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The quantum theory of the meson (Kemmer 1938; Fröhlich, Heitler and Kemmer 1938; Bhabha 1938; Yukawa, Sakata and Taketani 1938; Stueckelberg 1938), in spite of its great similarity to the quantum theory of radiation, differs from it in certain important respects. The total Hamiltonian for the system of protons or neutrons and mesons contains terms in the interaction energy of mesons with the heavy particles which increase with increasing energy of the mesons. This has brought physicists to the prevalent view that for high energies this theory leads to large probabilities for multiple processes, and to explosions of the type first investigated by Heisenberg (1936). Indeed, the possibility of such processes has recently caused Heisenberg (1938) to take the position that they set a limit to the applicability of quantum mechanics. Connected with the same behaviour of the interaction is the fact that many effects, such for example as the perturbation of the self-energy of a proton calculated to the second order in the interaction, diverge more acutely than in ordinary radiation theory, and this has led Heitler (1938) and others to doubt the correctness of the fundamental equations even for mesons of energy comparable with their rest mass.

All the above views are in essence based on the results of second-order perturbation calculations. There is, however, another approach to the problem. Using the commutation rules for the observables and the well-known equations of motion of quantum mechanics, we can derive exactly from the same Hamiltonian on the one hand the Dirac equation for the proton or neutron under the influence of a given meson field, and on the other hand the equations of the meson field influenced by the presence of neutrons. (For brevity we shall henceforth only speak of neutrons, whereas our remarks will apply equally well to protons, since the two are on the same footing as far as this theory is concerned.) Treating the Hamiltonian "classically", that is, treating all the observables occurring in it as commuting variables, we can as usual again derive the same two sets of equations. In this paper, as a first step in the problem, we shall deal with classical equations, since it is possible either to solve them exactly, or at least to give approximate solutions the errors of which can be strictly estimated.

Now meson theory differs from ordinary radiation theory in three particulars. First, the mesons have a finite rest mass μ. Secondly, they have a direct interaction with the spin of the neutron characterized by the constant g_2, besides the constant g_1 which corresponds to the charge in electromagnetic theory. Thirdly, the meson field itself may carry electromagnetic charge. It should be noticed at once that g_2 is of the dimensions of a length times charge, so that the meson theory introduces *two independent* lengths which may be defined by $\chi = \hbar/\mu$ and g_2/g_1, and although with the experimental values of the constants these are of the same order of magnitude, theoretically the two must not be confused as there is no connexion between them yet. As has already been mentioned the quantized meson theory seems to lead to results different in many essentials from those of radiation theory, and in order to determine to what extent this is due to each of the three differences mentioned above, we shall deal as a first step with a theory which differs from radiation theory in only one particular. In this paper we shall investigate the classical behaviour of an *uncharged* meson field with a characteristic length χ connected with the rest mass μ, while the constant g_2 will be put equal to zero. In the course of the paper the result will be established that *the finite rest mass of the meson introduces no essential difference in the behaviour of mesons at high energies* and hence that the fundamental length required by Heisenberg cannot be identified with χ. *The type of explosions investigated by Heisenberg, if they exist, cannot therefore be connected with the rest mass of the meson but would be due either to the interaction constant g_2 or to the fact that the meson field carries electric charge.*

We shall assume that the equations for the meson field influenced by the presence of neutrons are exact. On the other hand, we should not expect the Dirac equation describing the motion of the neutron under the influence of the meson field to be exact, since this neglects what is the equivalent of radiation damping on the neutron. We shall therefore derive new classical equations for a point neutron from the consideration that energy and momentum must be conserved. In this way we shall arrive at a self-consistent mathematical scheme which is entirely free from any difficulties about infinite self-energies. This scheme is a generalization of the scheme already obtained by Dirac (1938) for the classical behaviour of electrons and radiation, and goes over continuously into it as the mass of the meson tends to zero.

By using these equations we shall calculate the scattering of mesons by neutrons and obtain a formula which is a generalization of the Thomson formula as modified by Dirac. It will appear that as far as the classical theory is concerned, we can at any stage in the calculations put $\chi = 0$ and make a transition to electromagnetic theory without getting any infinities or inconsistencies. For mesons the classical theory is also of practical interest. The treatment of a field classically instead of quantum-mechanically is equivalent to a neglect of the momentum properties of the individual light quanta. We may therefore expect the classical theory to give correct results for all processes where the momenta of individual mesons is small compared with the rest mass of neutrons. This covers the whole field of nuclear phenomena and an important part of cosmic-ray phenomena. The classical theory will give correct results for mesons of a few hundred million volts, and will even give the correct order of magnitude for mesons of 10^9 eV.

The meson field of neutrons

We shall use tensor notation throughout, and for the purpose of raising and lowering suffixes use the fundamental tensor $g_{\mu\nu}$ defined by $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$ with all the other components vanishing. The velocity of light will be put equal to unity. We shall further use the notation of writing u for $+(u_\mu u^\mu)^{\frac{1}{2}}$, where u_μ is any vector. We assume that the meson field is described by potentials U_μ and field strengths $G_{\mu\nu}$ which satisfy the equations

$$\left.\begin{aligned} G_{\mu\nu} - 4\pi S_{\mu\nu} &= -\left(\frac{\partial}{\partial x^\mu} U_\nu - \frac{\partial}{\partial x^\nu} U_\mu\right), \\ \chi^2 U_\nu - 4\pi R_\nu &= \frac{\partial}{\partial x_\mu} G_{\mu\nu}, \end{aligned}\right\} \tag{1}$$

where $\chi = \mu/\hbar$, μ being the rest mass of the meson. In classical theory neither μ nor $\hbar$ will appear explicitly, but only through the constant χ which is now a fundamental constant belonging to the field. The limit $\mu = 0$ corresponds to the limit $\chi = 0$, and in this case the equations (1) become the Maxwell equations. Since we are dealing with neutral mesons, we may take all the quantities in (1) to be real. R_ν is a four vector describing the effect of the neutron "charge" on the field, and $S_{\mu\nu}$ is an antisymmetrical tensor describing the effect of the neutron dipole moment. From the second of equations (1) it follows that

$$\chi^2 \frac{\partial}{\partial x_\nu} U_\nu = \frac{\partial}{\partial x_\nu} R_\nu. \tag{2}$$

A proton remains a proton and a neutron a neutron when a neutral meson is absorbed or emitted. We may further assume for the purpose of this paper that the emission of no charged mesons takes place. The right-hand side of

(2) then vanishes, and hence also the left-hand side. It then follows from the equations (1) that

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}U_\nu+\chi^2 U_\nu = 4\pi R_\nu + 4\pi\frac{\partial}{\partial x_\mu}S_{\mu\nu}. \tag{3}$$

It is shown in the appendix that the solution of (3) can be given with the help of a Green's function in the form

$$U_\nu(x_\rho) = \iiiint_{-\infty}^{\infty} dx'_\rho\, G(x_\rho, x'_\rho)\left\{R_\nu(x'_\rho)+\frac{\partial}{\partial x'_\mu}S_{\mu\nu}(x'_\rho)\right\}, \tag{4}$$

where x'_ρ stands for the four variables x'_0, x'_1, x'_2, x'_3. The Green's function for this problem is

$$G(x_\rho, x'_\rho) = \left.\begin{pmatrix} 2\delta(u^2)-\frac{\chi}{u}J_1(\chi u) \\ 0 \\ 0 \end{pmatrix} \begin{matrix} u_0 > u_r \\ |u_0| < u_r \\ u_0 < -u_r \end{matrix}\right\} \tag{5a}$$

or

$$G(x_\rho, x'_\rho) = \left.\begin{pmatrix} 0 \\ 0 \\ 2\delta(u^2)-\frac{\chi}{u}J_1(\chi u) \end{pmatrix} \begin{matrix} u_0 > u_r \\ |u_0| < u_r \\ u_0 < -u_r \end{matrix}\right\} \tag{5b}$$

where we have written $u_\rho = x_\rho - x'_\rho$, and $u_r = \sum_1^3 (-u_k u^k)^{\frac{1}{2}}$. Here J_1 is the Bessel function of order 1. We shall frequently have to make use of the following properties of Bessel's functions:

$$s^{-n}J_{n+1}(s) = -\frac{d}{ds}\{s^{-n}J_n(s)\}, \tag{6a}$$

$$\operatorname*{Lt}_{s\to 0}\frac{1}{s^n}J_n(s) = \frac{1}{2^n n!}. \tag{6b}$$

We notice that the Green's function only has a singularity in the form of a δ-function, the second part dependent on χ only contributing a plain discontinuity on the light cone. This part tends to zero as $\chi\to 0$. This is the basis of all the results which will follow below, and shows at once that the worst singularities in the meson field are identical with those of the electromagnetic field, the additional singularities being of lesser order.

In the classical theory we must treat the neutron as a point with a dipole attached to it moving along a classical world line which we assume to be given. We denote its co-ordinates by z_μ which are functions of τ, the proper time along the world line from some arbitrary point on it. Consistently with the quantum-mechanical expression for R_ν as a charge-current density, we now assume

$$R_\nu = g_1\int_{-\infty}^{\infty} d\tau\, v_\nu(\tau)\,\delta(x_0-z_0)\,\delta(x_1-z_1)\,\delta(x_2-z_2)\,\delta(x_3-z_3), \tag{7}$$

where v_ν is the velocity $\dot{z}_\nu$. We shall denote a differentiation with respect to the proper time by a dot. g_1 is a constant playing the part of a charge. Similarly we may write for $S_{\mu\nu}$

$$S_{\mu\nu} = g_2\int_{-\infty}^{\infty} d\tau\, S'_{\mu\nu}(\tau)\,\delta(x_0-z_0)\,\delta(x_1-z_1)\,\delta(x_2-z_2)\,\delta(x_3-z_3), \tag{8}$$

the $S'_{\mu\nu}$ being a function of τ satisfying the relation S'^2 = constant along the world line. g_2 is another constant. The $S_{\mu\nu}$ term can be treated exactly like the R_ν term, but on account of the very much greater complication which it introduces, we shall omit it from this paper in order not to confuse the issue. Since g_2 does not occur any further, we shall simply write g instead of g_1.

Introducing (7) into (4) and using (5*a*)

$$U_\nu^{\text{ret}}(x_\rho) = g\frac{v_\nu}{\kappa} - g\chi\int_{-\infty}^{\tau_0} d\tau \frac{v_\nu}{s} J_1(\chi s), \tag{9}$$

where

$$\left.\begin{aligned} s_\mu &= x_\mu - z_\mu(\tau),\\ \kappa &= s_\mu v^\mu, \end{aligned}\right\} \tag{10}$$

and τ_0 is the proper time of the "retarded" point, i.e. the point on the world line such that $s_\mu s^\mu = 0$ and $s_0 > 0$. We at once make the convention that whenever any quantity dependent on τ does not occur inside an integral with respect to τ then it is to be taken at the retarded point $\tau = \tau_0$. Thus the first term on the right-hand side of (9) is $v_\nu(\tau_0)\,[\{x_\mu - z_\mu(\tau_0)\}\, v^\mu(\tau_0)]^{-1}$. This is the expression corresponding to the retarded potentials. There is a similar solution given by (5*b*), which gives one the advanced potentials. In this paper we shall only consider retarded potentials since the theory is quite symmetrical in retarded and advanced potentials.

From the first of equations (1) it then follows, using the rules that are given in the appendix for differentiating with respect to x_μ, that

$$G_{\mu\nu}^{\text{ret}} = G_{\mu\nu}^0 + G_{\mu\nu}^\chi, \tag{11}$$

where

$$G_{\mu\nu}^0 = -g\left\{\frac{s_\mu v_\nu - s_\nu v_\mu}{\kappa^3}(1-\kappa') + \frac{s_\mu \dot{v}_\nu - s_\nu \dot{v}_\mu}{\kappa^2}\right\}, \tag{12}$$

with $\kappa' = s_\mu \dot{v}^\mu$, and

$$G_{\mu\nu}^\chi = g\frac{\chi^2}{2}\frac{s_\mu v_\nu - s_\nu v_\mu}{\kappa} - g\chi^2\int_{-\infty}^{\tau_0} d\tau \frac{s_\mu v_\nu - s_\nu v_\mu}{s^2} J_2(\chi s). \tag{13}$$

The first term of $G^{\chi}_{\mu\nu}$ is singular on the world line, since its value at any point on it is not unique, but depends on the direction from which the point is approached. Both $G^{\chi}_{\mu\nu}$ and the part dependent on χ in U^{ret}_{ν} of (9) are, however, quite finite at all points of space including the world line of the neutron, and tend continuously to zero as $\chi \to 0$. The only infinities in the retarded field are therefore contained in $G^{0}_{\mu\nu}$ which is just the Maxwell field for a moving-point charge. The expressions (5*a*, *b*) and (11) show explicitly that the singularities in the meson field are no more acute than in ordinary Maxwell theory, and further, that the limit to the case $\chi = 0$ can be made continuously.

The energy-momentum tensor $T_{\mu\nu}$ corresponding to the equations (1) in a region not occupied by a neutron is (see for example Proca 1936)

$$4\pi T_{\mu\nu} = G_{\mu\sigma} G^{\sigma}{}_{\nu} + \tfrac{1}{4} g_{\mu\nu} G_{\sigma\rho} G^{\sigma\rho} + \chi^2 U_{\mu} U_{\nu} - \frac{\chi^2}{2} g_{\mu\nu} U_{\rho} U^{\rho}. \tag{14}$$

In the absence of neutrons, the solution of (1) can be written in the form of plane waves

$$U_{\mu} = \alpha_{\mu} \cos (w_{\nu} x^{\nu}), \tag{15}$$

where the direction of polarization α_{μ} and the direction of propagation ω_{ν} satisfy the relations which follow from (2) and (3)

$$\left.\begin{aligned} \alpha_{\mu}\omega^{\mu} &= 0, \\ \omega_{\mu}\omega^{\mu} &= \chi^2. \end{aligned}\right\} \tag{16}$$

The energy momentum tensor for this solution, using (14), is, on the average,

$$-\frac{1}{8\pi} \omega_{\mu} \omega_{\nu} \alpha^2. \tag{17}$$

We notice that since α_{μ} is perpendicular to the time-like vector ω_{μ}, α^2 is negative.

The classical equations of motion of a neutron

We have now to find the classical equations of motion of a neutron to take the place of the Dirac equation of the quantized theory, and we further wish to take account not only of the effect of an external meson field on the neutron, but also of the reaction of the neutron's own radiated meson field. The easiest way to do this is, following Dirac, to consider the world line of the neutron as given and to enclose it by a narrow tube, the radius of which will in the end be made to tend to zero. We now derive the equations of motion from the condition that the flow of energy and momentum out of a portion of the tube in the presence of an external field as calculated by using (14) shall only depend on the conditions at the two ends of the tube, that is, that it shall be a perfect differential. We shall thus have conservation of energy and momentum, and the reaction of the radiated meson field will be included in the equations.

It can be shown easily that our results will not depend on the actual shape of the world tube. It is convenient to take the world tube defined by

$$\kappa \equiv s_\mu v^\mu = \epsilon, \tag{18}$$

where ϵ is some constant which in the end will be made to tend to zero. s_μ, in accordance with (10), is the distance from a point x_μ on the world tube to the retarded point $z_\mu(\tau_0)$, so that $s^2 = 0$. If we fix our attention on any point on the world line in the Lorentz frame in which the electron is instantaneously at rest, then a sphere about this point of radius ϵ taken at a time ϵ later is a section of this world tube.

It is shown in the appendix that the flow of energy and momentum out of that portion of the world tube with ends defined by the proper times τ_1 and τ_2 is

$$\int_{\tau_1}^{\tau_2} d\tau \left[g^2 \left(\frac{1}{2} \frac{\dot{v}_\mu}{\epsilon} - \tfrac{2}{3} v_\mu \dot{v}^2 \right) + g(\tilde{G}^\chi_{\mu\sigma} + G^{\text{in}}_{\mu\sigma})\, v^\sigma \right], \tag{19}$$

where terms which vanish with ϵ are omitted. $G^{\text{in}}_{\mu\nu}$ is the incoming external field, defined, following Dirac, as the actual field $G^{\text{act}}_{\mu\nu}$ at the point minus the retarded field $G^{\text{ret}}_{\mu\nu}$ given by (11),

$$G^{\text{in}}_{\mu\nu} = G^{\text{act}}_{\mu\nu} - G^{\text{ret}}_{\mu\nu}. \tag{20}$$

$\tilde{G}^\chi_{\mu\nu}$ is the second term in (13) of $G^\chi_{\mu\nu}$,

$$\tilde{G}^\chi_{\mu\nu} = -g\chi^2 \int_{-\infty}^{\tau_0} d\tau \, \frac{s_\mu v_\nu - s_\nu v_\mu}{s^2} J_2(\chi s). \tag{21}$$

The first term of $G^\chi_{\mu\nu}$ in (13) is not single valued on the world line. Its value averaged over all directions of approach to a point on the world line is, however, zero. We may alternatively look on $\tilde{G}^\chi_{\mu\nu}$ as the value of $G^\chi_{\mu\nu}$ at any point averaged over a small region of space surrounding that point. $\tilde{G}^\chi_{\mu\nu}$ will then be identical with $G^\chi_{\mu\nu}$ everywhere except on the world line, where it will reduce to (21). In expression (20), the values of $G^{\text{in}}_{\mu\nu}$ and $\tilde{G}^\chi_{\mu\nu}$ are taken for any value of τ at the point on the world line corresponding to it.

In accordance with what we have said at the beginning of this section, the integrand of (19) must be a perfect differential, since it must only depend on conditions at the two ends of the tube. We must then have

$$g^2\left(\frac{1}{2}\frac{\dot{v}_\mu}{\epsilon}-\tfrac{2}{3}v_\mu\dot{v}^2\right)+g(\tilde{G}^\chi_{\mu\sigma}+G^{\text{in}}_{\mu\sigma})\,v^\sigma=\dot{B}_\mu, \tag{22}$$

where $\dot{B}_\mu$ is some tensor. Multiplying (22) by v_μ we see that $\dot{B}_\mu$ must satisfy

$$\dot{B}_\mu v^\mu=-\tfrac{2}{3}g^2\dot{v}^2. \tag{23}$$

Here we have made use of two of the relations

$$\left.\begin{aligned} v^2&=1,\\ v_\mu\dot{v}^\mu&=0,\\ v_\mu\ddot{v}^\mu+\dot{v}^2&=0.\end{aligned}\right\} \tag{24}$$

Therefore $\dot{B}_\mu$ must be of the form

$$\dot{B}_\mu=\dot{B}'_\mu+\tfrac{2}{3}g^2\ddot{v}_\mu,$$

where $\dot{B}'_\mu v^\mu=0$. The simplest assumption for $\dot{B}'_\mu$ is to put it equal to a constant times v_μ. Writing the constant in the form $(1/2\epsilon-M)$, we finally get for the equations of motion

$$M\dot{v}_\mu-\tfrac{2}{3}g^2\ddot{v}_\mu-\tfrac{2}{3}g^2v_\mu\dot{v}^2+g(\tilde{G}^\chi_{\mu\sigma}+G^{\text{in}}_{\mu\sigma})\,v^\sigma=0. \tag{25}$$

Dividing by $\tfrac{2}{3}g^2$ and introducing (21), we may write them in the form

$$a\dot{v}_\mu-\ddot{v}_\mu-v_\mu\dot{v}^2-\tfrac{3}{2}\chi^2v^\sigma\int_{-\infty}^{\tau}d\tau'\,\frac{s_\mu v_\sigma(\tau')-s_\sigma v_\mu(\tau')}{s^2}\,J_2(\chi s)=-\frac{3}{2g}G^{\text{in}}_{\mu\sigma}v^\sigma, \tag{26}$$

where we have written a for $3M/2g^2$. Explicitly, $s_\mu=z_\mu(\tau)-z_\mu(\tau')$. These are the fundamental equations of motion of a neutron in a meson field. They have a very similar form to the equations recently given by Dirac (1938) for the classical motion of an electron in an electromagnetic field. The only difference is the appearance of the term $\tilde{G}^\chi_{\mu\nu}$. It shows that the motion of the neutron at a time τ is influenced in principle by the whole previous history of the neutron. In practice, however, the integral (21) converges fairly rapidly, and only the behaviour at times not very much earlier than $1/\chi$ will be of importance. This is what we should expect, since meson waves are propagated with all velocities less than that of light so that a neutron may be influenced at a time τ by a disturbance it has emitted at an earlier instant. We notice again that the equations go over quite continuously as $\chi\to 0$ into the equation given by Dirac. We shall return to this point later.

The equations (26) with no ingoing field, that is with the right-hand side equal to zero, have solutions of two types, as in the case of the equations found by Dirac. We can see at once that v_μ constant is a solution, for the integral vanishes from antisymmetry since in this case s_μ is proportional to v_μ, and the other three terms also vanish. The second type consists of solutions in which the neutron by itself works up an acceleration at an ever-increasing rate. We shall not consider these in detail. The corresponding solutions when $\chi = 0$ have been given by Dirac. It is enough to say that following Dirac we must postulate here also that only those solutions occur in nature for which the velocity does not tend to infinity as time progresses. Boundary conditions restricting the allowed solutions to those which do not tend to infinity at large distances are common in physics. We have here for the first time a boundary condition in the time co-ordinate, restricting the allowed solutions to those where the velocity does not become infinite in the infinitely distant future.

Energy of the field associated with a neutron

Pryce (1938) has shown that it is possible to alter the definition of the electromagnetic energy-momentum tensor when a charge is present, so that the total energy associated with a point charge is finite. It is possible to do the same for the meson field. Since we have already shown that the singularities in the meson field are identical with those of the electromagnetic field, the additional terms being finite, it follows that we can at once take over the tensor which has to be subtracted from (14). The new definition of the energy-momentum tensor $T'_{\mu\nu}$ for the meson-field energy is then

$$T'_{\mu\nu} = T_{\mu\nu} - \frac{\partial}{\partial x_\sigma} \kappa_{\sigma\mu\nu}, \tag{27}$$

where the tensor $\kappa_{\sigma\mu\nu}$ is the one given in the paper of Pryce.

We now wish to calculate the meson-field energy associated with a neutron at rest. Instead of starting from $T'_{\mu\nu}$ it is more convenient to do it by a different way. For a neutron at rest at the origin

$$U_0 = g\frac{e^{-\chi r}}{r}; \quad U_1 = U_2 = U_3 = 0 \quad \left(r^2 = \sum_1^3 x_k^2\right).$$

In this case, using (14) and (1),

$$4\pi T_{00} = \tfrac{1}{2}\sum_{k=1}^{3}\left(\frac{\partial}{\partial x_k}U_0\right)^2 + \tfrac{1}{2}\chi^2 U_0^2.$$

We surround the neutron by a small sphere of radius ϵ. The meson-field energy outside this sphere given by (14) is

$$\int T_{00}\, d\text{ vol.} = \frac{1}{8\pi}\int\left\{\sum_1^3\left(\frac{\partial}{\partial x_k}U_0\right)^2+\chi^2 U_0^2\right\}d\text{ vol.}$$

$$= \frac{1}{8\pi}\int\left\{\sum_{k=1}^3\frac{\partial}{\partial x_k}\left(U_0\frac{\partial}{\partial x_k}U_0\right)-\sum_1^3 U_0\frac{\partial^2}{\partial x_k^2}U_0+\chi^2 U_0\right\}d\text{ vol.}$$

The last two terms cancel each other in free space by (3), and the first term can be transformed to an integral over the surface of the sphere

$$-\frac{1}{8\pi}\int\left(U_0\frac{\partial}{\partial r}U_0\right)dS = \frac{g^2}{2}\left[\frac{1}{\epsilon}+\chi\right]e^{-2\chi\epsilon} \doteqdot \frac{g^2}{2}\left\{\frac{1}{\epsilon}-\chi+O(\epsilon)\right\}.$$

If $\chi = 0$ this is just the electromagnetic energy of a point charge. The subtracting of the tensor $\kappa_{\sigma\mu\nu}$ has just the effect of cancelling this term, since Pryce has shown that the electromagnetic energy of a point charge is zero with a new definition of the type (27). Hence the meson-field energy of a point neutron consistent with the equations (26) as defined by the tensor (27) is just $-\frac{1}{2}g^2\chi$. The mass of the neutron which plays a part in nuclear phenomena is not the real mass M but an effective mass M' connected with M by

$$M' = M - \tfrac{1}{2}g^2\chi. \tag{28}$$

We shall come across this point again later. For the scattering of mesons, however, and for all processes where frequencies comparable with χ are involved, it is the real mass M that matters. Using the actual values of M, g and χ, we find that M' differs from M by about ten million volts, or roughly 1 %.

Scattering of mesons by neutrons

We now consider the solutions of (26) where the external force on the neutron has the nature of a harmonic oscillation along the x-axis. In other words, we put

$$\left.\begin{aligned} G_{10}^{\text{in}} &= \gamma\cos\omega_0 t,\\ G_{20}^{\text{in}} &= G_{30}^{\text{in}} = 0,\end{aligned}\right\} \tag{29}$$

where we have written t instead of z_0 to conform to the more usual notation. We shall solve these equations for the case where γ is small, so that the velocity of the neutron is always small. We choose our co-ordinates so that the neutron executes oscillations about the origin along the x-axis. We write the solution in the form

$$z_1 = \frac{\beta}{\omega_0} \sin(\omega_0 t + \delta); \quad z_2 = z_3 = 0, \tag{30}$$

where δ is a phase and β an amplitude, which have to be determined. We shall assume that $\beta \ll 1$, so that higher powers of β can be neglected. We have

$$\frac{dz_1}{dt} = \beta \cos(\omega_0 t + \delta),$$

and hence to the first order

$$\left.\begin{aligned} v_1 &= \frac{dz_1}{dt}\left\{1 - \left(\frac{dz_1}{dt}\right)^2\right\}^{-1} \fallingdotseq \beta \cos(\omega_0 t + \delta), \\ v_0 &= \left\{1 - \left(\frac{dz_1}{dt}\right)^2\right\}^{-1} \fallingdotseq 1. \end{aligned}\right\} \tag{31}$$

The right-hand side of (26) has then only one term, namely,

$$-\frac{3}{2g}\gamma \cos \omega_0 t. \tag{32}$$

To this approximation we may replace $d\tau$ by dt. Hence, in the integral in (26),

$$s_0 = z_0(\tau) - z_0(\tau') \fallingdotseq t - t',$$

$$s_1 = z_1(\tau) - z_1(\tau') = \frac{\beta}{\omega_0}\{\sin(\omega_0 t + \delta) - \sin(\omega_0 t' + \delta)\},$$

and

$$s = \sqrt{(s_0^2 - s_1^2)} \fallingdotseq t - t'.$$

The integral on the left-hand side of (26) then becomes

$$\int_0^\infty ds \frac{J_2(\chi s)}{s^2}\left[\frac{\beta}{\omega_0}\sin(\omega_0 t + \delta) - \frac{\beta}{\omega_0}\sin(\omega_0 t' + \delta) - s\beta\cos(\omega_0 t' + \delta)\right]$$

$$= \frac{\beta\chi}{\omega_0} A \sin(\omega_0 t + \delta) - \frac{\beta\chi}{\omega_0}\frac{1}{2i}[e^{i(\omega_0 t + \delta)} B - \text{conj. complex}]$$

$$-\frac{\beta}{2}[e^{i(\omega_0 t + \delta)} C + \text{conj. complex}].$$

Here

$$A = \int_0^\infty ds \frac{J_2(s)}{s^2} = \tfrac{1}{3},$$

$$B = \int_0^\infty ds \frac{J_2(s)}{s^2} e^{-i\nu s} = \begin{cases} \tfrac{1}{3}[(1-\nu^2)^{\frac{3}{2}} + i\nu^3 - \tfrac{3}{2}i\nu] & 0 < \nu \equiv \omega_0/\chi < 1, \\ \tfrac{1}{3}[-i(\nu^2-1)^{\frac{3}{2}} + i\nu^3 - \tfrac{3}{2}i\nu] & \nu > 1, \end{cases}$$

$$C = \int_0^\infty ds \frac{J_2(s)}{s} e^{-i\nu s} = \begin{cases} \tfrac{1}{2} - \nu^2 - i\nu\sqrt{(1-\nu^2)} & 0 < \nu < 1, \\ \tfrac{1}{2} - \nu^2 + \nu\sqrt{(\nu^2-1)} & \nu > 1, \end{cases}$$

with ν standing for ω_0/χ. The term containing the integral on the left-hand side of (26) can then be written

$$\beta\omega_0^2 P \sin(\omega_0 t + \delta) + \beta\omega_0^2 Q \cos(\omega_0 t + \delta), \tag{33}$$

with

$$\left.\begin{aligned} P &= \begin{cases} \dfrac{1}{2\nu^3} - \dfrac{1}{2}\dfrac{\sqrt{(1-\nu^2)}}{\nu^3} - \dfrac{\sqrt{(1-\nu^2)}}{\nu} & 0 < \nu < 1, \\ \dfrac{1}{2\nu^3} & \nu > 1, \end{cases} \\ Q &= \begin{cases} 1 & 0 < \nu < 1, \\ 1 - \dfrac{1}{2}\dfrac{\sqrt{(\nu^2-1)}}{\nu^3} - \dfrac{\sqrt{(\nu^2-1)}}{\nu} & \nu > 1. \end{cases} \end{aligned}\right\} \tag{34}$$

Introducing (29), (30) and (33) into (26), we get

$$-\beta a\omega_0 \sin(\omega_0 t + \delta) + \beta\omega_0^2 \cos(\omega_0 t + \delta) - \beta\omega_0^2 P \sin(\omega_0 t + \delta)$$
$$- \beta\omega_0^2 Q \cos(\omega_0 t + \delta) = -\frac{3}{2}\frac{\gamma}{g} \cos \omega_0 t.$$

From this we obtain at once

$$\beta = \frac{3}{2}\frac{\gamma}{g\omega_0[(a + \omega_0 P)^2 + \omega_0^2(Q-1)^2]^{\frac{1}{2}}}, \tag{35}$$

$$\cos\delta = \frac{\omega_0(Q-1)}{[(a + \omega_0 P)^2 + \omega_0^2(Q-1)^2]^{\frac{1}{2}}}. \tag{36}$$

Now it is shown in the appendix that the field at a very distant point with the co-ordinates X, Y, Z, T produced by a neutron executing the motion (30) is given by

$$\left.\begin{aligned} U_1 &= \begin{cases} \dfrac{g\beta}{R} e^{-R\sqrt{(\chi^2-\omega_0^2)}} \cos(\omega_0 T+\delta) & \dfrac{\omega_0}{\chi} \equiv \nu < 1, \\[2ex] \dfrac{g\beta}{R} \cos\{\omega_0 T - R\sqrt{(\omega_0^2-\chi^2)}+\delta\} & \nu > 1, \end{cases} \\ U_0 &= \begin{cases} \dfrac{g\beta X}{R^2} \dfrac{\sqrt{(\chi^2-\omega_0^2)}}{\omega_0} e^{-R\sqrt{(\chi^2-\omega_0^2)}} \sin(\omega_0 T+\delta) & \nu < 1, \\[2ex] \dfrac{g\beta X}{R^2} \dfrac{\sqrt{(\omega_0^2-\chi^2)}}{\omega_0} \cos\{\omega_0 T - R\sqrt{(\omega_0^2-\chi^2)}+\delta\} & \nu > 1. \end{cases} \end{aligned}\right\} \tag{37}$$

These naturally satisfy (2) and (3) for empty space.

When $\nu < 1$, i.e. if $\omega_0 < \chi$, the field falls off exponentially at large distances, and no energy is radiated by the neutron. It is interesting to note that in this case the whole field moves in phase with the neutron. We should therefore expect an extra mass due to the meson field to be added to the real mass M. This is in fact so. By (34)

$$Q - 1 = 0, \tag{38}$$

so that the amplitude of the oscillations in (35) is determined by $(a+\omega_0 P)$. For the case of very slow oscillations, $\omega_0 \ll \chi$, we find using (34) that the effective mass is

$$\tfrac{2}{3}g^2(a+\omega_0 P) \doteqdot M - \tfrac{1}{2}g^2\chi; \tag{39}$$

this agrees with what we found in the previous section.

The work done by the external force on the neutron in a cycle is by (29) and (31)

$$-g\int G_{10}^{\text{in}} v_1\, dt = -\tfrac{1}{2}g\gamma\beta\cos\delta.$$

This vanishes by (36) and (38) when $\omega_0 < \chi$, consistently with the result already mentioned, that no radiation takes place. It is thus possible even in the classical meson theory for a neutron to vibrate with frequency less than χ without radiating any energy. This is in contrast with the usual radiation theory.

We now consider the more interesting case $\omega_0 > \chi$. The field given by (37) for $\nu > 1$ has the character of a plane wave at large distances. It can at once be split into transverse and longitudinal waves, the amplitudes of the two parts being given by

$$\left.\begin{aligned} &\alpha_0^{\text{tr}} = \alpha_2^{\text{tr}} = \alpha_3^{\text{tr}} = 0; \quad \alpha_1^{\text{tr}} = \frac{g\beta}{R}\sqrt{\left(1-\frac{X^2}{R^2}\right)}; \\ &\alpha_0^{\text{long}} = \frac{g\beta X}{R^2}\frac{\sqrt{(\omega_0^2-\chi^2)}}{\omega_0}; \quad \alpha_1^{\text{long}} = \frac{g\beta X}{R^2}; \quad \alpha_2^{\text{long}} = \alpha_3^{\text{long}} = 0. \end{aligned}\right\} \tag{40}$$

Using the expression (17) for the energy-momentum tensor of a plane wave we find that the energy crossing unit area per unit time in the direction R is for transverse waves

$$\frac{1}{8\pi}\frac{g^2\beta^2}{R^2}\left(1-\frac{X^2}{R^2}\right), \tag{41a}$$

and for longitudinal waves $$\frac{1}{8\pi}\frac{g^2\beta^2X^2}{R^4}\frac{\chi^2}{\omega_0^2}. \tag{41b}$$

Thus, calling θ the angle between the direction R in which the meson radiation is observed and the x-axis, i.e. the direction of oscillation of the neutron, we find that the energy of the transverse waves has a distribution proportional to $\sin^2\theta$, while that of the longitudinal waves is $\cos^2\theta$, as we should expect. The total energy radiated in transverse waves is proportional to $\frac{1}{3}g^2\beta^2$, and in longitudinal waves $\frac{1}{6}g^2\beta^2\chi^2/\omega_0^2$. Thus, for low frequencies $\omega_0\sim\chi$, twice the energy is radiated in transverse as in longitudinal waves, whereas for higher frequencies, progressively less is radiated in longitudinal waves.

To find the scattering of meson waves by neutrons we have only to suppose that the external field (29) is due to a plane wave. If this wave is transverse, then it must travel perpendicular to the x-axis, and hence have a potential in essence of the form

$$U_0 = U_2 = U_3 = 0; \quad U_1 = \frac{\gamma}{\omega_0}\sin\{\omega_0 t - y\sqrt{(\omega_0^2-\chi^2)}\}.$$

By (17), the energy flow associated with this is $\gamma^2/8\pi\omega_0$. If the original plane wave is longitudinal, then it must travel along the x-axis and be described by the potential

$$\begin{aligned} U_0 &= \frac{\sqrt{(\omega_0^2-\chi^2)}}{\chi^2}\gamma\sin\{\omega_0 t - x\sqrt{(\omega_0^2-\chi^2)}\},\\ U_1 &= \frac{\omega_0}{\chi^2}\gamma\sin\{\omega_0 t - x\sqrt{(\omega_0^2-\chi^2)}\},\\ U_2 &= U_3 = 0. \end{aligned}$$

The energy flow associated with this is $\gamma^2\omega_0/8\pi\chi^2$. Dividing the total radiated energy $\frac{1}{3}g^2\beta^2(1+\frac{1}{2}\chi^2/\omega_0^2)$ by the inflowing energy, and using (35) and (34), we finally obtain for the effective "cross-section" for the scattering of transverse meson waves

$$6\pi\left(1+\frac{1}{2}\frac{\chi^2}{\omega_0^2}\right)\frac{1}{\left(a+\frac{1}{2}\frac{\chi^3}{\omega_0^2}\right)^2+\left(\omega_0^2-\frac{3}{4}\frac{\chi^4}{\omega_0^2}-\frac{1}{4}\frac{\chi^6}{\omega_0^4}\right)}, \tag{42}$$

and of longitudinal meson waves

$$6\pi\frac{\chi^2}{\omega_0^2}\left(1+\frac{1}{2}\frac{\chi^2}{\omega_0^2}\right)\frac{1}{\left(a+\frac{1}{2}\frac{\chi^3}{\omega_0^2}\right)^2+\left(\omega_0^2-\frac{3}{4}\frac{\omega_0^4}{\chi^2}-\frac{1}{4}\frac{\omega_0^6}{\chi^4}\right)}. \tag{43}$$

If we let $\chi\to 0$, the first just becomes $6\pi(a^2+\omega_0^2)^{-1}$ which is the generalization of the Thomson formula recently given by Dirac. On the other hand, the scattering cross-section of the longitudinal waves goes to zero as $\chi\to 0$ as we should expect. The relation of this theory to the quantum-mechanical cross-sections will be discussed below.

The sufficient condition for the correctness of the above cross-sections is that the maximum velocity acquired by the neutron shall always be small compared with unity, i.e. that $\beta \ll 1$. For longitudinal mesons the field varies in the direction of motion of the neutron, so that we have further to consider the approximation made in taking the field acting on the neutron as the field at the origin. This approximation will be a good one if the amplitude of the neutron β/ω_0 is small compared with the wave-length $(\omega_0^2-\chi^2)^{-\frac{1}{2}}$, in other words if $\beta \ll 1$.

The cross-section (42) in common with the corresponding cross-section given by Dirac has a very interesting property. For $\omega_0 \gg a \equiv 3M/2g^2$ it becomes proportional to ω_0^{-2}, since then a may be neglected. In other words, for extremely high frequencies the scattering ceases to depend either on the mass M or the change g of the neutron. It is clear that a behaviour of this sort cannot be approximated to by a series in ascending powers of g^2.

The above cross-sections neglect the effect of the spin $S_{\mu\nu}$ of the neutron, which was put equal to zero at the beginning. Its effect will be treated later. The effect of the spin term is to alter the angular distribution of the emitted mesons, but not to change the relative order of magnitude of the transverse and longitudinal cross-sections.

As we have already stated in the introduction, by treating a field classically instead of by quantum mechanics we neglect the momentum properties of the individual field quanta. Classical formulae will therefore be correct for those processes where the momenta of the field quanta concerned may be neglected. For the scattering that we have calculated above, this will be so if the momentum of a meson is small compared to M, the neutron mass. Hence the formulae (42) and (43) may be taken at once to describe the scattering of mesons by neutrons up to meson energies $\hbar\omega_0$ small compared to 10^9 eV. They will still give the correct order of magnitude for energies of 10^9 eV. They may be taken to supersede completely the formulae previously given by Heitler (1938) and myself, from which they differ by an order of magnitude. Our theory shows that the scattering of mesons is a small effect, the cross-section being of the order $8\pi g^4/3M^2 \sim 10^{-28}$ cm.², though scattering may be through large angles when it takes place.

The expressions (42) and (43) show another curious effect. Mesons of the same energies will have a very different scattering, dependent on their polarization. The scattering of longitudinally polarized mesons is less than that of transversely polarized ones by a factor χ^2/ω_0^2. When scattering does take place, however, the scattered mesons will be transversely polarized in most cases, so that their chance of being scattered again will be much greater.

Relation to the quantized meson field theory

We now wish to compare the results we have obtained above with those derivable from the quantized theory of uncharged and charged mesons. In the notation of a previous paper (Bhabha 1938) the interaction energy of neutral mesons with a neutron can be written in momentum space in the form

$$I_0 = \sum_p \frac{1}{\sqrt{V}\sqrt{(2E)}}\Bigg[-ig_1\Bigg\{\sum_{r=1,2}\alpha_{rp}(\overline{a_{rp}}-b_{rp})-\frac{p}{\mu}(\overline{a_{3p}}+b_{3p}) + \alpha_{3p}\frac{E}{\mu}(\overline{a_{3p}}-b_{3p})\Bigg\}\tau_3\exp\Bigg\{-\frac{i}{\hbar}(p,X)\Bigg\}+\text{conj. complex}\Bigg]. \tag{44}$$

This only differs from the corresponding expression (58*a*) in the above paper for charged mesons in having the isotopic operator τ_3 instead of τ_{NP} and τ_{PN}. For brevity we have omitted the g_2 term. The interaction (44) contains the mass of the meson μ, and has terms which increase proportionally to the momentum or energy of the meson. According to the prevalent view an interaction of this sort should lead for high energies to multiple processes and explosions of Heisenberg's type. This is however not so. The scattering of *neutral* mesons by neutrons calculated quantum-mechanically by Booth and Wilson* leads to a result which for meson energies small compared with the rest mass of the *neutron* agrees with the above cross-sections (42) and (43), if we neglect there the effects of damping, that is all the terms in square brackets except a. For higher energies the cross-sections *decrease* with increasing energy. This clearly demonstrates that interaction terms which increase with increasing energy of the particles, as in (44), are *not* sufficient to produce Heisenberg explosions. It becomes clear from our classical calculations and the above quantum-mechanical

* Dr Heitler has also calculated this result in the non-relativistic case.

result that the mass of the meson has nothing whatsoever to do with the fundamental length required by Heisenberg for the occurrence of explosions. It does not yet follow that no explosions involving neutral mesons will occur, for as we have pointed out in the introduction, the existence of the g_2 term introduces another length in the interaction independent of μ, and we have not yet shown that this will not lead to explosions.

One can also see that the mass of the meson will not lead to the occurrence of Heisenberg explosions by the general argument given by me in a previous note (1939). For according to the prevalent view the critical energy $\hbar\chi$ above which multiple processes become important is smaller the smaller χ. In the limiting form of our theory $\chi = 0$, which constitutes electrodynamics, all energies are therefore above the critical energy (which in this case is zero), and hence if this argument were correct, we would always expect explosions to occur. The well-known results of electrodynamics show that this argument is wrong, and that in the actual meson theory for energies large compared with χ we should expect the theory to behave progressively more like electrodynamics. This expectation is also confirmed by our treatment of the classical meson theory, which shows that for high frequencies $\omega_0 \gg \chi$ all processes approximate with increasing closeness to those on the classical Maxwell theory.

It is not possible to use the same argument for a charged meson field, for a limit to the case $\mu = 0$ is not then possible. This is due to the physically evident reason that a field with $\mu = 0$ must propagate itself with the velocity of light, and if such a field carried electric charge, the electromagnetic fields concerned would become infinite. In other words, a charge-bearing field cannot exist without having a fundamental length χ or a mass μ associated with it.

This is also put into evidence by the scattering of charged mesons by neutrons calculated in an earlier paper. In the non-relativistic case, with $g_2 = 0$, this cross-section may be written for longitudinal mesons in the form

$$\text{const.}\frac{g^4}{\mu^2}\frac{(\omega_0^2-\chi^2)^2}{\omega_0^2\chi^2}, \tag{45}$$

and is zero for transverse mesons, as has also been shown by Heitler (1938). (45) does not contain M but μ in the denominator. It is therefore much larger than (42) and (43), and goes to infinity as $\mu \to 0$. This behaviour is directly connected with the charge of the meson, for whereas on the present theory a positive meson may only be absorbed by a neutron and emitted by a proton, a neutral meson may be absorbed or emitted by either a neutron or a proton. There are thus twice as many intermediate states leading to the scattering of neutral mesons as of charged mesons, and these largely compensate each other, reducing the cross-section (45) to magnitudes of the order of (42) and (43).* Since however the *interaction* of charged mesons with neutrons and protons has no correspondence with any classical theory, I do not think it is yet possible to say to what extent this part of the theory is correct. The cross-section (45) is in any case too large to be reconciled with experiment.

The fundamental equations (26) have only two constants of the dimensions of a reciprocal length in them. Using the value of g given by $g^2/\hbar c \doteqdot \frac{1}{6}$, their magnitudes are

$$\left.\begin{aligned} a = \frac{3M}{2g^2} &\doteqdot 3\times 10^{14}\,\text{cm.}^{-1}, \\ \chi &\doteqdot 5\times 10^{12}\,\text{cm.}^{-1}. \end{aligned}\right\} \tag{46}$$

We see that with the actual magnitudes of the constants as they occur in nature

$$a \gg \chi.$$

Electromagnetic theory is the particular limit where $\chi = 0$. *Hence we should not expect the meson theory to differ much from electromagnetic theory for frequencies* $\omega_0 \gg \chi$. *Characteristic differences will become marked only for frequencies* $\omega_0 \lesssim \chi$. This is all borne out by our previous calculations and is contrary to the prevalent view. The formulae (42) and (43) show that the effects of the reaction of the emitted meson field on the motion of neutrons becomes important when $\omega_0 \gtrsim a$. Its effect is small for $\omega_0 \ll a$, for its neglect is equivalent to the neglect of all the terms in the square brackets in (42) and (43) except a. The energies concerned are very large, being

$$\hbar\omega_0 \gtrsim \hbar a \sim 10\,M \sim 10^{10}\,\text{eV}.$$

Since the quantized theory of the neutral meson field neglects this reaction, we have good reason for believing that it will be correct to a good approximation up to this energy, and only go wrong above it.

Appendix

To solve the equation (3) we first solve the equation

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} + \chi^2\right) G = \delta(x_0 - x_0')\,\delta(x_1 - x_1')\,\delta(x_2 - x_2')\,\delta(x_3 - x_3'). \tag{47}$$

* I am indebted to Professor Pauli for drawing my attention to this point in a letter.

The solution of this can be written in the form of a quadruple integral with respect to four variables p_0, p_1, p_2, p_3, and for brevity we shall write all formulae as if these constituted the components of a four-vector:

$$G(x_\rho, x'_\rho) = -\frac{1}{(2\pi)^4}\iiiint_{-\infty}^{\infty} dp_0\,dp_1\,dp_2\,dp_3\,\frac{e^{ip_\mu(x^\mu - x'^\mu)}}{p_\mu p^\mu - \chi^2}. \tag{48}$$

The denominator in (48) has two roots given by

$$p_0 = \pm\sqrt{(p^2+\chi^2)},$$

where $p = \sum_1^3 p_k^2$. We write $u_\mu = x_\mu - x'_\mu$ for brevity. In order that the solution shall not tend to infinity for small u, we choose the path of the p_0 integration to go from $-\infty$ to $+\infty$ and pass in the complex plane below both the singularities $\pm\sqrt{(p^2+\chi^2)}$, or above both of them. The p_1, p_2 and p_3 integrations go along the real axis from $-\infty$ to $+\infty$. In order to see that (48) is a solution, introduce it into the left-hand side of (47) and interchange the orders of integration and differentiation We get at once

$$\frac{1}{(2\pi)^4}\iiiint_{-\infty}^{\infty} dp_0\,dp_1\,dp_2\,dp_3\,e^{ip_\mu(x^\mu - x'^\mu)} = \delta(x_\rho - x'_\rho),$$

which proves the result. G is the Green's function of this equation.

To evaluate G, we first introduce polar co-ordinates for p_k and carry out the angular integrations.

$$G = \frac{1}{(2\pi)^3}\frac{1}{u_r}\frac{\partial}{\partial u_r}\iint_0^\infty dp\,dp_0\,\frac{e^{ip_0u_0 + ipu_r}}{p_0^2 - p^2 - \chi^2}.$$

Now carrying out the p_0 integration, we are left with the residues at the two poles $\pm\sqrt{(p^2+\chi^2)}$,

$$G = \frac{i}{(2\pi)^2}\frac{\partial}{u_r\partial u_r}\left[\int_0^\infty dp\,\frac{e^{iu_0\sqrt{(p^2+\chi^2)}+iu_rp}}{\sqrt{(p^2+\chi^2)}} - \text{conj. complex}\right],$$

which, writing $p = \chi\sinh q$, becomes

$$-\frac{1}{(2\pi)^2}\frac{\partial}{u_r\partial u_r}\int_{-\infty}^{\infty} dq\sin\{\chi\sqrt{(u_0^2-u_r^2)}\cosh q\} \quad \text{if } u_0 > |u_r|.$$

If $|u_0| < |u_r|$, we get $\sinh q$ instead of $\cosh q$, and the integral vanishes. For $u_0 > |u_r|$ the integral is just equal to $\pi J_0\{\chi\sqrt{(u_0^2-u_r^2)}\}$.

If $u_0 < 0$, the path of the p_0 integration in (48) can be deformed into an infinite semicircle below the real axis with centre at the origin, and hence G vanishes. We thus arrive at the result

$$G(x_\rho, x'_\rho) = -\frac{1}{4\pi}\frac{\partial}{u_r \partial u_r} F, \tag{48a}$$

with

$$F = \begin{pmatrix} J_0\{\chi\sqrt{(u_0^2-u_r^2)}\} \\ 0 \\ 0 \end{pmatrix} \begin{matrix} u_0 > u_r \\ |u_0| < u_r \\ u_0 < -u_r. \end{matrix} \tag{48b}$$

If we choose the path of the p_0 integration in (48) to run *above* both the singularities $\pm\sqrt{(p^2+\chi^2)}$, we obtain the Green's function (48*a*) with F given by

$$F = \begin{pmatrix} 0 \\ 0 \\ J_0\{\chi\sqrt{(u_0^2-u_r^2)}\} \end{pmatrix} \begin{matrix} u_0 > u_r \\ |u_0| < u_r \\ u_0 < -u_r. \end{matrix} \tag{48c}$$

The differentiation with respect to u_r give the δ-functions at $u_0 = u_r$ due to the discontinuities in F. Our final result can then be put in the form given in the text.

In differentiating U_ν with respect to the co-ordinates, we notice the following points.

$$s_\mu = x_\mu - z_\mu(\tau).$$

Hence, when s_μ occurs inside an integral,

$$\frac{\partial s_\mu(\tau)}{\partial x_\nu} = \delta^\nu_\mu \tag{49}$$

and

$$\frac{\partial s(\tau)}{\partial x_\nu} = \frac{s^\nu}{s}.$$

When it occurs outside an integral

$$\frac{\partial s_\mu(\tau_0)}{\partial x_\nu} = \delta^\nu_\mu - v_\mu(\tau_0)\frac{\partial \tau_0}{\partial x_\nu}.$$

Since

$$s_\mu(\tau_0)\, s^\mu(\tau_0) = 0,$$

$$s^\mu(\tau_0)\frac{\partial s_\mu(\tau_0)}{\partial x_\nu} = 0.$$

Hence

$$\frac{\partial \tau_0}{\partial x_\nu} = \frac{s^\nu}{\kappa}, \tag{50}$$

where

$$\kappa \equiv s_\mu v^\mu.$$

Similarly
$$\frac{\partial v_\mu(\tau_0)}{\partial x_\nu} = \dot{v}_\mu \frac{\partial \tau_0}{\partial x_\nu} = \frac{\dot{v}_\mu s^\nu}{\kappa},$$
$$\frac{\partial \kappa(\tau_0)}{\partial x_\nu} = v^\nu - \frac{s^\nu}{\kappa}(1-\kappa'),$$
with
$$\kappa' = s_\mu \dot{v}^\mu.$$

In differentiating the integral in (9) it should be remembered that we get a contribution from the change in s_μ according to (49), the limits being kept fixed, and a contribution due to the change of the upper limit τ_0 according to (50).

We now calculate the flow of energy and momentum out of the world tube. This is given by
$$-\int T_{\mu\nu}\, dS^\nu,$$
where dS^ν are the components of an element of the three-dimensional surface of the world tube. The world tube is determined by
$$\left.\begin{aligned} s_\mu s^\mu &= 0. \\ \kappa = s_\mu v^\mu &= \epsilon. \end{aligned}\right\} \tag{51}$$

Hence, for variations dx_μ on the surface,
$$\left.\begin{aligned} s^\mu dx_\mu &= s^\mu v_\mu d\tau = \epsilon\, d\tau, \\ v^\mu dx_\mu &= (1-\kappa')\, d\tau. \end{aligned}\right\} \tag{52}$$

Multiplying the first equation by $(1-\kappa')$ and the second by ϵ and subtracting
$$\{s^\mu(1-\kappa') - v^\mu \epsilon\}\, dx_\mu = 0.$$

Hence, $s_\mu(1-\kappa') - v_\mu \epsilon$ is the normal to the world tube. Its absolute length is $\epsilon\sqrt{(1-2\kappa')}$. The normalized normal is therefore
$$N_\mu = \frac{s_\mu(1-\kappa') - v_\mu \epsilon}{\epsilon\sqrt{(1-2\kappa')}}. \tag{53}$$

The surface of the world tube at any point can be determined by three vectors $\epsilon^{(1)}$, $\epsilon^{(2)}$ and $\epsilon^{(3)}$. Two, say $\epsilon^{(1)}$ and $\epsilon^{(2)}$, can be taken as pure space vectors lying in the surface of the two-dimensional sphere in the rest system, i.e. the system in which the retarded point is instantaneously at rest. The space components of $\epsilon^{(3)}$ are then along the normal to the sphere, i.e. proportional to s_k. The time component is determined by (52). In the rest system v_μ has the components (1, 0, 0, 0) and therefore $s_0 = \epsilon$. Hence
$$\epsilon_0^{(3)} = (1-\kappa')\, d\tau.$$

It is also normal to N_μ, so that

$$-s^0\kappa'(1-\kappa')\,d\tau + s^k(1-\kappa')\,\epsilon^{(3)}_k = 0,$$

i.e.

$$\epsilon^{(3)}_k = -\frac{s_k}{\epsilon}\kappa' d\tau.$$

The length of $\epsilon^{(3)}$ is $\surd(1-2\kappa')\,d\tau$. Thus an element of volume corresponding to a displacement $d\tau$ of the retarded point is

$$dS = \surd(1-2\kappa')\,d\tau\,d\sigma,$$

where $d\sigma$ is an element of the surface of the sphere round the retarded point. It finally follows that

$$dS^\nu = N^\nu dS = \{s_\mu(1-\kappa') - v_\mu \epsilon\}\,\epsilon\,d\tau\,d\omega, \tag{54}$$

$d\omega$ being an element of solid angle in the rest system.

From (12) and (13), on the world tube

$$-\frac{1}{g}G^0_{\mu\nu} = \frac{s_\mu v_\nu - s_\nu v_\mu}{\epsilon^3}, + \frac{s_\mu \dot{v}_\nu - s_\nu \dot{v}_\mu}{\epsilon^2} - \frac{s_\mu v_\nu - s_\nu v_\mu}{\epsilon^3}\kappa' \tag{55}$$

and

$$-\frac{1}{g}G^\chi_{\mu\nu} = -\frac{\chi^2}{2}\frac{s_\mu v_\nu - s_\nu v_\mu}{\epsilon} + \chi^2\int_{-\infty}^{\tau_0} d\tau\,\frac{s_\mu v_\nu - s_\nu v_\mu}{s^2}J_2(\chi s), \tag{56}$$

where we have separated terms of different order in ϵ by a comma. The first term in (55) is of the order ϵ^{-2}, the other two being of order ϵ^{-1}. $G^\chi_{\mu\nu}$ is of order 1. dS^ν is of order ϵ^2. The energy tensor (14) may be written

$$T_{\mu\nu} = T^0_{\mu\nu} + T^{\text{mix}}_{\mu\nu} + T^{\text{in}}_{\mu\nu},$$

where $T^0_{\mu\nu}$ is the tensor which only consists of $G^0_{\mu\nu}$ and U^0_ν. Since $G^\chi_{\mu\nu}$ is of order 1, it may be treated together with $G^{\text{in}}_{\mu\nu}$. $T^{\text{in}}_{\mu\nu}$ when integrated over the tube is of order ϵ^2 and may be neglected. In $T^{\text{mix}}_{\mu\nu}$, we need only consider the first term of order ϵ^{-2} in (55), and neglect the κ' term in (54), since these give contributions of higher order.

Now from (55) and (54)

$$-\frac{1}{g}G^0_{\mu\nu}dS^\nu = \{v_\mu(1-\kappa') + \epsilon\dot{v}_\mu\}\,d\omega\,d\tau. \tag{57}$$

Therefore

$$\frac{1}{g^2}G^0_{\mu\sigma}G^{0\sigma}{}_\nu dS^\nu = \left[\frac{s_\mu - \epsilon v_\mu}{\epsilon^3},\ -2\frac{s_\mu - \epsilon v_\mu}{\epsilon^3}\kappa' - \frac{v_\mu}{\epsilon^2}\kappa' - \frac{\dot{v}_\mu}{\epsilon},\ +\frac{s_\mu}{\epsilon^3}\kappa'^2 + \frac{s_\mu \dot{v}^2}{\epsilon}\right]d\omega\,d\tau,$$

$$\frac{1}{g^2}G_{\sigma\rho}\,G^{\sigma\rho} = -\frac{2}{\epsilon^4}.$$

For the purposes of integration we note the following formulae, which can be derived easily in the rest system, and then written at once in tensor form. A_μ is any tensor not a function of position on the sphere:

$$\left.\begin{aligned}
&\frac{1}{4\pi}\int (s_\mu - \epsilon v_\mu)\, d\omega = 0,\\
&\frac{1}{4\pi}\int s_\mu A^\mu d\omega \;\; = A_0 = v_\mu A^\mu,\\
&\frac{1}{4\pi}\int s_\mu s_\nu A^\nu d\omega = \begin{cases} v_\nu A^\nu & \text{for} \quad \mu = 0,\\ -\frac{1}{3}A_k & \text{for} \quad \mu = k\end{cases}\\
&\qquad\qquad\qquad = -\tfrac{1}{3}A_\mu + \tfrac{4}{3}v_\mu v_\nu A^\nu.
\end{aligned}\right\} \tag{58}$$

With the help of these

$$-\frac{1}{4\pi}\int \{G^0_{\mu\sigma}G^{0\sigma}{}_\nu + \tfrac{1}{4}g_{\mu\nu}G^0_{\rho\sigma}G^{0\rho\sigma}\}\, dS^\nu = g^2\left[\frac{1}{2}\frac{\dot{v}_\mu}{\epsilon} - \tfrac{2}{3}v_\mu \dot{v}^2\right] d\tau. \tag{59}$$

In calculating the flow due to $T^{\text{mix}}_{\mu\nu}$ we notice that the first term in (56) will contribute nothing. For this could only give a contribution when multiplied by the first term of (55). But the first term of (55) when multiplied by itself should have given a contribution of order ϵ^{-2} to (59). (59) shows that this term vanishes. Since the first term in (56) is just $-\chi^2\epsilon^2/2$ times the first term of (55), it will give no contribution to the energy flow. The second term in (56) is continuous everywhere and hence may be just considered as a part of $G^{\text{in}}_{\mu\nu}$. For brevity we will not write it explicitly in the calculations and add it in the final formula. Using (54), (55) and (57)

$$\begin{aligned}
&-\frac{1}{4\pi}\int \{G^0_{\mu\sigma}G^{\text{in}\,\sigma}{}_\nu + G^{\text{in}}_{\mu\sigma}G^{0\,\sigma}{}_\nu\}\, dS^\nu\\
&\quad = g\int \frac{s_\mu v_\sigma - s_\sigma v_\mu}{\epsilon^2}\, G^{\text{in}\,\sigma}{}_\nu (s_\nu - v_\nu \epsilon)\, d\omega\, d\tau + g\int G^{\text{in}}_{\mu\sigma}v^\sigma d\omega\, d\tau\\
&\quad = g[-\tfrac{1}{3}v_\sigma G^{\text{in}\,\sigma}{}_\mu + G^{\text{in}}_{\mu\sigma}v^\sigma]\, d\tau\\
&\quad = \tfrac{4}{3}g\, G^{\text{in}}_{\mu\sigma}v^\sigma d\tau
\end{aligned} \tag{60}$$

Next,

$$\begin{aligned}
&-\frac{1}{4\pi}\int \tfrac{1}{2}g_{\mu\nu}\{G^0_{\rho\sigma}\, G^{\text{in}\rho\sigma}\}\, dS^\nu\\
&\quad = \frac{g}{8\pi}\int \frac{s_\rho v_\sigma - s_\sigma v_\rho}{\epsilon^2}\, G^{\text{in}\rho\sigma}(s_\mu - \epsilon v_\mu)\, d\omega\, d\tau\\
&\quad = -\tfrac{1}{3}g\, G^{\text{in}}_{\mu\sigma}v^\sigma d\tau.
\end{aligned} \tag{61}$$

We have still to consider the U_μ terms in (26). The first term of (9) is of the order ϵ^{-1}, so that only the product of this term with itself will make a contribution to the energy flow:

$$-\frac{\chi^2}{4\pi}\int U_\mu U_\nu dS^\nu = -g^2\frac{\chi^2}{4\pi}\int\frac{v_\mu v_\nu}{\epsilon}(s^\nu - v^\nu\epsilon)\,d\omega\,d\tau = 0,$$

$$\frac{\chi^2}{4\pi}\int\tfrac{1}{2}g_{\mu\nu}U_\rho U^\rho dS^\nu = g^2\frac{\chi^2}{8\pi}\int\frac{s_\mu - \epsilon v_\mu}{\epsilon}d\omega\,d\tau = 0.$$

Adding (59), (60) and (61), we finally find for the energy flow out of a length $d\tau$ of the tube

$$\left[g^2\left(\frac{1}{2}\frac{\dot v_\mu}{\epsilon} - \tfrac{2}{3}v_\mu \dot v^2\right) + g(\tilde G^\chi_{\mu\sigma} + G^{\text{in}}_{\mu\sigma})\,v^\sigma\right]d\tau,$$

which is the result quoted in (19).

We now derive the expressions for the potentials U_μ at a large distance R due to a neutron oscillating according to

$$z_1 = \frac{\beta}{\omega_0}\sin\omega_0 t;\quad z_2 = z_3 = 0.$$

For this purpose it is convenient to write the expression (9) for the potential U_ν by partial integration in the form

$$U_\nu = g\int_{-\infty}^{\tau_0} d\tau\, J_0(\chi s)\frac{d}{d\tau}\left(\frac{v_\nu}{\kappa}\right). \tag{62}$$

To the first order in β $\quad s^2 \fallingdotseq (T-t)^2 - R^2,$

$$\kappa \fallingdotseq (T-t) - X\beta\cos\omega_0 t,$$

Therefore $\quad \dfrac{d}{d\tau}\left(\dfrac{v_1}{\kappa}\right) \fallingdotseq -\dfrac{\beta\omega_0\sin\omega_0 t}{T-t} + O\left(\dfrac{1}{(T-t)^2}\right) + O(\beta^2),$

$$\frac{d}{d\tau}\left(\frac{v_0}{\kappa}\right) = -\frac{\beta\omega_0 X\sin w_0 t}{(T-t)^2}.$$

Hence from (62), neglecting higher orders,

$$U_1 = g\int_0^\infty ds\frac{s}{(s^2+R^2)^{\frac{1}{2}}}J_0(\chi s)\left\{-\frac{\beta\omega_0\sin\omega_0 t}{(s^2+R^2)^{\frac{1}{2}}}\right\}$$

$$= -g\frac{\beta\omega_0}{2i}[e^{i\omega_0 T}I - \text{conj. complex}],$$

where $\quad I = \displaystyle\int_0^\infty ds\frac{s}{s^2+R^2}J_0(\chi s)\,e^{-i\omega_0\sqrt{(s^2+R^2)}}$

$$= \int_{i\omega_0}^{\infty+i\omega_0} d\omega_0\int_0^\infty ds\frac{s}{\sqrt{(s^2+R^2)}}J_0(\chi s)\,e^{-\omega_0\sqrt{(s^2+R^2)}}.$$

Now by a well-known theorem of Bessel functions (see Watson)

$$\int_0^\infty ds \frac{s}{\sqrt{(s^2+R^2)}} J_0(\chi s)\, e^{-\omega_0\sqrt{(s^2+R^2)}} = \frac{e^{-R\sqrt{(\chi^2+\omega_0^2)}}}{\sqrt{(\chi^2+\omega_0^2)}}.$$

Therefore

$$I = \int_{i\omega_0}^{\infty+i\omega_0} d\omega_0 \frac{e^{-R\sqrt{(\chi^2+\omega_0^2)}}}{\sqrt{(\chi^2+\omega_0^2)}} = \begin{cases} \dfrac{e^{-R\sqrt{(\chi^2-\omega_0^2)}}}{i\omega_0 R} & \text{for } |\omega_0| < \chi, \\ \dfrac{e^{-iR\sqrt{(\omega_0^2-\chi^2)}}}{i\omega_0 R} & \text{for } |\omega_0| > \chi, \end{cases}$$

neglecting higher powers in $1/R$.

Hence, at large distances

$$U_1 = \begin{cases} \dfrac{g\beta}{R} e^{-R\sqrt{(\chi^2-\omega_0^2)}} \cos(\omega_0 T) & |\omega_0| < \chi, \\ \dfrac{g\beta}{R} \cos\{\omega_0 T - R\sqrt{(\omega_0^2-\chi^2)}\} & |\omega_0| > \chi, \end{cases}$$

which is the result quoted in the text.

We similarly find

$$U_0 = g\int_0^\infty ds \frac{s}{(s^2+R^2)^{\frac{1}{2}}} J_0(\chi s) \left\{-\frac{\beta\omega_0 X \sin\omega_0 t}{s^2+R^2}\right\}.$$

Here we have to integrate twice with respect to ω_0. After some easy analysis we get the result quoted in the text in (37).

Summary

The vector equations for the meson field and their associated energy tensor are taken as exact, and the meson-field quantities are taken as commuting classical variables. A self-consistent classical scheme is developed for treating the meson-field and point neutrons (or protons) moving along classical world lines. The scheme takes account of the reaction of the emitted meson field on the motion of the neutron. The mass of the neutron (or proton) which occurs in nuclear phenomena is shown not to be the real mass but less than it by about ten million volts. For high-energy phenomena the real mass counts. Formulae are given for the scattering of mesons by neutrons which are valid to energies comparable with the mass of the neutron, and supersede the previous formulae. These show that for the same high energies, longitudinally polarized mesons are scattered much less than transversely polarized mesons, giving them much greater penetrating power. It is shown that classically the reaction of the emitted radiation is not important till energies of ten times the neutron mass and hence that the quantized neutral

meson theory which neglects this is valid up to about these energies. It is proved that the rest mass of the meson is not connected with Heisenberg explosions, but that if these exist, they must be due either to the spin interaction g_2 or to the fact that the meson field is electrically charged.

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