some easy calculation we get the result given in the text by (59).

We now come to the radiation reaction terms. In the notation of (80),

$$T^{(\epsilon)}_\mu = g_1^2 T^{(\epsilon)}_\mu(0) + g_1 g_2 T^{(\epsilon)}_\mu(1) + g_2^2 T^{(\epsilon)}_\mu(2) \tag{121}$$

and

$$T^{(0)}_\mu = g_1^2 T^{(0)}_\mu(0) + g_1 g_2 T^{(0)}_\mu(1) + g_2^2 T^{(0)}_\mu(2). \tag{122}$$

The terms in g_1^2 are the well-known reaction terms for a pure point charge and have been calculated in A. It was shown there that

$$T^{(\epsilon)}_\mu(0) = -\frac{1}{2}\frac{\dot{v}_\mu}{\epsilon}, \tag{123}$$

$$T^{(0)}_\mu(0) = \tfrac{2}{3}v_\mu \dot{v}^2. \tag{124}$$

The terms in $g_1 g_2$ can be calculated in a similar manner. Introducing (113) and (114) into (117) and using the relations (116), we find after some calculation that

$$T^{(\epsilon)}_\mu(1) = \frac{1}{\epsilon}\frac{d}{d\tau}[\tfrac{1}{3}v_\mu(\not{S}\dot{v}) + \tfrac{2}{3}S'_\mu + \tfrac{1}{3}S_{\mu\sigma}\dot{v}^\sigma], \tag{125}$$

and

$$T^{(0)}_\mu(1) = \frac{d}{d\tau}[\tfrac{1}{15}\not{S}_\mu\dot{v}^2 - \tfrac{2}{3}v_\mu(S'\dot{v}) - \tfrac{8}{15}\dot{v}_\mu(\not{S}\dot{v})]$$
$$+ \tfrac{1}{3}(S'_\mu - S_{\mu\sigma}\dot{v}^\sigma)\dot{v}^2 - \tfrac{2}{3}\ddot{S}_{\mu\sigma}\dot{v}^\sigma + \tfrac{2}{3}\dot{v}_\mu(\not{S}\ddot{v})$$
$$+ v_\mu\{\tfrac{2}{3}(S'\ddot{v}) + \tfrac{2}{3}(\ddot{v}S\dot{v}) - \tfrac{4}{3}(S''\dot{v}) - 2(\not{S}\dot{v})\dot{v}^2\}. \tag{126}$$

We finally come to the g_2^2 terms. The calculation of $T_\mu^{\text{ret.}}(2)$ is carried out in exactly the same way. (114) is introduced into (117) and then the relations (116) are used to calculate each integral. Since $F_{\mu\nu}^{(2)}$ given by (114) contains no less than 18 terms, $T_{\mu\nu}$ which is quadratic in $F_{\mu\nu}$ contains some 324 terms. Some of these, of course, vanish at once from symmetry, or due to (18), but nevertheless the calculation is very lengthy and tedious. We have not found a way of shortening it. It is interesting to notice that with the expression (114) for $F_{\mu\nu}^{(2)}$, there are no terms of order ϵ^{-3} and ϵ^{-2} in $F_{\rho\sigma}^{(2)}F^{(2)\rho\sigma}$, so that the contribution to $T_\mu^{(0)}(2)$ comes only from the first product in (117). Moreover, $F_{\rho\sigma}^{(2)}F^{(2)\rho\sigma}$ contributes nothing to $M_{\lambda\mu}^{(0)}(2)$ since the only other terms which could do this are those of order ϵ^{-4}, and (120) shows that these are multiplied by the factor $(s_\lambda s_\mu - s_\mu s_\lambda)\kappa'$ which vanishes. We find

$$\begin{aligned}
T_\mu^{(\epsilon)}(2) = \frac{d}{d\tau}\Bigg[&\frac{1}{\epsilon^3}\{-\tfrac{1}{6}v_\mu S^2+\tfrac{1}{3}v_\mu \not{S}^2-\tfrac{1}{3}S_{\mu\sigma}\not{S}^\sigma\} \\
&+\frac{1}{\epsilon^2}\{\tfrac{2}{3}v_\mu(\not{S}S')+\tfrac{2}{5}v_\mu(\not{S}S\dot{v})+\tfrac{8}{15}\dot{v}_\mu\not{S}^2-\tfrac{1}{5}\dot{v}_\mu S^2-\tfrac{1}{3}S_{\mu\sigma}S'^\sigma \\
&-\tfrac{4}{15}S_{\mu\sigma}S^{\sigma\nu}\dot{v}_\nu-\tfrac{1}{3}\dot{S}_{\mu\sigma}\not{S}^\sigma\}+\frac{1}{\epsilon}\{v_\mu[-\dot{v}^2\not{S}^2+\tfrac{1}{5}\dot{v}^2S^2 \\
&-\tfrac{1}{3}\dot{S}^2+\tfrac{4}{3}S'^2+\tfrac{4}{15}(S'S\dot{v})-\tfrac{1}{15}(\dot{v}SS\dot{v})+\tfrac{14}{15}(\not{S}\dot{S}\dot{v})] \\
&+\dot{v}_\mu[\tfrac{14}{15}(\not{S}S')+\tfrac{4}{15}(\not{S}S\dot{v})]+\tfrac{4}{15}S_{\mu\sigma}\dot{S}^{\sigma\nu}\dot{v}_\nu-\tfrac{2}{15}S_{\mu\sigma}\not{S}^\sigma\dot{v}^2 \\
&-\tfrac{2}{5}\dot{S}_{\mu\sigma}S^{\sigma\nu}\dot{v}_\nu+\tfrac{2}{3}\not{S}_\mu(S'\dot{v})-\tfrac{2}{3}S'_\mu(\not{S}\dot{v})\}\Bigg],
\end{aligned} \tag{127}$$

$$\begin{aligned}
T_\mu^{(0)}(2) = \frac{d}{d\tau}[&v_\mu\{\tfrac{12}{5}(S'\dot{S}\dot{v})-\tfrac{32}{35}(\not{S}S\dot{v})\dot{v}^2-\tfrac{2}{5}(\dot{v}S\dot{S}\dot{v})\} \\
&+\dot{v}_\mu\{-\tfrac{16}{35}\not{S}^2\dot{v}^2+\tfrac{2}{5}(\not{S}\dot{S}\dot{v})+\tfrac{2}{35}S^2\dot{v}^2-\tfrac{4}{35}(\dot{v}SS\dot{v})+\tfrac{2}{15}\dot{S}^2 \\
&+\tfrac{4}{5}(\not{S}S'')+\tfrac{2}{5}(S'S\dot{v})\}+\tfrac{2}{5}\dot{S}_{\mu\sigma}\dot{S}^{\sigma\nu}\dot{v}_\nu-\tfrac{4}{35}S_{\mu\sigma}S^{\sigma\nu}\dot{v}_\nu\dot{v}^2] \\
&+v_\mu\{-\tfrac{1}{3}\ddot{S}^2+2S''^2+\tfrac{4}{5}(S''S\ddot{v})-\tfrac{2}{5}(\dot{v}S\ddot{S}\dot{v})+\tfrac{4}{5}(\not{S}\ddot{S}\ddot{v}) \\
&+\tfrac{16}{5}(\not{S}S'')\dot{v}^2+\dot{S}^2\dot{v}^2-\tfrac{12}{5}(S'\dot{S}\ddot{v})+\tfrac{8}{5}(\dot{v}\dot{S}\dot{S}\dot{v})-\tfrac{24}{5}S'^2\dot{v}^2-\tfrac{2}{15}(\ddot{v}SS\ddot{v}) \\
&-\tfrac{16}{5}(\not{S}S')(\dot{v}\ddot{v})+\tfrac{1}{15}S^2\ddot{v}^2+\tfrac{32}{15}(\not{S}S\ddot{v})\dot{v}^2-\tfrac{1}{3}S^2\dot{v}^4-\tfrac{8}{15}\not{S}^2\ddot{v}^2 \\
&-\tfrac{4}{5}(\dot{v}SS\dot{v})\dot{v}^2+\tfrac{10}{3}\not{S}^2\dot{v}^4\}+\dot{v}_\mu\{-\tfrac{2}{3}(\dot{S}\ddot{S})+\tfrac{8}{5}(S'S'')-\tfrac{2}{5}(S''S\dot{v}) \\
&-\tfrac{2}{5}(\not{S}\ddot{S}\dot{v})-\tfrac{4}{5}(S'\dot{S}\dot{v})+\tfrac{2}{5}(\dot{v}S\dot{S}\dot{v})+\tfrac{8}{5}(\not{S}S\dot{v})\dot{v}^2-\tfrac{4}{5}(\not{S}S''')\} \\
&+\ddot{v}_\mu\{-\tfrac{2}{15}S^2\dot{v}^2+\tfrac{4}{15}(\not{S}S\ddot{v})+\tfrac{16}{15}\not{S}^2\dot{v}^2\}+\tfrac{2}{15}S_{\mu\sigma}\ddot{S}^{\sigma\nu}\ddot{v}_\nu+\tfrac{2}{5}S_{\mu\nu}S''^\nu\dot{v}^2 \\
&-\tfrac{2}{5}S_{\mu\nu}S'^\nu(\dot{v}\ddot{v})-\tfrac{2}{15}S_{\mu\nu}\not{S}^\nu\ddot{v}^2+\tfrac{4}{15}S_{\mu\sigma}S^{\sigma\nu}\ddot{v}_\nu\dot{v}^2+\tfrac{2}{3}\dot{S}_{\mu\nu}\not{S}^\nu\dot{v}^4-\tfrac{6}{5}\dot{S}_{\mu\nu}S'^\nu\dot{v}^2 \\
&-\tfrac{2}{5}\dot{S}_{\mu\nu}\not{S}^\nu(\dot{v}\ddot{v})-\tfrac{2}{5}\dot{S}_{\mu\nu}\dot{S}^{\nu\rho}\ddot{v}_\rho+\tfrac{2}{3}\ddot{S}_{\mu\nu}S''^\nu+\tfrac{2}{15}\ddot{S}_{\mu\sigma}S^{\sigma\rho}\ddot{v}_\rho+\tfrac{2}{5}\ddot{S}_{\mu\nu}\not{S}^\nu\dot{v}^2.
\end{aligned} \tag{128}$$

The calculation of $M^{\text{ret.}}_{\lambda\mu}$ can be carried out in exactly the same way by introducing (113) and (114) into (120). We give only the result. As before (Bhabha 1939*b*), $M^{(\epsilon)}_{\lambda\mu}(0) = 0$, while

$$M^{(0)}_{\lambda\mu}(0) = -\tfrac{2}{3}(v_\lambda \dot{v}_\mu - v_\mu \dot{v}_\lambda), \tag{129}$$

$$M^{(\epsilon)}_{\lambda\mu}(1) = [\tfrac{1}{2}v_\lambda(\tfrac{2}{3}S'_\mu + \tfrac{1}{3}S_{\mu\sigma}\dot{v}^\sigma) + \tfrac{1}{3}\dot{S}_{\lambda\mu}]_-, \tag{130}$$

$$\begin{aligned} M^{(0)}_{\lambda\mu}(1) = {} & \frac{d}{d\tau}[\tfrac{1}{3}\dot{S}_{\lambda\mu} + \tfrac{1}{3}\dot{v}_\lambda \mathcal{S}_\mu - \tfrac{1}{3}v_\lambda S_{\mu\sigma}\ddot{v}^\sigma]_- + [v_\lambda\{\tfrac{1}{15}\mathcal{S}_\mu \dot{v}^2 - \tfrac{8}{15}\dot{v}_\mu(\mathcal{S}\dot{v})\}]_- \\ & + [v_\lambda\{2\dot{S}_{\mu\sigma}\ddot{v}^\sigma + \tfrac{4}{3}S''_\mu + \tfrac{2}{3}S_{\mu\sigma}\dddot{v}^\sigma + \tfrac{1}{3}\mathcal{S}_\mu \ddot{v}^2 + \tfrac{4}{3}\dot{v}_\mu(\mathcal{S}\ddot{v})\}]_-. \end{aligned} \tag{131}$$

Finally,

$$\begin{aligned} M^{(\epsilon)}_{\lambda\mu}(2) = \Bigg[& \frac{1}{\epsilon^3}(-\tfrac{1}{3}v_\lambda S_{\mu\sigma}\mathcal{S}^\sigma) + \frac{1}{\epsilon^2}\Big\{\tfrac{8}{15}v_\lambda \dot{v}_\mu \mathcal{S}^2 - \tfrac{1}{5}v_\lambda \dot{v}_\mu S^2 \\ & - \tfrac{1}{3}v_\lambda S_{\mu\sigma}S'^\sigma - \tfrac{4}{15}v_\lambda S_{\mu\sigma}S^{\sigma\nu}\dot{v}_\nu - \tfrac{1}{3}v_\lambda \dot{S}_{\mu\sigma}\mathcal{S}^\sigma) - \tfrac{1}{3}\frac{d}{d\tau}(v_\lambda S_{\mu\sigma}\mathcal{S}^\sigma)\Big\} \\ & + \frac{1}{\epsilon}\Big\{v_\lambda[\tfrac{14}{15}\dot{v}_\mu(\mathcal{S}S') + \tfrac{4}{15}\dot{v}_\mu(\mathcal{S}S\dot{v}) + \tfrac{4}{15}S_{\mu\sigma}\dot{S}^{\sigma\nu}\dot{v}_\nu - \tfrac{2}{15}S_{\mu\sigma}\mathcal{S}^\sigma \dot{v}^2 \\ & - \tfrac{2}{5}\dot{S}_{\mu\sigma}S^{\sigma\nu}\dot{v}_\nu + \tfrac{2}{3}\mathcal{S}_\mu(S'\dot{v}) - \tfrac{2}{3}S'_\mu(\mathcal{S}\dot{v})] + \frac{d}{d\tau}[-\tfrac{2}{15}v_\lambda \dot{v}_\mu S^2 \\ & + \tfrac{4}{5}v_\lambda \dot{v}_\mu \mathcal{S}^2 - \tfrac{4}{3}\mathcal{S}_\lambda S'_\mu - \tfrac{2}{3}S_{\lambda\sigma}\dot{S}^\sigma{}_\mu - \tfrac{4}{3}v_\lambda S_{\mu\sigma}S'^\sigma \\ & - \tfrac{2}{5}v_\lambda S_{\mu\sigma}S^{\sigma\nu}\dot{v}_\nu - \tfrac{2}{3}\dot{v}_\lambda S_{\mu\sigma}\mathcal{S}^\sigma]\Big\}\Bigg]_-, \end{aligned} \tag{132}$$

$$\begin{aligned} M^{(0)}_{\lambda\mu}(2) = {} & \frac{d}{d\tau}[v_\lambda\{\tfrac{4}{5}\dot{v}_\mu(\mathcal{S}S') - \tfrac{2}{5}\dot{v}_\mu(\mathcal{S}S\dot{v}) - \tfrac{4}{3}S_{\mu\sigma}S''^\sigma \\ & + \tfrac{2}{15}S_{\mu\nu}\dot{S}^{\nu\rho}\dot{v}_\rho - \tfrac{8}{15}\dot{S}_{\mu\nu}S^{\nu\rho}\dot{v}_\rho + \tfrac{1}{5}S_{\mu\sigma}\mathcal{S}^\sigma \dot{v}^2 + \tfrac{4}{5}\mathcal{S}_\mu(S'\dot{v}) - \tfrac{4}{5}S'_\mu(\mathcal{S}\dot{v})\} \\ & - \dot{v}_\lambda\{\tfrac{2}{15}S_{\mu\sigma}S'^\sigma + \tfrac{8}{15}\dot{S}_{\mu\sigma}\mathcal{S}^\sigma\} - \tfrac{2}{3}S_{\lambda\sigma}\ddot{S}^\sigma{}_\mu + \tfrac{2}{15}S_{\lambda\mu}(S'\dot{v}) \\ & - \tfrac{2}{15}\dot{S}_{\lambda\mu}(\mathcal{S}\dot{v}) + \tfrac{2}{15}\mathcal{S}_\lambda \dot{S}_{\mu\sigma}\dot{v}^\sigma - \tfrac{2}{15}S'_\lambda S_{\mu\sigma}\dot{v}^\sigma - \tfrac{4}{3}\mathcal{S}_\lambda S''_\mu]_- \\ & + [v_\lambda\{\dot{v}_\mu[-\tfrac{16}{35}\mathcal{S}^2\dot{v}^2 + \tfrac{2}{5}(\mathcal{S}\dot{S}\dot{v}) + \tfrac{2}{35}S^2\dot{v}^2 - \tfrac{4}{35}(\dot{v}SS\dot{v}) + \tfrac{2}{15}\dot{S}^2 \\ & + \tfrac{4}{5}(\mathcal{S}S'') + \tfrac{2}{5}(S'S\dot{v})] + \tfrac{2}{5}\dot{S}_{\mu\nu}\dot{S}^{\nu\sigma}\dot{v}_\sigma - \tfrac{4}{35}S_{\mu\nu}S^{\nu\sigma}\dot{v}_\sigma \dot{v}^2\}]_- \\ & + [v_\lambda\{\tfrac{4}{3}S_{\mu\nu}S'''^\nu + \tfrac{16}{15}S_{\mu\nu}\ddot{S}^{\nu\rho}\dot{v}_\rho + \tfrac{2}{5}\ddot{S}_{\mu\nu}S^{\nu\rho}\dot{v}_\rho - \tfrac{2}{3}\dot{v}_\mu \dot{S}^2 - \tfrac{12}{5}\dot{v}_\mu(\mathcal{S}S'') \\ & - \tfrac{6}{5}\dot{S}_{\mu\nu}\dot{S}^{\nu\rho}\dot{v}_\rho + \tfrac{8}{5}\dot{v}_\mu S'^2 - \dot{S}_{\mu\nu}\mathcal{S}^\nu \dot{v}^2 + S_{\mu\nu}S'^\nu \dot{v}^2 + \tfrac{1}{5}S_{\mu\nu}S^{\nu\sigma}\dot{v}_\sigma \dot{v}^2 \\ & + \tfrac{1}{5}\dot{v}_\mu S^2\dot{v}^2 + \tfrac{4}{5}\ddot{v}_\mu(\mathcal{S}S\dot{v}) - \tfrac{8}{5}\dot{v}_\mu \mathcal{S}^2\dot{v}^2 - \tfrac{2}{5}\dot{v}_\mu(\dot{v}SS\dot{v})\} \\ & + \dot{v}_\lambda\{\tfrac{4}{3}S_{\mu\nu}S''^\nu + \tfrac{2}{3}\ddot{S}_{\mu\nu}\mathcal{S}^\nu - \tfrac{4}{3}\dot{S}_{\mu\nu}S'^\nu + \tfrac{4}{15}\dot{S}_{\mu\nu}S^{\nu\rho}\dot{v}_\rho - \tfrac{2}{5}S_{\mu\nu}\dot{S}^{\nu\rho}\dot{v}_\rho \\ & + \tfrac{3}{5}S_{\mu\nu}\mathcal{S}^\nu \dot{v}^2 - \tfrac{1}{15}\ddot{v}_\mu S^2 + \tfrac{2}{5}\ddot{v}_\mu \mathcal{S}^2\} - \tfrac{2}{15}\ddot{v}_\lambda S_{\mu\nu}S^{\nu\sigma}\dot{v}_\sigma \\ & + \tfrac{4}{3}\mathcal{S}_\lambda S'''_\mu + \tfrac{2}{3}S_{\lambda\nu}S^{\text{iii}\,\nu}{}_\mu - \tfrac{2}{3}S''_\lambda S_{\mu\sigma}\dot{v}^\sigma + \tfrac{2}{3}\mathcal{S}_\lambda \ddot{S}_{\mu\nu}\dot{v}^\nu \\ & - \tfrac{2}{3}S_{\lambda\sigma}\dot{v}^\sigma \dot{S}_{\mu\rho}\dot{v}^\rho + \tfrac{2}{3}S_{\lambda\sigma}\dot{S}^\sigma{}_\mu \dot{v}^2 + 2\mathcal{S}_\lambda S'_\mu \dot{v}^2]_-. \end{aligned} \tag{133}$$

The singular terms in all the above expressions have already been dealt with in the text. We now have to determine $A^{(0)}_{\mu}$ and $B^{(0)}_{\lambda\mu}$. We can see from (126) and (128) that certain terms of $T^{(0)}_{\mu}(1)$ and $T^{(0)}_{\mu}(2)$ are perfect differentials. It simplifies the calculation to add these terms straight away to $A^{(0)}_{\mu}(1)$ and $A^{(0)}_{\mu}(2)$, for these terms then fall out of the translational equation. They cancel one another on the right-hand side of (69a). Their inclusion in $A^{(0)}_{\mu}$, however, alters the left-hand side of (79) and is equivalent to subtracting certain terms from $M^{(0)}_{\lambda\mu}$. It appears that this actually effects a simplification in $M^{(0)}_{\lambda\mu}$. The terms that are so cancelled in $M^{(0)}_{\lambda\mu}(1)$ and $M^{(0)}_{\lambda\mu}(2)$ have been written separately in the second square brackets in (131) and (133) respectively. Further we see that the terms in the first square brackets in (131) and (133) are perfect differentials. These can therefore be eliminated at once by including them in $B^{(0)}_{\lambda\mu}$. These terms then also drop out of the rotational equations. We may therefore work as if the terms in the first square brackets in $T^{(0)}_{\mu}(1)$ and $T^{(0)}_{\mu}(2)$, and the terms in the first and second square brackets in $M^{(0)}_{\lambda\mu}(1)$ and $M^{(0)}_{\lambda\mu}(2)$, were absent. This effects some simplification in the subsequent calculation, and we henceforth assume that these terms have been eliminated.

Consider first $A^{(0)}_{\mu}(0)$. This must not contain $S_{\lambda\mu}$ and must have one differentiation with respect to τ. It can easily be seen that it must be $-\frac{2}{3}\dot{v}_{\mu}$. With this substitution the right-hand side of (69a) vanishes at once, and the left-hand side of (79) also vanishes by (129), provided that $B^{(0)}_{\lambda\mu}(0) = 0$. This is in any case inevitable since no term in $B^{(0)}_{\lambda\mu}(0)$ is possible which only contains v_{μ} but none of its derivatives. We thus find that

$$T^{\text{react.}}_{\mu}(0) = \tfrac{2}{3}(v_{\mu}\dot{v}^2 + \ddot{v}_{\mu}), \tag{134}$$

while $M^{\text{react.}}_{\lambda\mu}(0) = 0$. These are the well-known radiation reaction terms for a point charge.

Now consider $A^{(0)}_{\mu}(1)$ and $B^{(0)}_{\lambda\mu}(1)$. Expressions (126) and (131) show that the terms of these must be linear in $S_{\lambda\mu}$ and contain respectively two and one differentiations with respect to τ. The most general expressions for these quantities are then of the following form:

$$A^{(0)}_{\mu}(1) = f_1 v_{\mu}(\mathcal{S}\ddot{v}) + f_2 v_{\mu}(S'\dot{v}) + f_3 \dot{v}_{\mu}(\mathcal{S}\dot{v}) + f_4 \mathcal{S}_{\mu}\dot{v}^2 + f_5 S''_{\mu} + f_6 \dot{S}_{\mu\sigma}\dot{v}^{\sigma} + f_7 S_{\mu\sigma}\ddot{v}^{\sigma}, \tag{135}$$

and

$$B^{(0)}_{\lambda\mu}(1) = [h_1 \dot{S}_{\lambda\mu} + h_2 v_{\lambda} S'_{\mu} + h_3 v_{\lambda} S_{\mu\sigma}\dot{v}^{\sigma} + h_4 \dot{v}_{\lambda}\mathcal{S}_{\mu}]_{-}. \tag{136}$$

The f's and h's are arbitrary constants which have to be determined. Using (131) (without the terms in the first two brackets), (135) and (136), we build the expression on the left-hand side of (79). The coefficients of those terms in this which are not of the form of the right of (79) must be put equal to zero. Thus, the coefficient of $v_\lambda \dot{v}_\mu(\mathcal{S}\dot{v})$ is $-(f_3 - \frac{4}{3})$, and this must be zero; whence $f_3 = \frac{4}{3}$. Similarly, the coefficient $v_\lambda \mathcal{S}_\mu \dot{v}^2$ is $-(f_4 - \frac{1}{3})$; whence $f_4 = \frac{1}{3}$. The coefficient of $v_\lambda S_{\mu\sigma} \ddot{v}^\sigma$ is $-(f_7 + h_3 - \frac{2}{3})$. This need not be zero, but it must be equal to minus the coefficient of $\ddot{v}_\lambda \mathcal{S}_\mu$ which is $-h_4$. Hence $h_4 = -(f_7 + h_3 - \frac{2}{3})$. These two terms then combine to give

$$h_4[S_{\mu\sigma}(\ddot{v}^\sigma v_\lambda - v^\sigma \ddot{v}_\lambda)]_-,$$

which is of the form of the right-hand side of (79) and is permissible. Proceeding in this way we can show that the left-hand side of (79), which is equal to $M^{\text{react.}}_{\lambda\mu}(1)$ by (88), must be reduced to

$$M^{\text{react.}}_{\lambda\mu}(1) = h_2[S_{\lambda\sigma}(\ddot{v}^\sigma v_\mu - v^\sigma \ddot{v}_\mu)]_-. \tag{137a}$$

Finally, using (126) and (135), we form the right-hand side of (69*a*) and equate it to the left-hand side, which in this case becomes just $h_2(S'\ddot{v})$. The coefficients of each term on the two sides of (69*a*) must be equal, and this gives us another set of equations determining the coefficients. It is found that all the coefficients can be determined in terms of one of them, namely h_2. We call this h below. The result is given in the table:

$f_1 = \frac{2}{3}$	$f_2 = -3h + \frac{8}{3}$	$f_3 = \frac{4}{3}$	$f_4 = \frac{1}{3}$
$f_5 = -h + \frac{4}{3}$	$f_6 = -2h + 2$	$f_7 = \frac{2}{3}$	
$h_1 = 0$	$h_2 = h$	$h_3 = h$	$h_4 = -h$

If we use (126) and (135) with these values of the coefficients, $T^{\text{react.}}_\mu(1)$ given by (85) can easily be worked out. Similarly, by (56) and with h for h_2, the equation (137*a*) leads to

$$C^{\text{react.}}_{\rho\mu}(1) = h(\ddot{v}_\rho v_\mu - \ddot{v}_\mu v_\rho). \tag{137b}$$

$T^{\text{react.}}_\mu(2)$ and $M^{\text{react.}}_{\lambda\mu}(2)$ can be found in the same way, but due to the fact that they are quadratic in $S_{\lambda\mu}$ and contain respectively four and three derivatives with respect to τ, they are very much more complicated. $A^{(0)}_\mu(2)$ contains 56 terms which may be grouped in the following way. One group has terms of the type $v(SS)v^2$ with three derivatives (dots) distributed over the different symbols. By (4), (5) and (7), there are five independent terms in this group, namely $v^{\text{iii}}_\mu S^2$, $\dot{v}_\mu \dot{S}^2$, $\dot{v}_\mu S^2 \dot{v}^2$, $v_\mu(\dot{S}\ddot{S})$, and $v_\mu(\dot{v}\ddot{v})$. Another group has terms of the type $v_\mu(vSSv)v^2$ and contains 23 terms. The other groups are characterized by terms of the type $v_\mu(vSv)(vSv)$, $(S_{\mu\sigma}S^{\sigma\nu}v_\nu)v^2$ and $(S_{\mu\sigma}v^\sigma)(vSv)v^2$, and contain 3, 13 and 12 terms respectively. There are thus in all 56 ir pendent terms with arbitrary coefficients in $A^{(0)}_{\mu}(2)$. Similarly, the terms o $B^{(0)}_{\lambda\mu}(2)$ can be classified in the following groups:

$$[v_\lambda S_{\mu\sigma}S^{\sigma\nu}v_\nu]_- v^2, \quad [v_\lambda S_{\mu\sigma}v^\sigma]_-(vSv), \quad (v_\lambda v_\mu]_-(vSSv),$$
$$[S_{\lambda\sigma}v^\sigma S_{\mu\rho}v^\rho]_-, \quad \text{and} \quad S_{\lambda\mu}(vSv),$$

with two dots distributed over the symbols in each term, plus two other terms, $[v_\lambda \ddot{v}_\mu S^2]_-$ and $[S_{\lambda\rho}\ddot{S}^\rho{}_\mu]_-$, which do not belong to these groups. There are 11, 5, 3, 4 and 3 terms respectively in each group, making 28 terms in all. It is remarkable that the conditions (79) and (69a) are so stringent that the coefficients of all these terms can be determined in terms of six arbitrary constants which we denote by k_1 to k_6. We give only the result here. $T^{\text{react.}}_\mu(2)$ is an extremely cumbersome expression. However, when $\dot{v}_\mu$, $\ddot{v}_\mu$, $v^{\text{iii}}{}_\mu$ and $\dot{S}_{\lambda\mu}$, $\ddot{S}_{\lambda\mu}$ and $S^{\text{iii}}{}_{\lambda\mu}$ vanish, it reduces to

$$-\tfrac{1}{15}v^{\text{iv}}_\mu S^2 + \tfrac{2}{5}v^{\text{iv}}_\mu \not{S}^2 + \tfrac{2}{15}v_\mu(\not{S}Sv^{\text{iv}}) + (k_1+k_3+\tfrac{2}{3})\,v_\mu(\not{S}S'''')$$
$$+(k_1+\tfrac{2}{3})\,S^{\text{iv}}_{\mu\nu}\not{S}^\nu + k_3 S_{\mu\nu}S''''^{\,\nu}. \qquad (138)$$

Finally, $M^{\text{react.}}_{\lambda\mu}(2)$ can be written

$$M^{\text{react.}}_{\lambda\mu}(2) = [S_\lambda{}^\rho C^{\text{react.}}_{\rho\mu}(2)]_-, \qquad (139)$$

with
$$C^{\text{react.}}_{\rho\mu}(2) = \tfrac{2}{3}S^{\text{iii}}_{\rho\mu} + \tfrac{2}{3}\dot{S}_{\rho\mu}\dot{v}^2 + [\tfrac{4}{3}v_\rho S'''_\mu + 2v_\rho S'_\mu \dot{v}^2$$
$$-k_1 v^{\text{iii}}_\rho \not{S}_\mu - (k_1+k_2+\tfrac{2}{15})\,\ddot{v}_\rho S_{\mu\sigma}\dot{v}^\sigma - (k_1-k_3-k_4+\tfrac{2}{3})\,\ddot{v}_\rho S'_\mu$$
$$+(k_1-k_3+k_4+\tfrac{2}{3})\,v_\rho \dot{S}_{\mu\sigma}\ddot{v}^\sigma - (k_2+k_3-\tfrac{8}{15})\,\dot{v}_\rho S_{\mu\sigma}\ddot{v}^\sigma$$
$$-(k_3-\tfrac{2}{3})\,v_\rho S_{\mu\sigma}v^{\text{iii}\,\sigma} + (k_4+\tfrac{2}{3})\,\{v_\rho \ddot{S}_{\mu\sigma}\dot{v}^\sigma + \dot{v}_\rho S''_\mu\}$$
$$+(2k_4-\tfrac{2}{3})\,\dot{v}_\rho \dot{S}_{\mu\sigma}\dot{v}^\sigma - 2k_5 v_\rho \not{S}_\mu(\dot{v}\ddot{v}) - (k_5-\tfrac{3}{5})\,\{v_\rho S_{\mu\sigma}\dot{v}^\sigma\dot{v}^2$$
$$+\dot{v}_\rho \not{S}_\mu \dot{v}^2\} + k_6\{v_\rho \ddot{v}_\mu(\not{S}\dot{v}) + v_\rho \dot{v}_\mu(\not{S}\ddot{v})\}]_-. \qquad (140)$$

We see from (138) that $S^{\text{iv}}_{\lambda\mu}$ appears in the translational equation, but the arguments given in the text show that this does not prevent the equations from being used to determine the motion of the particle. The only terms which contain $S^{\text{iv}}_{\lambda\mu}$ are in fact just the last three terms in (138). (No term containing $S^{\text{iv}}_{\lambda\mu}$ vanishes by putting any of the lower derivatives of v_μ or $S_{\lambda\mu}$ equal to zero since the terms in $T^{\text{react.}}_\mu(2)$ have in all four derivatives, and in those terms where four derivatives appear in $S_{\lambda\mu}$ no dots can appear over any other symbol.) (138) shows that by putting $k_1+\frac{2}{3}=0$, $k_3=0$, we could eliminate $S_{\lambda\mu}$ from the equations of motion altogether. This specialization simplifies the coefficients in (140). Out of the remaining four coefficients, $k_6=0$ effects an obvious simplification.

To obtain the specialized equation when either $S_{\mu\rho}v^\rho = 0$ or $S^*_{\mu\rho}v^\rho = 0$, we have to replace $C^{\text{react.}}_{\rho\mu}$ by

$$D_{\rho\mu} \equiv C^{\text{react.}}_{\rho\mu} - (C^{\text{react.}}_{\rho\sigma} v^\sigma v_\mu - C^{\text{react.}}_{\mu\sigma} v^\sigma v_\rho), \tag{141}$$

as shown in the text. This expression is in general no simpler than (140). However, in the particular case $\mathcal{S}_\mu = 0$ we can use the relations (94). The expression for $D_{\rho\mu}$ then reduces to

$$D_{\rho\mu} = \tfrac{2}{3}S^{\text{iii}}_{\rho\mu} + \tfrac{2}{3}\dot{S}_{\rho\mu}\dot{v}^2 - [-\tfrac{2}{3}v_\rho S'''_\mu + (d-\tfrac{2}{3})\,v_\rho S'_\mu \dot{v}^2$$
$$+ (d-\tfrac{2}{3})\,\dot{v}_\rho S''_\mu + d\ddot{v}_\rho S'_\mu + 2(d+\tfrac{1}{3})\,\dot{v}_\rho \dot{S}_{\mu\sigma}\dot{v}^\sigma]_-, \tag{142}$$

where we have written d in place of $-(k_2+k_3+k_4-\frac{8}{15})$. It is interesting to note that in this case three of the six arbitrary constants automatically fall out, while the other three only appear in the above combination, so that in effect there is only one arbitrary constant in this case. (142) is exactly the radiation reaction term found for this case in a previous paper (B, (46)) by a direct method. $T^{\text{self.}}_\mu(2)$ is given explicitly in that paper (B, appendix).

Forming the expression on the right-hand side of (141) from (137 *b*), we see that it vanishes, which shows that there are no $g_1 g_2$ terms in $D_{\rho\mu}$. We thus get the result quoted in the text that even in the case $g_1 \neq 0$ the charge g_1 adds nothing to the radiation reaction terms in the rotational equation.

In the case $\mathcal{S}_\mu = 0$, the radiation reaction terms in the translational equation (101) are then

$$T^{\text{self.}}_\mu = g_1^2 T^{\text{self.}}_\mu(0) + g_1 g_2 T^{\text{self.}}_\mu(1) + g_2^2 T^{\text{self.}}_\mu(2).$$

$T^{\text{self.}}_\mu(0)$ is just given in (134), while

$$T^{\text{self.}}_\mu(1) = \tfrac{4}{3}v_\mu(S'\ddot{v}) + \tfrac{2}{3}S'_\mu \dot{v}^2 - \tfrac{2}{3}S'''_\mu - 2\ddot{S}_{\mu\sigma}\dot{v}^\sigma - \tfrac{4}{3}\dot{S}_{\mu\sigma}\ddot{v}^\sigma, \tag{143}$$

and thus contains no arbitrary constant. $T^{\text{self.}}_\mu(2)$ contains the one arbitrary constant d which occurs in (142) and is given explicitly in B (appendix). We see clearly that, whereas the reaction terms in the general equations (72) and (73) contain no less than seven arbitrary constants, in the special case $\mathcal{S}_\mu = 0$ (but not in the case $S^*_{\mu\rho}v^\rho = 0$) there is only one arbitrary constant, and the radiation reaction terms both in the translational and rotational equations are vastly simpler.

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General classical theory of spinning particles in a meson field

By H. J. Bhabha, Ph.D.

Department of Physics, Indian Institute of Science, Bangalore

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An exact classical theory of the motion of a point dipole in a meson field is given which takes into account the effects of the reaction of the emitted meson field. The meson field is characterized by a constant $\chi = \mu/\hbar$ of the dimensions of a reciprocal length, μ being the meson mass, and as $\chi \to 0$ the theory of this paper goes over continuously into the theory of the preceding paper for the motion of a spinning particle in a Maxwell field. The mass of the particle and the spin angular momentum are arbitrary mechanical constants. The field contributes a small finite addition to the mass, and a negative moment of inertia about an axis perpendicular to the spin axis.

A cross-section (formula (88 *a*)) is given for the scattering of *transversely* polarized neutral mesons by the rotation of the spin of the neutron or proton which should be valid up to energies of 10^9 eV. For low energies E it agrees completely with the old quantum cross-section, having a dependence on energy proportional to p^4/E^2 (p being the meson momentum). At higher energies it deviates completely from the quantum cross-section, which it supersedes by taking into account the effects of radiation reaction on the rotation of the spin. The cross-section is a maximum at $E \sim 3{\cdot}5\mu$, its value at this point being 3×10^{-26} cm.^{2}, after which it decreases rapidly, becoming proportional to E^{-2} at high energies. Thus the quantum theory of the interaction of neutrons with mesons goes wrong for $E \gtrsim 3\mu$. The scattering of longitudinally polarized mesons is due to the translational but not the rotational motion of the dipole and is at least twenty thousand times smaller.

With the assumption previously made by the present author that the heavy partilesc may exist in states of any integral charge, and in particular that protons of charge $2e$ and $-e$ may occur in nature, the above results