Cooperative Ring Exchange and
Quantum Melting of Vortex Lattices in Atomic Bose-Einstein Condensates

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Cooperative ring-exchange is suggested as a mechanism of quantum melting of vortex lattices in a rapidly-rotating quasi two dimensional atomic Bose-Einstein condensate (BEC). Using an approach pioneered by Kivelson *et al.* [Phys. Rev. Lett. **56**, 873 (1986)] for the fractional quantized Hall effect, we calculate the condition for quantum melting instability by considering large-correlated ring exchanges in a two-dimensional Wigner crystal of vortices in a strong 'pseudomagnetic field' generated by the background superfluid Bose particles. BEC may be profitably used to address issues of quantum melting of a pristine Wigner solid devoid of complications of real solids.

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I. INTRODUCTION

The creation and observation of the triangular vortex lattices in a rapidly-rotating atomic Bose-Einstein condensate (BEC) [1–3] has opened a new direction for the study of quantum vortex matter. Theoretical predictions [4] for the existence of fractional quantum Hall like states at even higher rotational speeds in quasi two dimensional atomic BEC has given a further impetus to this fascinating field. The quantum melting of an ordered vortex lattice to an exotic quantum fluid of atoms at very low temperatures is a quantum phase transition, where one would like to understand the mechanism of melting and nature of phase transition.

Melting of classical solids with short range inter atomic potential in 2D is a well studied subject, where topological defects play a fundamental role. In the presence of long range interaction, such as one component coulomb plasma in 2D, melting is dominated by ring exchanges [5] rather than topological defects. From this point of view the logarithmic repulsion among the imposed vortices in a rotating BEC provides an opportunity to study quantum melting of a 'pristine' Wigner solid with long range forces, that is free from the complications of solid state systems.

In this letter we write down an effective Hamiltonian for the vortex degrees of freedom, motivated by an analogy [6,7] between the Magnus force acting on a vortex moving on a two dimensional neutral superfluid fluid and the Lorenz force acting on a charged particle in a magnetic field. We develop a theoretical approach, borrowing heavily from pioneering ideas of Kivelson, Kallin, Arovas and Schrieffer (KKAS) [8], developed in the context of fractional quantized Hall effect (FQHE), and suggest a cooperative ring exchange (CRE) mechanism for quantum melting of vortex lattices in quasi 2D atomic BEC and indicate a possible direction for a microscopic understanding of the quantum liquid of molten vortices.

In contrast to many recent theoretical works on atomic BEC which exploits an analogy between the Hamiltonian of a rotating neutral boson atoms and charge particle in an external magnetic field in two dimensions, our work uses the vortex (collective) coordinates directly and provides another microscopic approach to understand quantum melting and the quantum Hall-like state that may be formed in these atomic system. Existing theoretical works focus on exact diagonalization [9] of small number of atoms to get some idea about quantum melting and the possible quantum Hall like melted states. A recent interesting work [10] that studies melting of vortex lattices in a rapidly-rotating 2D BEC, also shows that BEC is destroyed by the vortex lattices.

Experimentally, at the present moment it is a challenging task to produce vortex liquid state in a rapidly-rotating atomic BEC, in contrast to the formation of an incompressible liquid state of electrons in a high magnetic field at higher filling fractions. With the rapid advances in the field of laser cooled atomic gases one can anticipate to get 'snapshots' of the melted configurations of the vortex lattices, where CRE should leave its unique signatures as we mention at the end.

In the cooperative ring exchange approach to FQHE, KKAS view the Laughlin quantum Hall state as a Wigner solid of electrons in 2D in strong magnetic field, that has been quantum melted by cooperative ring exchange processes. Briefly, ring exchange, as the name suggests, is a cooperative shift of a ring of contiguous particles in an ordered lattice (figure 1) resulting in a cyclic permutation within the ring. While the amplitude for a quantum tunneling event of a specific ring of size L sites is exponentially small $\sim \alpha^L$ (with α, the single particle tunneling amplitude being < 1), the number of rings of size L is exponentially large $\sim e^{bL}$. Thus the total amplitude $\sim \alpha^L e^{bL}$ may exponentially diverge, if $-\ln \alpha < b$, leading to a proliferation of ring exchanges and a consequent quantum melting.

This melting depends on the electronic filling fraction, the ratio between the density of conduction electrons and the density of flux quanta. For very low filling fraction, electrons are expected to form a Wigner crystal. At higher filling fraction, electrons forms an incompressible liquid state and exhibits quantized Hall effect. Similarly, we could also expect the quantum melting of the vortex lattices depends on the vortex filling factor, the ratio of the total number of vortex to the total number of boson.

II. HAMILTONIAN OF THE VORTICES IN A ROTATING QUASI-2D BEC

We consider a large number of vortices in a rapidly rotating quasi-2D BEC; the condition for the condensate in a quasi-2D trap is $\mu = \rho_0 g_2 < \hbar \omega_z$ and the atoms in the condensate should be in the atomic lowest Landau level generated by the fast external rotation is $\rho_0 g_2 < 2\hbar\Omega$. Here, ρ_0 is the boson density, $g_2 = 2\sqrt{2\pi}\hbar\omega_z a_z a$ is the effective interaction strength in quasi-2D Bose system [11], where $a_z = \sqrt{\frac{\hbar}{m_o \omega_z}}$ is harmonic oscillator length along the z-direction with ω_z is trap frequency in the axial (z) direction and Ω is the trap rotational frequency. Also, a is the s-wave atomic scattering length. A vortex in a fluid is an excitation in which each fluid particle is given an angular momentum m relative to the vortex center. Here, we treat a vortex as a point particle moving under the influence of the Magnus force. The Magnus force is an effective interaction between superfluid particles and vortices in relative motion [6,7]. The force acting on a single vortex [6] is then

$$\mathbf{F} = \mathbf{v} \times \hat{z}(2\pi\hbar\rho_0). \tag{1}$$

Here, $\mathbf{v}$ is the vortex velocity relative to the superfluid particles and ρ_0 is the superfluid particle density. The Magnus force is equivalent to the Lorentz force acting on a charge particle (e) in a magnetic field. Hence, $eB_{\text{eff}} = 2\pi\hbar\rho_0$ is the pseudo magnetic field.

The interaction potential between two vortices separated by a distance r is

$$V(r) = -\frac{2\pi\hbar^2\rho_0}{m_0}\ln\left(\frac{r}{\xi}\right), \tag{2}$$

where $\xi \sim \sqrt{\frac{a_z}{\rho_0 a}}$ is the coherence length of the vortex core and m_0 is the mass of a superfluid particle. The above potential is valid only when the distances between two vortices is greater than the coherence length. Notice that the interaction strength between two vortices depends on the superfluid density as well as the s-wave scattering length a.

The Hamiltonian of a rotating BEC containing vortices can be written in terms of center of vortices (collective coordinate) as [6]

$$H_v = \sum_{i=1}^{N_v} \frac{(\mathbf{p}_i - \pi\hbar\rho_0\hat{z} \times \mathbf{r}_i)^2}{2m_v} - \frac{2\pi\hbar^2\rho_0}{m_0}\sum_{i<j}\ln\left[\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\xi}\right], \tag{3}$$

where N_v is the total number of vortices. The effective vortex mass $m_v = \pi\rho_0\xi^2$ can be in principle derived from a microscopic approach [6]. Since the coherence length is very small, the vortex mass also becomes small. This Hamiltonian is similar to that of a charged particles moving under the influence of the Lorentz force by a magnetic field B_{eff}. The pseudo vector potential due to the Magnus force is $\mathbf{A}_{\text{eff}} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}_{\text{eff}}$. For N_v number of vortices in an area A, one gets the vortex filling factor, $\nu_v = \frac{N_v}{A}\frac{h}{eB_{\text{eff}}} = \frac{N_v}{N}$. Notice that the vortex filling factor ν_v is just inverse of the bosonic filing factor $\nu_b = \frac{N}{N_v}$. For large N_v the vortex density is approximately uniform and $N_v = \frac{2m\omega A}{\hbar}$. N is the number of the superfluid particles. The effective magnetic length is $l_0 = \sqrt{\frac{\hbar}{eB_{\text{eff}}}} = \frac{1}{\sqrt{2\pi\rho_0}}$. The pseudomagnetic field generated by the background superfluid particles leads to the quantization of the cyclotron motion and producing Landau levels of the vortices. The eigen spectrum of the single vortex Hamiltonian are uniformly spaced with energy gap $\hbar\omega_{\text{eff}}$, where $\omega_{\text{eff}} = \frac{2\pi\hbar\rho_0}{m_v}$ is the effective cyclotron frequency. The limit of $m_v \rightarrow 0$ and/or large superfluid density (ρ_0) is equivalent to the vortices are in the lowest Landau level (LLL). We can project the Hamiltonian onto the LLL and the corresponding normalized wave functions are degenerate eigenfunctions of the angular momentum m is,

$$\psi(z) = \frac{1}{\sqrt{(2l_0^2)^{m+1}\pi m!}}z^m e^{-\frac{|z|^2}{4l_0^2}}, \qquad m = 0, 1, 2, ... \tag{4}$$

where $z = x + iy$ and (x, y) are the position coordinate of a vortex. When the vortices are confined to the lowest Landau level,(i.e. the cyclotron degrees of freedom are confined to the LLL), the kinetic degrees of freedom of the vortices are frozen, since the spacing between Landau levels, $\hbar\omega_{\text{eff}}$, is large compared with all other energies in the problem. The remaining degrees of freedom are the vortex guiding-center coordinates, $\mathbf{R} = \frac{\mathbf{r}}{2} + \frac{l_0^2}{\hbar}(\mathbf{p} \times \hat{z})$. The guiding center coordinate $\mathbf{R}$ specify the center of a Gaussian-localized probability amplitude of width l_0. These coordinates have no kinetic energy. Hence the vortices in the LLL will remain localized about a given guiding center coordinate $\mathbf{R}$ indefinitely.

III. COHERENT STATE PATH INTEGRATION

In this section, we would like to review the coherent state path integral formalism. For detailed derivations,

please consult the references [12,13]. In symmetric gauge, the wave function of a vortex in the LLL with guiding-center position $\mathbf{R}$ is

$$\phi_{\mathbf{R}}(\mathbf{r}) = \frac{1}{\sqrt{2\pi l_0^2}} exp \left[-\frac{|\mathbf{r} - \mathbf{R}|^2}{4l_0^2} + \frac{i(\mathbf{r} \times \mathbf{R}).\hat{z}}{2l_0^2} \right]. \quad (5)$$

It has the same form as a coherent state in a two-dimensional phase space [12]. Here, the state label $\mathbf{R}$ is a continuous variable. The coherent state overlap is given by,

$$< \mathbf{R}_1 | \mathbf{R}_2 > = exp \left[-\frac{|\mathbf{R}_1 - \mathbf{R}_2|^2}{4l_0^2} + \frac{i(\mathbf{R}_1 \times \mathbf{R}_2).\hat{z}}{2l_0^2} \right]. \quad (6)$$

This coherent state $|\mathbf{R} >$ forms a nonorthogonal, over-complete basis. Nevertheless, the projection operator P onto the LLL is given by,

$$P = \int \frac{d^2 R}{2\pi l_0^2} |\mathbf{R} > < \mathbf{R}| \quad (7)$$

which is unity within the LLL since $< \mathbf{R}_1 | P | \mathbf{R}_2 > = < \mathbf{R}_1 | \mathbf{R}_2 >$.

We use the coherent state path integral [12,13] expression for the partition function to calculate the tunneling coefficient of a vortex. The partition function for $2D$ interacting vortices in a pseudomagnetic field due to the Magnus force is

$$Z(\nu_v) = Tr e^{-\beta H_v}. \quad (8)$$

Here, we discuss the main features of this formalism for a single vortex in the LLL in the complex plane. This can be generalized for many vortex system very easily. The coherent state in the complex plane is

$$|R > = \frac{1}{\sqrt{2\pi l_0^2}} exp \left[\frac{zR^* - z^* R}{4l_0^2} \right] exp \left[-\frac{|z - R|^2}{2l_0^2} \right], \quad (9)$$

where $R = X + iY$ is the guiding center coordinate of a vortex in the complex plane and the asterisk denotes the complex conjugation. The coherent state overlap in the complex plane is

$$< R_j | R_k > = exp \left[(\frac{R_j R_k^* - R_j^* R}{4l_0^2}) - \frac{|R_j - R_k|^2}{2l_0^2} \right] \quad (10)$$

$$= exp \left[-\frac{[R_j^*(R_j - R_k) - (R_j^* - R_k^*)R_k]}{4l_0^2} \right]. \quad (11)$$

Now the path integral representation of the partition function $Z(\nu_v)$ can be obtained in the usual way. First, we split the inverse temperature β into a large number of equal intervals $\epsilon = \beta/n$, i.e., $e^{-\beta V}$ is written as $[e^{-\epsilon V}]^n$, and then insert the projection operator P at each infinitesimally small interval. Then,

$$< R_f | e^{-\beta V} | R_i > = \int \prod_{k=1}^{n} \frac{d^2 R_k}{(2\pi l_0^2)^n} \prod_{j=0}^{n} < R_{j+1} | e^{-\epsilon V} | R_j >, \quad (12)$$

where $R_0 = R_i, R_{n+1} = R_f$. In general, the Hamiltonian can be written

$$V_{j,k} = V(R_j^*, R_k) = \frac{< R_j | V | R_k >}{< R_j | R_k >}. \quad (13)$$

The matrix element can be written as,

$$< R_f | e^{-\beta H} | R_i > = \int \prod_{k=1}^{n} \frac{d^2 R_k}{(2\pi l_0^2)^n} \prod_{j=0}^{n} < R_{j+1} | R_j >$$
$$\times \left[1 - i\epsilon V(R_{j+1}^*, R_j) + O(\epsilon^2) \right] \quad (14)$$

We are neglecting terms of $O(\epsilon^2)$ and higher order terms by standard procedure. Using Eq.(10), and

$$\frac{dR_j}{dt} = \frac{R_{j+1} - R_j}{\epsilon} \quad (15)$$

we obtain

$$Z(\nu_v) = \int \prod_{k=1}^{n} \frac{d^2 R_k}{(2\pi l_0^2)^n} exp \left[i\epsilon \sum_{j=0}^{n} \frac{i}{4l_0^2} S_j \right], \quad (16)$$

where

$$S_j = \left(R_j^* \frac{dR_j}{dt} - \frac{dR_j^*}{dt} R_{j+1} \right) - V(R_j^*, R_{j+1}). \quad (17)$$

The above path integral can be written as

$$Z = \int D[\mathbf{R}] e^{iS[\mathbf{R}]}, \quad (18)$$

where

$$S = \int_0^\beta dt \left[\frac{1}{4l_0^2} \left(R^* \frac{dR}{dt} - \frac{dR^*}{dt} R \right) + V(R^*, R) \right]. \quad (19)$$

This action is linear in time derivatives and hence discontinuous paths have finite action. It implies that the coherent state path integral is dominated by discontinuous paths and the limits is ill defined. Despite these difficulties, the continuum version of the path integral can be used to develop a saddle-point approximation for the partition function [8]. We are interested in the semi-classical limit when $V(R)$ is a slowly varying function of its argument over the length scale l_0 and we can use the saddle point approximation to evaluate the path integral.

The single vortex path integral can be generalized to many vortex path integral directly. The action for many vortex is

$$S(\mathbf{R}) = \int_0^\beta dt \left[\frac{1}{2} \sum_{j=1}^{N_v} \left(\dot{\mathbf{R}}_j \times \mathbf{R}_j \right) . \hat{z} + \sum_{j<k} V \left(\mathbf{R}_j - \mathbf{R}_k \right) \right],$$

(20)

where

$$V(R) = \langle \phi_{\mathbf{R}}(\mathbf{r}) | V(\mathbf{r}) | \phi_{\mathbf{R}}(\mathbf{r}) \rangle \qquad (21)$$

is two-body interaction potential in coherent states representation.

In the saddle point-approximation, the classical path is obtained by minimizing the action, $\frac{\delta S}{\delta R_j(\tau)}|_{R=R_c} = 0$. The classical paths satisfy the following equations of motion

$$\dot{\mathbf{R}}_j = \frac{l_0^2}{\hbar} (\nabla_j V_j) \times \hat{z}, \text{ where } V_j = \sum_{k \neq j} V(\mathbf{R}_j - \mathbf{R}_k). \quad (22)$$

The path integral can be expressed as a sum over saddle point contribution in which the contribution of paths in the neighborhood of each classical path is evaluated by expanding the action to quadratic order in $R - R_c$. The partition function Z is calculated within the semiclassical approximation. The partition function can be expressed as a sum over classical paths, assuming the vortices to be bosons, $Z = \sum_c D[R_c] e^{-S[R_c]}$, where $D[R_c]$ is the fluctuation determinant and $S[R_c]$ is the action evaluated along the classical path. There are interesting issues [14] about the statistics of vortices in a compressible superfluid such as ours, and in the present paper we assume them to be bosons. The partition function Z can be written by considering only the leading order contribution as [8] $Z = Z_0 \sum_c \tilde{D}[R_c] e^{-S_0[R_c]}$, where $S = S_0 + \tilde{S}$. Let us consider the contribution of a single large exchange ring to Z. The real part of the action would be $\alpha_0 L$ (see the sec. V), where L is the number of vortices in the ring and α_0 is independent of path. The fluctuation determinant [8] is $\tilde{D}[R_c] \sim exp[-\delta \alpha L + O(\ln L)]$. Here, $\delta \alpha$ is a real constant which renormalize α_0. The imaginary part, the phase change as a cooperative motion along a ring is $\theta = \oint e \mathbf{A}_{\text{eff}}.d\mathbf{l} = 2\pi N$, where N is the number of the superfluid particles enclosed by the ring [14]. This is the analog of the Bohm-Aharanov phase factor for a charged particle moving in a magnetic field. So the partition function becomes, $Z \sim exp[-\alpha L \pm i2\pi N]$, where $\alpha = \alpha_0 + \delta \alpha$.

IV. COOPERATIVE RING EXCHANGE MECHANISM

How does the vortex lattice melt? To understand the melting of the vortex lattices, the Lindemann criteria can not be used here since it is used in the melting of classical solids. The vortices are not executing almost independent thermal motions as in a classical solid. The

dynamics of the present problem is governed by a Hamiltonian with only first-order time derivatives, which give rise to its own peculiar properties. If we consider a rigid Wigner solid and allow one ring of vortices to tunnel coherently they see a periodic potential with the periodicity of the lattice (Fig 1). If we observe the coherent motion of one chain over a long time compared to the tunneling time (τ_0), the potential that it sees will not be periodic. The physically important rings being one-dimensional and long, this can result in the destruction of the long-range order along the chain rather easily. This in turn will feed back and affect the rest of the neighborhood, resulting possibly in a molten state. This will also result in the path of the wave packets of vortices being displaced away from the edges of the triangle of the lattice. This means that the self-consistent potential seen by a vortex no longer has a component which has long-range order.
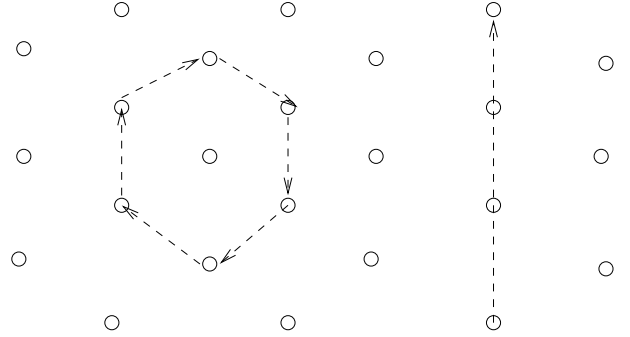

FIG. 1. A schematic diagram of cooperative ring exchange events on a ring and line. The vortices are indicated by the small circles.

V. CALCULATION OF THE TUNNELING COEFFICIENT

The numerical value of the tunneling coefficient $\alpha(\nu_v)$ determines whether the vortices form the liquid state or Wigner crystal. To estimate this tunneling coefficient we consider the following simple exchange path which is shown in figure 1. Consider the path in which one row of vortices exchanges one step in the X direction in the background of the static potential of all other vortices, $X_i(\beta) = X_i(0) + d$ and $Y_i(\beta) = Y_i(0)$, where $d = \sqrt{\frac{4\pi}{\sqrt{3}\nu_v}} l_0$ is the lattice constant of the Wigner crystal for a given density ν_v. There is no net phase changes since this straight path does not enclose any area. We are imposing the periodic boundary conditions in the X direction, $X_i(\tau) = X_{i+L}(\tau)$.

For $|Y_j| \ll d$, the two-body interaction potential in the coherent state representation (given in the eq. 21) can be approximated by

$$\frac{V}{\hbar\Omega} = \frac{4\pi}{\sqrt{3}\nu_v}\{\sum_{j=1}^{L}\left[\frac{Q_y}{2}Y_j^2 + \frac{Q_x}{(2\pi)^2}(1-\cos(2\pi X_j))\right] \quad (23)$$

$$+ \frac{1}{2}\sum_{j>k}[K_x(j-k)(X_j-X_k)^2 + K_y(j-k)(Y_j-Y_k)^2]\},$$

where X_j's and Y_j's are in units of the lattice constant d. Here, $K_x(j-k) = \frac{\partial^2 V_{jk}}{\partial X_j \partial X_k}|_{R_C}$ and $K_y(j-k) = \frac{\partial^2 V_{jk}}{\partial Y_j \partial Y_k}|_{R_C}$ are evaluated along the classical paths. The best fit to the actual potential is obtained with $Q_x/Q_y \sim 0.6$ for $\nu_v \sim 1/2$ and Q_x, Q_y and $K_x(j), K_y(j)$ are weakly dependent on ν_v. The calculation of the fitting parameters Q_x and Q_y is given in the Appendix. Notice that $Q_x/Q_y < 1$ implies that when one-dimensional chain moves coherently in the X direction, the potential barrier is much less than in the X direction compared to that of the Y direction.

The dimensionless Euclidean action is given by

$$S = \frac{4\pi}{\sqrt{3}\nu_v}\int_0^\beta d\tau[\sum_j[\frac{i}{2}\dot{X}_j Y_j + \frac{Q_y}{2}Y_j^2$$

$$+ \frac{Q_x}{(2\pi)^2}(1-\cos(2\pi X_j))]$$

$$+ \frac{1}{2}\sum_{j>k}[K_x(j-k)(X_j-X_k)^2 + K_y(j-k)(Y_j-Y_k)^2]].$$

Since S is a quadratic form in Y_j, the motion in the Y direction can be integrated out exactly. After doing the Y_j integration, we get an effective action S_{eff} for the X motion, with a quadratic kinetic energy,

$$S_{\text{eff}} = \frac{1}{\sqrt{3}\pi\nu_v}\int_0^\beta d\tau[\sum_{j<k}[\frac{1}{2}\dot{\phi}_j M(j-k)\dot{\phi}_k \quad (24)$$

$$+ \frac{1}{2}[K_x(j-k)(\phi_j-\phi_k)^2] + \sum_j \frac{Q_x}{(2\pi)^2}(1-\cos(\phi_j))],$$

where τ is the imaginary time variable, $(M(j-k))^{-1} = \frac{1}{8}[(\frac{Q_y}{2}+\sum_j K_y(j)\delta_{jk}) - \sum_{j<k}K_y(j-k)]$ and $\phi_j = 2\pi X_j$. S_{eff} is the effective action for a one-dimensional sine-Gordan chain. The classical path satisfying the boundary conditions are $\phi_j(0) = 0$ and $\phi_j(\beta) = 2\pi$, corresponds to the simultaneous coherent motion of all the vortices, i.e. $\phi_j(\tau) = \phi_0(\tau) + 2\pi j$. Due to the simultaneous coherent motion of all the vortices, the above effective action becomes,

$$S_{\text{eff}} = \frac{1}{\sqrt{3}\pi\nu_v}\int d\tau \sum_i^L \left[\frac{\dot{\phi}_i^2}{8Q_y} + Q_x(1-\cos(\phi_i))\right]. \quad (25)$$

By using the Euler-Lagrange equation of motion, one can calculate the $\dot{\phi}$. Hence, the above effective action along the classical path becomes, $S[R_c] = \alpha_0 L$, where $\alpha_0(\nu_v) = \frac{4}{\sqrt{3}\pi\nu_v}\sqrt{\frac{Q_x}{Q_y}}$. The α_0 is independent of $K's$. To evaluate the fluctuation determinant we have to take the continuum limit of the effective action S_{eff}. To take a continuum limit of the effective action, $(\phi_j - \phi_k)$ is replaced by $(j-k)\partial_x\phi$, but $\sum_j j^2 K_x(j)$ is diverging linearly since $K_x(j) \sim 1/j^2$. This is an infrared divergence and the continuum model must be constructed by taking the upper cut-off limit carefully. Here, we do not calculate the $\delta\alpha$ which is a non-trivial task.

Kivelson et al. [8] has given an extensive discussion of how to map Wigner crystal of electrons in a magnetic field into the discrete Gaussian model. Following the ref. [8], one can map the sum over all classical paths to a sun over classical spin configurations. All the contributions of ring exchanges happening in a time interval τ_0 are summed by modeling the change in the action by a discrete Gaussian model in an imaginary field [15],

$$H_{DG} = \alpha_0(\nu_v)\sum_{<\lambda,\gamma>}(S_\lambda - S_\gamma)^2 + ih(\nu_v)\sum_\lambda S_\lambda, \quad (26)$$

where $<\lambda,\gamma>$ denotes a nearest-neighbor pair on the dual lattice and S_λ is an integer variable associated to every triangle in the lattice. S_λ counts the number of clockwise minus counterclockwise ring exchanges that surround a plaquette λ. The function $\alpha_0(\nu_v)$ is a measure of the tunneling barrier. The function $h(\nu_v)$ is the phase factor which arises as a result of the pseudo magnetic flux enclosed by the exchange rings. This model is known to have a phase transition [15] at a critical value of $\alpha = \alpha_c(\nu_v) \sim 1.1$ [8]. For $\alpha(\nu_v) > \alpha_c(\nu_v)$, the ground state is a vortex Wigner crystal and for $\alpha(\nu_v) < \alpha_c(\nu_v)$, the ground state is a quantum mechanical vortex liquid state. In our calculation we find that the quantum melting will occur at $\nu_v \sim \frac{1}{2}$. The current experiments [1,2] with $\nu_v \ll \frac{1}{2}$ are in the regimes of vortex lattice ground state. So our result is consistent with the experimental results, but it does not match very well with the other theoretical results [9,10]. The numerical calculation is based on the small number of atomic bosons as well as small number of vortices. In our approach we assumed a large number of atomic bosons and vortices. We believe that this discrepancies is related to the system size. One would say that a large system has been considered in the ref. [10] and calculated the melting condition which is comparable to the numerical result [9]. In the ref. [10], first they have calculated the root mean square of the displacement from the equilibrium position of a vortex in terms of the filling factor ν_v. Then they have used the Lindemenn criterion and assumed that the melting will occur when the fluctuation of the vortex position is $0.15d$ to get the melting condition which is close to the numerical result [9]. Although the Lindemann criterion gives a reasonable description of the melting of a classical solid, there is little evidence that it can be applied to the melting of a vortex lattice. The vortices are intrinsically quantum objects whose equation of motion are quite different from those of atoms in a harmonic crystal.

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VI. SUMMARY

In this paper, we treated the vortices as a new degrees of freedom and considered a model Hamiltonian of interacting vortices. Later, we assumed the vortices are in the lowest Landau level due to the low mass of the vortices and the high densities of the superfluid Bose particles. The concept of cooperative ring exchange is introduced to explain the mechanism of quantum melting of the Wigner crystals. Finally, we estimated the tunneling coefficient which determines the condition for quantum melting instability of the vortex lattices. Latest experiments [1] with $N \sim 10^5$, $N_v \sim 10$ ($\nu_v \sim 10^{-4}$) and [2] with $N \sim 10^7$, $N_v \sim 100$ ($\nu_v \sim 10^{-5}$) are in the regime in which the ground state is a vortex lattice. It is a challenge for experimentalists to produce a vortex liquid state in a rotating Bose condensed state.

Our present work, resulting in a discrete Gaussian model (equation 10) predicts Laughlin like even denominator bosonic vortex filling fraction $\nu_v = \frac{1}{2}$, to emerge on quantum melting. We can also determine the asymptotic form of the wave functions [16]. Along with a rich phase structure the discrete Gaussian model also determines the nature of the quantum melting transition. To the extent the vortex degrees of freedom retain their identity, the results of CRE approach may remain valid in the quantum melted region. This needs to be investigated further.

As mentioned earlier CRE processes should leave its finger print as specific fluctuation patterns (figure 1) that preempts quantum melting. It should be interesting to look for snapshots of such displaced large rings in the actual vortex lattice imaging.

APPENDIX A: CALCULATION OF THE PARAMETERS Q_X AND Q_Y

Here, we describe how to calculate the parameters Q_x and Q_y. We consider the simplest possible exchange path, namely one line of vortices shifting coherently within the Wigner crystal. When the line $\mathcal{L}$ is displaced, we have $\mathbf{R}_i = \mathbf{T}_i + \mathbf{d}\delta_{i\in\mathcal{L}}$, with $\delta_{i\in\mathcal{L}}$ unity if and only if lattice site i lies on the line in question. The matrix element of the potential between two vortices in coherent basis state is

$$V(\mathbf{R}) = \langle \phi_{\mathbf{R}}(\mathbf{r})|V(\mathbf{r})|\phi_{\mathbf{R}}(\mathbf{r}).\rangle \qquad (A1)$$

Accordingly, the energy of the displaced line configuration relative to that of the perfect Wigner crystal is

$$\Delta E = \sum_{\langle i,j\rangle} \left[V(\mathbf{R}_i - \mathbf{R}_j) - V(\mathbf{T}_i - \mathbf{T}_j)\right]. \qquad (A2)$$

This sum can be broken up into three terms. The first term includes all pairs (i,j) in which both sites i and j lie off the line. This contribution to ΔE is zero. The second term involves all pairs (i,j) where one of the sites, say i, is on the line and the other, j, is off the line:

$$\Delta E_2 = \sum_{i\in\mathcal{L},|\notin\mathcal{L}} \left[V(\mathbf{T}_i + \mathbf{d} - \mathbf{T}_j) - V(\mathbf{T}_i - \mathbf{T}_j)\right]. \qquad (A3)$$

Clearly the line energy is extensive, hence the energy per tunneling of the vortex can be written

$$U(\mathbf{d}) = \Delta E_2/L = \sum_{j\notin\mathcal{L}} \left[V(\mathbf{T}_j - \mathbf{a}) - V(\mathbf{T}_j)\right], \qquad (A4)$$

where we have chosen the origin to lie on the line. The third and final term is that arising from both i and j on the line. Since the tunneling is cooperative, this contribution to the classical action vanishes.

By allowing one line of vortices to tunnel coherently along the line, one can fit the change in energy ΔE into a periodic potential with the appropriate choice of the parameter Q_x. On the other hand, by allowing one line of vortices to tunnel coherently perpendicular to the line, one can fit the change in energy into a quadratic potential with appropriate choice of the parameter Q_y.

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