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We discuss recent work on the static and dynamical properties of the asymmetric exclusion process, generalized to include the effect of disorder. We study in turn: random disorder in the properties of particles; disorder in the spatial distribution of transition rates, with a single easy direction, and with random reversals of the easy direction; dynamical disorder, where particles move in a disordered landscape which itself evolves in time. In every case, the system exhibits phase separation; in some cases, it is of an unusual sort. The time-dependent properties of density fluctuations are in accord with the argument that the dynamical universality class is unaffected by disorder if the kinematic wave velocity is nonzero.

INTRODUCTION

Simple models of particles moving on a lattice have shed a good deal of light on the general question of collective effects in transport. Even the simplest sorts of interactions between particles can induce collective phenomena involving large numbers of particles, ranging from the motion of density fluctuations as a kinematic wave, to the occurrence of phase transitions to states with macroscopically inhomogeneous density (jams). Models which display these phenomena include the asymmetric simple exclusion process (ASEP), the simplest model which incorporates directed motion and mutual exclusion [1–3], and its generalizations which include more realistic features which mimic vehicular traffic [4].

Here we are concerned with the effects of extensive disorder on interacting-particle transport in one dimension. As in equilibrium systems, disorder has a strong effect on the properties of driven systems which reach a nonequilibrium steady state. However, in contrast to equilibrium systems, relatively little is known, in a general way, about the effects of quenched disorder on nonequilibrium systems, as the absence of detailed balance, together with the breaking of translational invariance, makes even the determination of the steady state weights difficult in general. In this backdrop, the study of simple models of disordered nonequilibrium systems has proved valuable, as it reveals how interactions, drive and disorder combine to produce new types of states; the simplicity of the models allows for a detailed characterization and understanding.

In this paper, we will review some results obtained for a disordered system of particles moving stochastically on a one-dimensional lattice. We restrict ourselves to the disordered ASEP, in which interactions between particles enter through rules for the hopping rates – the hard-core constraint is modelled by forbidding any move which would lead to more than one particle per site. We will consider three types of disorder. The first is *particle-wise* quenched disorder, in which different particles have different properties which do not change in time. Secondly, we will consider *space-wise* quenched disorder, in which particles do not differ from each other, but are subject to hopping rates which are randomly distributed in space. Finally, we will turn to *dynamical disorder* where particles move in a disorder-induced landscape, which is itself evolving in time. Interestingly, there are inter-relations between the effects of the three types of disorder, which we will discuss below.

It is clear that extensive disorder at the microscopic scale would induce inhomogeneities in the particle density on a similar scale. Furthermore, disorder can also induce variations of the density on a *macroscopic* scale, associated with phase separation [5] . The models considered here share this feature, but in some cases the phase separated state has quite unusual properties. We also discuss the time-dependent properties of these systems, both in the steady state and while approaching it.

PARTICLE-WISE DISORDER

Consider a system of cars on a single-lane road, too narrow for overtaking to be possible. Each car has an intrinsic maximum speed, which however may not be achieved due to a slowly moving car in front of it. It is then evident that the slowest car in the system will be trailed by a number of intrinsically faster cars, and will see a larger than average headway in front of it. The question arises: In the limit of an infinitely long road with a finite density of cars, will this headway be finite or infinite? Is there a phase transition between these two possible types of behaviour as the density is changed?

This question motivates the study of a collection of particles with a different intrinsic rate of motion for each. In a simple lattice model, studied in [6] and [7], N particles reside on the sites of a one-dimensional ring with L sites. The dynamics involves random sequential updating of configurations. An elementary move consists of an attempted rightward hop of a particle, say the kth, with a particle-dependent hopping rate $u(k)$; the move is actually implemented only if the site immediately to the right is unoccupied. A configuration is specified by the set of particle locations $\{y_k\}$. One may instead consider the set $\{m_k\}$, where $m_k = y_{k+1} - y_k - 1$ is the headway or gap between the kth particle and the one ahead of it. On a periodic ring, a typical set $\{m_k\}$ represents N equivalent configurations which are translational shifts of each other. Evidently the overall density $\rho = N/L$ is conserved by the dynamics.

Disorder enters through the selection of hopping rates $u(k)$, each of which is drawn independently from a distribution $Prob(u)$. Especially interesting is the case with a power law variation near a minimum cutoff speed: $Prob(u) = [(n+1)/(1-u_o)^{n+1}](u-u_o)^n$ with u in the range $[u_o, 1]$. Here n is an index which characterizes the decay of the distribution function near the cutoff. As we shall see below, this system shows a phase transition as the density is decreased, from a state with finite headways ahead of each particle, to one in which the headway ahead of the slowest particle is infinitely long, or more accurately, a finite fraction of the of the lattice size L.

Mapping to Zero Range Process

The analysis of this system is aided by exactly mapping this system with particle-wise disorder to another system with space-wise disorder [6, 7]. View the particle index k as labelling urns arranged in a one-dimensional sequence (Fig. 1). The occupancy of urn k is taken to be the length of headway m_k in front of the kth particle in the original particle problem. Now, a single rightward hop of a particle reduces the headway in front of it, and augments it behind. In the urn representation this corresponds to an elementary move of a particle from urn k to urn $k-1$ (Fig. 1).

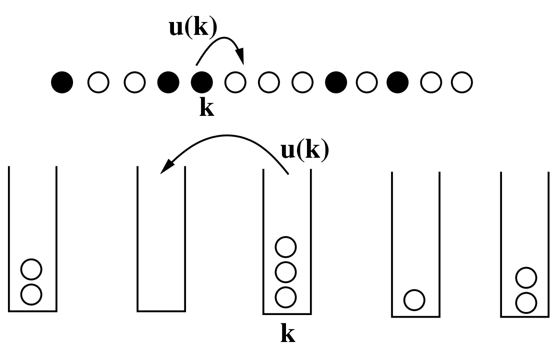

FIG. 1: Particle-wise disordered model and the Zero-range process

This model is a special case of the zero range process (ZRP) which describes the motion of particles between wells or urns, with (in general), occupancy dependent, site-dependent hopping rates $u(k, m_k)$ [1, 8]. The ZRP has the virtue that the steady state measure $P_s(C)$ is known. In our case, where the rates depend on sites but not on occupancies, the steady state weights are particularly simple. In a grand canonical formulation of the problem we have

$$P_s(C) = \frac{1}{Z} \prod_{k=1}^{N} \left(\frac{z}{u(k)} \right)^{m_k} . \qquad (1)$$

Here Z is the grand partition function, m_k is the mass at site k, and z is the fugacity which is determined by requiring $\sum_k m_k = M$. For a given realization of disorder $\{u(k)\}$, the density $\tilde{\rho} = M/N$ is given by $N^{-1} \sum_k z/(u(k) - z)$. As $\tilde{\rho}$ is increased, the model undergoes a phase transition [6, 7, 9]. We use the self-averaging property to replace the sum over k by an average over $Prob(u)$, separating out the lowest u:

$$\tilde{\rho} = \frac{1}{N} \frac{z}{u_{min} - z} + \int_{u_o}^{1} du \, \frac{z}{u - z} \; f(u). \qquad (2)$$

where u_{min} is the lowest realized value of $u(k)$; u_{min} approaches u_o in the thermodynamic limit. The fugacity z increases with $\tilde{\rho}$ until one reaches a critical density $\tilde{\rho}_c = u_o(n+1)/n(1-u_o)$ beyond which z gets pinned to u_o. Beyond this point, the system enters a high-density phase in which the excess mass $M - \tilde{\rho}_c N$ is 'condensed' at the site with the lowest hopping rate in the ZRP. The transition is analogous to Bose-Einstein condensation in the ideal Bose gas or a system of bosons in a random potential [10].

In the particle-wise disordered model, the condensate represents an infinite headway in front of the slowest car, implying an infinitely long jam behind it. Such a jam implies spatial phase separation, and exists only for low enough density; at higher densities, blocking effects between cars reduce the contrast in speeds to such an extent that headways are interspersed throughout the system and there is no phase separation.

Time-dependent Properties

It is interesting to ask about the dynamical properties of the system, both in the steady state, and also describing relaxation to the steady state. In steady state, it is evident that the mean current between every pair of neighbouring sites in the ZRP is the same. It is less obvious that the full distribution of time intervals between particle transfer events is also identical on each site. This property follows from the application of Burke's theorem from queueing theory [11, 12] to the problem at hand. This theorem states that in a reversible birth-death process, a Poisson input of births results in an identical Poisson distribution of deaths. Identifying the occupancy of each urn of the ZRP with a population undergoing such a process then implies the result, as the output from one urn constitutes the input for the next. In the original particle problem, this means that the time history of every particle follows an identical Poisson distribution despite the fact that each particle has different intrinsic speeds. This holds in the thermodynamic limit in both the disordered and the phase separated state [13].

Next, let us ask about the behaviour of statistical fluctuations of the density in the steady state. In a homogeneous driven system with a density-dependent current $\tilde{J}(\tilde{\rho})$, density fluctuations are transported around the system as a kinematic wave with speed $\tilde{c} = \partial\tilde{J}/\partial\tilde{\rho}$, in time $N/\tilde{c}$ [3, 14]. This wave of fluctuations decays on a time scale $\sim N^z (z > 1)$, implying that it circulates several times around a system with periodic boundary conditions before it dissipates. In the disordered ZRP, there are different sorts of behaviour in different phases. For $\tilde{\rho} < \tilde{\rho}_c$ and for $\tilde{\rho} = \tilde{\rho}_c, \quad n > 1$, the wave speed $\tilde{c}$ is nonzero. There is clear evidence of the wave in simulation results for the fluctuations in the integrated number of particles which move past a site. This correlation function shows pronounced oscillations with period $N/\tilde{c}$, which ultimately damp down [15].

In [16] it has been argued that quenched disorder does not have a significant effect on the damping of fluctuations in driven systems if the kinematic wave speed $\tilde{c}$ is nonzero. This 'kinematic wave criterion' implies that there is then *no* change in the universality class of the decay of density fluctuations, which should continue to follow the behaviour in the absence of disorder. The reason is that in such a case, each density fluctuation does not stay long in the vicinity of a local patch of disorder. To that extent, the effect of quenched disorder on a ballistically moving density fluctuation of spatial extent Δx is akin to temporal noise, and we should expect the lifetime of the density fluctuation would continue to be $\sim (\Delta x)^z$, where z is the dynamical exponent in the absence of disorder. Monte Carlo simulations [15] confirm that when $\tilde{c} \neq 0$, z equals 3/2, the value for a driven system in the absence of disorder [19]. At $\tilde{\rho} = \tilde{\rho}_c$, the speed $\tilde{c}$ vanishes for $n \leq 1$. In this case, disorder is expected to have a strong effect, and perhaps change the universality class away from that in its absence. The determination of this class remains an open problem.

For $\tilde{\rho} > \tilde{\rho}_c$, the system has density $\tilde{\rho}_c$ in the bulk, with the surplus mass $(M - \tilde{\rho}N)$ being on the slowest site. Thus, density fluctuations would be expected to move as a kinematic wave in the bulk so long as $n > 1$. However, this wave would be absorbed at the condensate site and not be able to circulate, implying that correlation functions in this phase would be non-oscillatory.

Now let us turn to the kinetics of approach to the steady state, starting from a state with uniform density. In the disordered phase ($\tilde{\rho} < \tilde{\rho}_c$), the kinetics is governed by the same dynamical exponent z that operates in steady state, and the relaxation time is of order L^z, with $z = 3/2$. In the condensate phase, however, a different physical process, namely coarsening, governs the approach. Initially, particles hop out of relatively fast sites quite quickly, and get trapped at slow sites in the immediate neighbourhood. At moderate times, there is thus a finite density of noticeably large aggregates at such slow sites, which then relax by releasing their mass to yet slower sites [17]. Thus the mass at all such slow sites, except for the very slowest, shows a nonmonotonic variation in time. The mass at the slowest (condensate) site grows as $M_1 \sim t^\beta$, with $\beta = (n+1)/(n+2)$, a result first derived for a model with deterministic dynamics [18], but which remains true even in the stochastic model under consideration [5, 6, 15]. This translates into a statement about the growth of headway lengths ξ in the particle model

$$\xi(t) \sim t^{\frac{n+1}{n+2}}. \tag{3}$$

A similar coarsening description for headways is also found to hold in a lattice gas model of traffic, which incorporates features of acceleration and slowing down, and in which different drivers have different propensities for random braking [20].

Furthermore, the problem has also been studied using open boundary conditions, with injection and ejection rates specified [21]. In the absence of disorder, it is known that the open system exhibits phase transitions between a low-density phase, a high-density phase, and a high-current phase, as the injection and ejection rates are varied [22, 23]. With particle-wise disorder, there is a shift of the phase boundaries owing to the breaking of particle-hole symmetry. Within the low-density phase, there is a regime in which the drift velocity of all particles is determined by the slowest [21], as with perodic boundary conditions.

Finally, we discuss some recent work on two-way traffic [26, 27]. Consider particle-wise disorder in which the majority of particles move preferentially rightward, while a minority (a fraction f) move peferentially leftward. For each particle, the ratio of the larger hopping rate to the smaller one, which is a measure of bias, is taken to be fixed, but the direction of hopping is random. The problem was mapped onto a ZRP, which inherits the random easy-direction property. This system has much in common with the spatially disordered ASEP with bidirectional bonds, to be discussed in the next section. In particular, as in that case, the current decays as an inverse power of the size, with the power depending continuously on both f and the bias.

SPACE-WISE DISORDER

Consider the transport of interacting particles which are driven through a randomly disordered one-dimensional path. To address the question of what sort of macroscopic states result, we restrict ourselves to the disordered asymmetric simple exclusion process. Disorder enters through the transition rates, which are assigned randomly to the bonds between neighbouring sites, e.g.

$u_{i,i+1}$ describes the rate of attempted hopping from site i to $i+1$, while $u_{i+1,i}$ is the attempt rate in the reverse direction. A hop to a site actually occurs only if the site in question is unoccupied. Results obtained for a particular disorder realization $R \equiv \{u_{i,i+1}, u_{i+1,i}\}$ are then to be averaged over R using a pre-specified probability distribution, taken to be the product of identical and independent distributions across all bonds. The larger of the two rates on each bond defines the easy direction on that bond. Below we will distinguish between two situations: *unidirectional*, in which the easy direction of every bond is the same, but the strength is a random variable ((a) and (b) in Fig. 2) and *bidirectional*, in which the easy direction is itself a random variable ((c) and (d) in Fig. 2).

In the unidirectional case, we will further specialize to only the case of no backward hopping ($u_{i+1,i} = 0$):

$$Prob(u_{i,i+1}) = (1-f)\delta(u_{i,i+1}-u_1)+f\delta(u_{i,i+1}-u_2) \quad (4)$$

with $u_2 < u_1$. The fraction f and relative strength $r = u_2/u_1$ of weak bonds specify the extent of disorder.

In the bidirectional case, the easy direction of each bond is taken to vary randomly, but the relative hopping rate λ against and along the easy direction on each bond is taken to be the same.

$$Prob(b_{i,i+1} = \frac{u_{i+1,i}}{u_{i,i+1}}) = (1-f)\delta(b_{i,i+1}-\lambda)+f\delta(b_{i,i+1}-\lambda^{-1}). \quad (5)$$

Here f is the fraction of backward pointing bonds.

The large scale behaviours that result depend on the type of disorder as well as on the overall density ρ. Figure 2 shows a schematic depiction of a segment of the disordered lattice, the corresponding variation of the current J with system size L, and the schematic spatial variation of the mean occupancies $\rho_i \equiv < n_i >$ in a particular realization of disorder in different cases. These are discussed below.

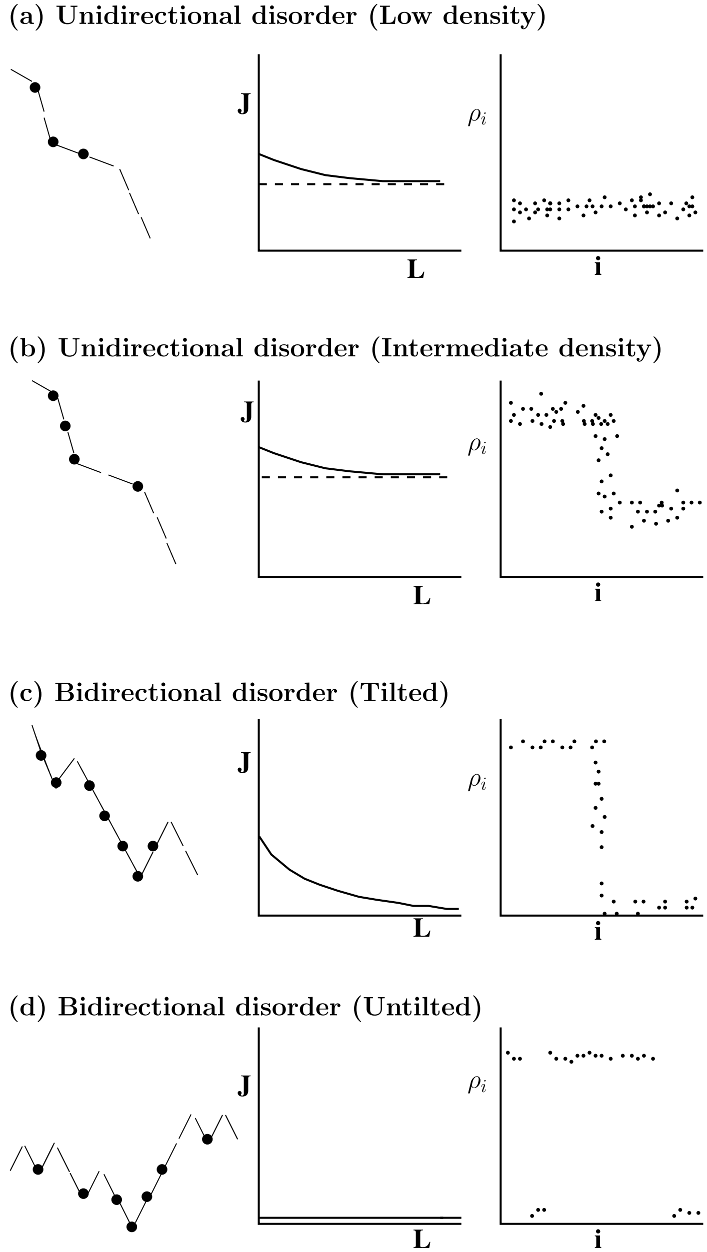

FIG. 2: Space-wise disorder: depiction of possible behaviours

(a) With unidirectional disorder and a particle density that is sufficiently far from 1/2, i.e. $|\rho - 1/2| > \Delta$, the density is homogeneous on the macroscopic scale. The value of Δ depends on the concentration f and relative strength $r = u_2/u_1$ of the weak bonds. The current J approaches a nonzero value in the thermodynamic limit.

(b) With unidirectional disorder and ρ close to 1/2, i.e. $|\rho - 1/2| < \Delta$, the state exhibits phase separation; the system is characterized by two distinct values of the density, each extending over a macroscopic region. As in (a), J has a finite value in the thermodynamic limit. This value of the current is the same for all ρ in the regime $|\rho - 1/2| < \Delta$.

(c) With bidirectional disorder, but with an asymmetry in the number of forward and backward bonds ($f \neq 1/2$), the potential has local minima and an overall tilt. On a macroscopic scale, the system has two distinct values of the density, close to 1 in one region and close to 0 in the other. The current J falls with increasing system size L as an inverse power of L.

(d) With bidirectional disorder and equal numbers of forward and backward bonds ($f = 1/2$), the potential has local minima, but no overall tilt. Consequently the mean current is zero. The density profile shows large sample-to-sample variations, but every sample has a high-density region that extends over a region that is a finite fraction of the system size.

Let us turn to an explanation of the features described above. Central to the discussion is the observation that while there are strong spatial variations in the local density $< n_i >= \rho_i$, the steady state value of the current J

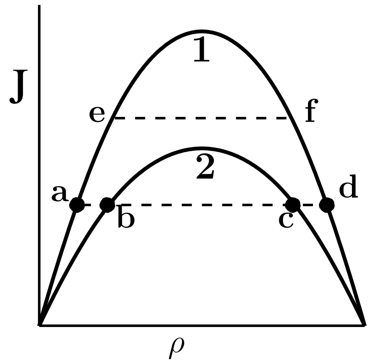

FIG. 3: Current-density plots

is uniform across all bonds.

Unidirectional disorder

The time averaged current in bond $(i, i+1)$ is $J_{i,i+1} = u_{i,i+1} < n_i(1 - n_{i+1} >$. Since the exact steady state weights of particle configurations $\{n_i\}$ are not known, one cannot compute the two-point correlation function involved. To proceed, we employ a mean field approximation which ignores correlations betwen site densities in the steady state, and replaces $< n_i n_j >$ by $< n_i >< n_j >= \rho_i \rho_j$, but does allow for spatial variation of ρ_i, which is crucial. To find the mean field solution $\{\rho_i\}$ we may proceed in two ways. (i) For a given value of J, iterate the set of equations $\rho_{i+1} = 1 - J/u_{i,i+1}\rho_i$ around the periodic chain, until convergence is achieved. (ii) For a given value of the overall density, evolve local densities (in fictitious time) through $\rho_i(t+1) = \rho_i(t) + j_{i-1,i}(t) - j_{i,i+1}(t)$ where the instantaneous currents are given by $j_{i,i+1}(t) = u_{i,i+1}\rho_i(t)[1-\rho_{i+1}(t)]$. The densities then converge to time-invariant values, and the corresponding current is uniform across bonds. As discussed in [24], method (ii) is preferable, as values of J above a threshold value are not allowed, and method (i) does not converge well near the threshold. The resulting mean field density profile can be tested against the results of direct Monte Carlo simulation. The agreement is remarkably good, in that the mean field approximation tracks every one of a macroscopic number of 'minishocks' and correctly predicts their positions, though it does not accurately reproduce their shapes [24].

The principal result is the existence of a density range Δ in which the current has a constant, density-independent value (regime (b)), and where the system exhibits phase separation of densities [24]. A simple qualitative understanding of both these features can be obtained by referring to Fig. 3. The argument relies on a conjectured maximum current principle, which states that for a given value of the overall density, the system settles into a state which maximizes the mean current. In our system, let us suppose that the density in each stretch of like bonds is uniform, and label by 1 and 2 the stretches of strong and weak bonds respectively. The two parabolas in Fig. 3 are the corresponding J versus ρ curves for the pure 1 and pure 2 systems. Since the current is uniform in the disordered system, a value of current such as shown by by the upper dashed line ef in the figure is ruled out, as such a value cannot be sustained in a 2-stretch. By contrast, a value of J corresponding to say ad is quite possible, as the (1,2) densities can assume the values (ρ_a, ρ_b) if the density is relatively low, or alternatively (ρ_c, ρ_d) if the density is relatively high. Such choices correspond to the regime $|\rho - 1/2| > \Delta$, and result in a density profile that is homogeneous on a macroscopic scale as strong and weak bonds are interspersed at the microscopic level. If the density is too close to half-filling $|\rho - 1/2| < \Delta$, the system adjusts by keeping the current constant at its maximum allowed value, and replacing (ρ_a, ρ_b) in (1,2) stretches by (ρ_c, ρ_d), in a finite fraction of the full lattice. The resulting state then shows phase separation of the density. The location of the high density region is decided by the longest stretch of weak bonds in the system, as this stretch (whose length is of order lnL) mimics the effect of a single defect weak bond in an otherwise pure system, which is known to induce phase separation in a finite range of density [28, 29].

Interestingly, it is possible to obtain upper and lower bounds [5] on the value of the threshold density for phase separation $\rho_c = 1/2 - \Delta$ in terms of the ratio of bond strengths $r = u_2/u_1 < 1$ and the fraction f of weak bonds. An upper bound is obtained on noting that the maximum current that can flow through a stretch of slow bonds decreases as the stretch length increases. It is then plausible that the current would be the least in the fully segregated limit where all slow bonds form a single large stretch [24]. In such a system consisting of two connected homogeneous parts, it is easy to figure out the way a given overall density would distribute itself over the two portions. A lower bound on ρ_c was obtained [5] by arguing that the current in the disordered ASEP is bounded above by the current in a corresponding ZRP with the same disorder distribution and an equal number of particles.

The results discussed above hold for the case of binary disorder in the transition rates. Similar phenomena, e.g. the plateau in the $J - \rho$ curve over a finite range Δ of density, have been found also for more general, continuous distributions of the transition rates [30, 31]. The problem has also been studied using open boundary conditions, specifying injection and ejection rates. An argument based on moving shock fronts [32] suggests that in the presence of disorder, the high-current phase extends over a larger region of the phase diagram [31] than in the pure system [22]. Further, a numerical study of the first order transition between the low and high density phases shows that the transition remains first order, but the location of the transition shows strong sample to sample fluctuations which do not seem to damp down in the thermodynamic limit [33].

Going beyond the ASEP, the effect of slow sites on a lattice model of traffic was studied by numerical simulation in [34]. Interestingly, a regime with two current plateau regions was observed, and the system was observed to switch between the two branches.

To conclude this section, we mention an intriguing symmetry that holds for the disordered ASEP when no backward hopping is allowed. A typical configuration of quenched disorder is not symmetric under space inversion, and consequently nor is the steady state density profile. In spite of this, the magnitude of the steady state current was observed to be invariant under reflection [24]. Subsequently, this invariance property has been proved for both periodic [35, 36] and open [35] systems.

Bidirectional disorder: With tilt

When the easy direction on a bond is a random variable, one needs to distinguish between the cases $f < 1/2$ and $f = 1/2$, corresponding to a tilted and untilted potential respectively (figs. 2(c) and (d)). We may imagine assigning arrows to denote the easy direction on the bonds of a periodic ring; a fraction $(1 - f)$ point rightward, while a fraction f point leftward. A particle-hole exchange occurs in the direction of the arrow with rate u, and in the opposite direction with rate $u\lambda$ with $\lambda < 1$.

In the tilted case, there is an overall tendency for the particles to move rightward, but the question is whether there is a nonzero current in the thermodynamic limit. Local barriers to the rightward motion occur in the form of backbends which consist of a chance agglomeration of successive left-pointing arrows. An upper bound J_l on the current that can be carried by a backbend of length l is obtained by considering the boundary conditions, $\rho = 1$ at the leftmost end of the backbend and $\rho = 0$ at the rightmost end, which force the largest possible current through it Within a mean field approximation, the density profile, and hence the current, can be calculated [37] with the result $J_l \sim \lambda^{\frac{1}{2}l}$. The factor $\frac{1}{2}$ is a consequence of particle-hole symmetry which implies that the particle density extends up to half-way through the stretch, so that the topmost particle needs to be activated across $\frac{1}{2}l$ sites. An exact calculation [25] confirms the exponential decay of current with l. The current in the full system of size L is limited by the length $l^*(L)$ of the longest backbend. Since the probability of the occurrence of l successive backward bonds is f^l, we may estimate l^* from $Lf^{l^*} \simeq O(1)$. The estimate of the current $\lambda^{l^*/2}$ can be written as

$$J(L) \sim L^{-\theta/2} \tag{6}$$

where the exponent

$$\theta = \ln\lambda / \ln f \tag{7}$$

depends explicitly on the bias and fraction of reverse bonds. This dependence has been confirmed by Monte Carlo simulations [24].

Bidirectional disorder: Untilted

When there is no overall tilt, the mean current is identically zero. The problem then reduces to the equilibrium problem of hard core particles at temperature T in a static random potential corresponding to a height profile $\{h_i\}$. The Hamiltonian is

$$\mathcal{H} = \sum_i h_i n_i. \tag{8}$$

Height differences are related to the hopping rates of the model through the condition of detailed balance

$$exp[-(h_{i+1} - h_i)/T] = \frac{u_{i,i+1}}{u_{i+1,i}}. \tag{9}$$

Since each local slope of the height is equal in magnitude but random in sign, the full height profile is isomorphic to the trail of a random walker, with the height being the displacement of the walker and i being time. Thus the problem reduces to that of hard-core particles in a Sinai potential. In the limit of large bias ($T \to 0$), the system approaches the ground state, which can be found as follows. Make a constant-height cut of the height profile, fill particles up to that point, and then raise the level of the cut step by step and fill in particles till they are exhausted. The lengths ℓ of particle clusters (which are present below the topmost filled level) are then determined by returns to the origin of the walker, and their distribution follows an asymptotic power law $Prob(\ell) \sim \ell^{-3/2}$. This distribution has an infinite mean, and in a system of size L an estimate based on extremal statistics shows that the largest cluster size is gL. The prefactor g shows strong fluctuations from one sample to another, but is always of order unity. Thus a typical configuration consists of at least one $O(L)$-sized cluster in addition to $\sim \sqrt{L}$ clusters of size $\sim \sqrt{L}$.

In [38], a similar model (the Coarse-grained Depth or CD model) was analysed, and the disorder-averaged two-point correlation function $C_L(x) \equiv < n_i n_{i+x} >_L - \rho^2$ was computed, using the cluster size distribution $Prob(\ell)$ as input. The result is that $C_L(x)$ is a scaling function of argument x/L:

$$C_L(x) = Y(x/L) \approx m_o^2[1 - a(x/L)^\alpha], \quad x/L << 1 \tag{10}$$

with intercept $m_o = 1$ and $\alpha = 1/2$.

The argument of the scaling function indicates that it describes a phase-separated state [39]. The finite intercept m_o^2 is a measure of long-range order. The cusp singularity is a manifestation of ill-defined interfaces, as it is known that a phase-separated system with well defined interfaces has $\alpha = 1$, in concordance with the Porod law [39]. We conclude that the state exhibits an unusual sort of phase separation, with unusual interfacial characteristics and strong sample to sample fluctuations.

Time-dependent properties

In this subsection we discuss both the steady state dynamics and the coarsening dynamics of systems with unidirectional and bidirectional disorder.

With unidirectional disorder, the behaviour of density fluctuations in steady state resembles that in the zero-range process discussed in the previous section. For $|\rho - \rho_c| > \Delta$, there is a kinematic wave which circulates through the system, and induces long-period oscillations in the density [16]. The dissipation was monitored by performing a Galilean shift to keep up with the wave. As expected on the basis of the kinematic wave criterion discussed earlier, disorder was found to be irrelevant, and $z = \frac{3}{2}$ still holds [16]. For $\rho = \frac{1}{2}$, on the other hand the system is phase separated, and the system supports two kinematic waves which move in oppsite directions in the two coexisting phases. Simulations suggest that the kinematic waves emanate from one interface, and terminate at another, leading to a different universality class [16].

In the phase-separated state, the kinetics of approach to the steady state involves a coarsening process [5]. Large-density stretches accumulate behind bottleneck regions which are stretches of consecutive weak bonds. If ΔJ is the difference of currents exiting from two such bottlenecks separated by a distance ξ, the region in between would fill up in time $t \sim \xi/\Delta J$. Estimating the probability of occurrence of stretches of length l and recalling that they carry a current $J_l = u_2/4(1 + a/l)$ [40] leads to the estimate

$$\xi(t) \sim \frac{t}{lnt}. \tag{11}$$

Turning to bidirectional disorder with tilt ($f < \frac{1}{2}$), recall that in steady state, particles are held back behind the longest backbend, while the region ahead of it is empty. The coarsening properties of this model were obtained by Krug [5] who pointed out that bottleneck stretch lengths are much smaller than their separations, implying that we may mentally replace stretches by single bonds that allow a maximum current $J_l \sim \lambda^{\frac{1}{2}l}$ to flow through. Following through, we find that the coarsening length is given by [5]

$$\xi(t) \sim t^{\frac{1}{1+\theta/2}}. \tag{12}$$

In the untilted case, the steady state is the equilbrium state of hard core particles in a random (Sinai) potential. Not much is known about the dynamics in this case, though the single particle problem is well studied [41]. The existence of large ($\sim O(\sqrt{L})$ potential barriers would make the approach to equilibrium logarithmically slow, and it would be interesting to see how much slower the coarsening process is than the familiar diffusion of a single particle in a Sinai landscape.

DYNAMICAL DISORDER

In this section, we study the properties of an ASEP subjected to bidirectional bond disorder, where the disorder is itself time-dependent. Specifically, we consider an untilted potential (as in Fig. 2(d)) which is evolving stochastically in time through the single-step model [42] which preserves the overall no-tilt condition under time evolution. condition. The basic dynamical move is the interchange of nearest neighbour links of the height field at a certain rate if they represent opposite slopes. If L and R stand for left-leading and right-leading links respectively, then a link configuration ....LR.... can evolve into ....RL.... at a certain rate, and the reverse move is allowed at an equal rate. The large space and time properties of the height field of this model are described by the continuum Edwards-Wilkinson model [43]. The ASEP particles are passive sliders on the surface, in that they are influenced by, but do not back-influence, the dynamics of the fluctuating surface.

This problem has been studied in [38]. Particles tend to form clusters when they fall into valleys, and to decluster when valleys overturn and form hills. However, declustering involves activation and is a slow process, while clustering is relatively rapid. This asymmetry leads to overall clustering. Numerical simulations show that this tendency to cluster persists at all length scales, and leads finally to a steady state that exhibits phase separation. However, this phase separated state is unusual in that is characterized by large fluctuations of the order parameter, which do not damp down in the thermodynamic limit. However, despite the presence of strong fluctuations, the system never loses its ordered character [44]. Fluctuations carry the system relatively rapidly from one ordered configuration to another macroscopically distinct one. However, the probability for the system to leave this attractor of ordered states vanishes exponentially in the system size.

A quantitative measure of the ordering is provided by the two point correlation function. Results of simulations show that Eq. 10, derived for the disordered problem with an untilted, bidirectional potential describes the form of $C_L(x)$ in this case as well. When the update rates for the landscape and particles are equal, the value of m_o^2 is close to 0.8, indicating long-range order, consistent with the existence of a phase ordered state. Moreover, the value of the cusp exponent α is close to $\frac{1}{2}$, its value for the problem discussed in the previous section. This quantitative similarity between the scaling properties of phase ordering induced by quenched disorder, and by dynamical disorder corresponding to an evolving Edwards-Wilkinson landscape, is not completely understood.

Time-dependent properties

The time dependence of the particles is dictated by fluctuations of the driving height field. Except at very small times, the autocorrelation function $A_L(t) \equiv < n_i(0)n_i(t) >$ is found numerically to be a scaling function,

$$A_L(t) = X(t/L^z) \approx m_o^2[1 - d(t/L^z)^\beta], \qquad t/L^z << 1 \tag{13}$$

with $z = 2$ being the dynamical exponent governing the height-field dynamics and d a constant of order unity. Simulations show that $m_o^2 \simeq 0.8, \beta \simeq 0.2$. Like its counterpart for static correlations (Eq. 10), the scaling function X shows an approach with a cusp singularity to a finite value at small argument, once again signalling long range order of an unusual sort. Finally, to understand the coarsening properties of this model, recall that clustering occurs by collecting particles in valleys of the driving height field. Since the largest valleys that form in time t are typically of size $t^{1/z}$, we expect the coarsening length in this case to follow

$$\xi(t) \sim t^{\frac{1}{2}}. \tag{14}$$

The time-evolving two-point correlation function $< n_i(t)n_{i+x}(t) > -\rho^2$ is expected to be similar to the form of Eq. 10, except that the system size L is replaced by the coarsening length $\xi(t)$. The results of numerical simulations agree well with this.

CONCLUSION

We have reviewed the properties of the one-dimensional ASEP with various types of disorder — particle-wise, space-wise (unidirectional and bidirectional), and dynamical. All forms of disorder bring about phase separation, though under different conditions, and of different types.

With particle-wise disorder, phase separation occurs below a critical density $\rho_c = (1+\tilde{\rho}_c)^{-1}$. The macroscopic stretch ahead of the slowest particle is totally devoid of particles ($\rho = 0$), whereas the stretch behind is mixed and has $\rho = \rho_c$.

With unidirectional space-wise disorder, the system supports phase separation in the region $\rho_c < \rho < 1 - \rho_c$, with coexisting phases that have densities ρ_c and $(1-\rho_c)$.

With bidirectional disorder with tilt, the system phase separates at all values of ρ into phases with densities close to 0 and 1. The deviations from these values are proportional to the value of the current (Eq. 6), and approach zero as $L \to \infty$.

With bidirectional disorder and no tilt, we have the problem of hard core particles in a Sinai potential. The phases are again characterized by $\rho = 0$ or 1, as evidenced by $m_o = 1$ in Eq. 10, but have some unusual properties: the interfacial region is very broad, and there are strong sample to sample variations in the length of the longest ordered stretch.

Finally, the state reached through dynamical disorder is surprisingly similar to the previous case, with the difference that $m_o \neq 1$ in this case.

Turning to dynamical properties in the steady state, we have seen that results are consistent with the kinematic wave criterion, namely that a finite velocity of the kinematic wave renders disorder irrelevant; the universality class characterizing the decay of fluctuations remains unchanged. The determination of the disorder-determined universality class when the wave speed vanishes remains an open problem, for instance, for particle-wise disorder with $n \leq 1$.

The coarsening behaviour characterizing the approach to the steady state is well understood in several cases (Eqs. 3, 11, 12, 14). However, the coarsening properties of a system of hard core particles in the Sinai potential, which corresponds to the untilted bidirectional case, remains an interesting open problem.

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