On the frequency of Titchmarsh's phenomenon for $\zeta(s)$—III

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Abstract. We obtain a lower bound for max $|\zeta(\tfrac{1}{2}+it)|$ as t varies over $T < t < T + Y$, where $(\log T)^{1/100} \leqslant Y \leqslant T$, as a function of Y ($1/100$ is unimportant). Our lower bound is exp $\{D(\log Y)^{\frac{1}{2}} (\log \log Y)^{-\frac{1}{2}}\}$ where D is a positive constant. (After submitting this paper for publication we came to know through a preprint of H L Montgomery that he had proved our result in the case $Y = T$. In his proof an essential assumption is Riemann hypothesis and our result is independent of any such unproved hypothesis. However he has other new results which are free from any hypothesis).

1. Introduction and statement of the result

The object of this paper is to prove the following:

Theorem. Let C be any positive constant, $T \geqslant 200$, $\log T \geqslant (200)^{1/C}$, and $(\log T)^C \leqslant Y \leqslant T$. Then there exists a positive constant D depending only on C such that

$$\max_{T < t \leq T + Y} |\zeta(\tfrac{1}{2}+it)| > \exp\left\{ D\left(\frac{\log Y}{\log \log Y}\right)^{\frac{1}{2}} \right\}.$$

Remark 1. Levinson (1972) proved in his paper, Ω-theorems for the Riemann-zeta function that

$$\max_{1 < t < 2T} |\zeta(\tfrac{1}{2}+it)|$$

exceeds

$$\exp\left\{ \frac{D_1(\log T)^{\frac{1}{2}}}{\log \log T} \right\}.$$

Our result gives an improvement of this result in 2 ways. First when $Y = T$ and next we have a new result with the parameter Y. By taking $Y = T^\theta$ with a constant θ $0 < \theta < 1$ we see that between T and $2T$ there are $\gg T^{1-\theta}$ points t (no two of which, are at a distance $\leqslant 1$) at which $|\zeta(\tfrac{1}{2}+it)|$ is large.

Remark 2.   Our proof runs closely along the lines of Ramachandra's (1974) paper. The new ideas are embodied in lemmas 1 to 4 below.

Remark 3.   Let $\{a_n\}$ be a sequence of complex numbers with the following properties.   (i) The functions

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad (s = \sigma + it),$$

convergent somewhere in the complex plane, can be continued in $\sigma \geqslant \frac{1}{2}$, $T \leqslant t \leqslant T + Y$ analytically and there $|F(s)| \leqslant T^A$ where A is a constant, (ii) There exists an infinite set S of primes and an integer constant q such that a_n are real and of the same sign (0 can be interpreted of either sign) when n runs through integers composed entirely of the prime factors in S (to the first power) and those of q to any power, (iii) whenever n is of the form $q \prod_{p \in S} p^{b(p)}$ with $b(p) = 0$ or 1 and $b(p) = 0$ for all but finitely many p, $|a_n|$ is bounded below, (iv) There exists a constant D_2 such that for all $x \geqslant 10$, $\sum_{x < p < D_2 x}^{*} 1$ (* denotes the restriction to the primes in S) lies between two constant multiples of $x / \log x$ (note: upper bound is always satisfied).   Under those conditions we can assert that

$$\max_{T < t < T+Y} |F(\tfrac{1}{2}+it)| \geqslant \exp\left\{ D_3 \left(\frac{\log Y}{\log \log Y} \right)^{\frac{1}{2}} \right\}$$

where the notation is as in the theorem.   It is also possible to prove that for every constant σ in $\frac{1}{2} < \sigma < 1$ we have

$$\max_{T < t < T+Y} |F(\sigma+it)| \geqslant \exp\left\{ D_4 \frac{(\log Y)^{1-\sigma}}{\log \log Y} \right\}$$

and that for $(\log \log T)^C \leqslant Y \leqslant T$ with $\log \log T \geqslant (200)^{1/C}$.

$$\max_{T < t < T+Y} |F(1+it)| > D_5 \log \log Y.$$

The proof of these generalisations are left to the reader. $F(s)$ can be taken for instance $\sum_{n=0}^{\infty} (an+b)^{-s}$ (where a and b are positive integers) or the zeta function of a ray class in an algebraic number field and so on.   It is also possible to formalize the result $[\,|\zeta(1+it)|\,]^{-1}$ is infinitely often bigger than $\log \log t$ so as to include the reciprocals of zeta functions, L series of number fields and so on.

2. Proof of the theorem

We will denote the positive constants by E, C_1, C_2, ..., 0 constants will be absolute and p will denote primes.   Let k be a positive integer $\geqslant 10$.   Put

$$(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s} \quad (\text{Re } s > 1)$$

and then for $\sigma > \frac{1}{2}$ put $f_k(\sigma) = \sum\limits_{n=1}^{\infty} (d_k(n))^2 \, n^{-2\sigma}$. Our first object is to obtain sharp upper and lower bounds for $f_k(\sigma)$. This is done in lemmas 1 to 4.

Lemma 1.

Put $\qquad A_p = 1 + \sum\limits_{r=1}^{\infty} (d_k(p^r))^2 \, p^{-2r\sigma}$ $\qquad\qquad$ where of course

$$d_k(p^r) = \frac{k(k+1)\ldots(k+r-1)}{r!} \leqslant k d_{k+1}(p^{r-1}) \qquad\qquad \text{then we have}$$

$$1 + k^2 p^{-2\sigma} < A_p < \left(\sum\limits_{r=0}^{\infty} d_k(p^r) \, p^{-\sigma} \right)^2 = (1 - p^{-\sigma})^{-2k}.$$

Also

$$A_p < 1 + k^2 p^{-2\sigma} \sum\limits_{r=1}^{\infty} (d_{k+1}(p^{r-1}))^2 \, p^{-2(r-1)\sigma} < 1 + k^2 p^{-2\sigma}(1 - p^{-\sigma})^{-2k+1}$$

Proof. Trivial.

Lemma 2. For $p > k^{1/\sigma}$ we have $(1 - p^{-\sigma})^{-2(k+1)} < (1 - 1/k)^{-2(k+1)} < 1000$, and so for $p > k^{1/\sigma}$ we have $A_p < 1 + 1000 \, k^2 p^{-2\sigma}$.

Proof. Trivial.

Lemma 3. We have
$$\prod\limits_{p} (1 + k^2 p^{-2\sigma}) < f_k(\sigma) < \prod\limits_{p > k^{1/\sigma}} (1 + 1000 k^2 p^{-2\sigma}) \prod\limits_{p \leqslant k^{1/\sigma}} (1 - p^{-\sigma})^{-2k}$$

Proof. Follows from lemmas 1 and 2.

Lemma 4. Put $\delta = \sigma - \frac{1}{2}$ and assume that $\delta \leqslant (\log k)^{-1}$ (we can also assume that $\delta = O((\log k)^{-1})$. Then there exist positive constants C_1, C_2 such that

$$\exp \{ C_1 k^2 L \} < f_k(\sigma) < \exp \{ C_2 k^2 L \},$$

where $L = \log (e (\delta \log k)^{-1})$.

Proof. For $p > (2k)^{1/\sigma}$, $1 + k^2 p^{-2\sigma} > \exp (\frac{1}{2} k^2 p^{-2\sigma})$ since for $0 < x < \frac{1}{2}$, $e^{x/2} < 1 + x$.

We have only to check that $\sum\limits_{p > (2k)^{1/\sigma}} p^{-2\sigma} \gg (\delta \log k)^{-1}$.

To see this put $U_n = 2^n (2k)^{1/\sigma}$ $(n = 1, 2, 3, \ldots)$. We have

$$\sum\limits_{U_n < p < U_{n+1}} p^{-2\sigma} \gg (\log U_{n+1})^{-1} \, U_{n+1}^{-2\delta} \quad \text{and if} \quad U_{n+1} \leqslant k^{C_3 (\delta \log k)^{-1}}$$

then

$$U_{n+1} \leqslant e^{C_3 \delta^{-1}}$$

and so

$$U_{n+1}^{-2\delta} \geqslant e^{-2C_3}.$$

If C_3 is a large constant we see that $\Sigma\,(\log U_{n+1})^{-1} \gg (\delta\,\log\,k)^{-1}$, where U_{n+1} runs over all possible values with $U_2 \leqslant U_{n+1} \leqslant e^{C_3\delta^{-1}}$. This proves the lower bound in the lemma.

To prove the upper bound we see that

$$\log \prod_{p\,\leq\,k^{1/\sigma}} (1-p^{-\sigma})^{-2k}$$

is $O(k)^2$ and also that

$$\log \pi\,(1+1000\,k^2 p^{-2\sigma})$$

$$\leqslant \log\,(\prod_{p\,\geq\,k^{1/\sigma}} (1-p^{-2\sigma})^{-1})^{1000k^2} = \log\,(\zeta(2\sigma) \prod_{p\,\leq\,k^{1/\sigma}} (1-p^{-2\sigma})^{1000k^2}).$$

Note that

$$\prod_{p\,\leq\,k^{1/\sigma}} (1-p^{-2\sigma}) = O\left(\frac{1}{\log\,k}\right).$$

This proves the lemma completely.

Lemma 5. Let E be a large positive constant and $\sigma = \tfrac{1}{2}+(\log\,k)^{-1}$. Fix k to be the largest integer satisfying $\exp\,(E\,k^2\,\log\,k) \leqslant Y$. Put $X = Y^{1/10}$. Then

$$\sum_{n=1}^{\infty} (d_k(n))^2\,n^{-2\sigma}\,e^{-2n/X} \gg \exp\,(C_1 k^2).$$

Also the error in breaking the series at $n = [X\,(\log\,X)^3]$ is $O(100^k)$.

Proof. By Lemma 4, $f_k(\sigma)$ lies between $e^{C_1 k^2}$ and $e^{C_2 k^2}$. It suffices to prove that $\underset{n>X}{\Sigma}\,(d_k(n))^2\,n^{-2\sigma}$ does not exceed 1. This is clear since an upper bound for this sum is $X^{-(\log\,k)^{-1}}\,f_k\,(\sigma-(2\,\log\,k)^{-1})$ which is by lemma 4 less than 1 if E is large. The last remark follows from the fact that if $n \geqslant X\,(\log\,X)^3$, $e^{-2n/X} = O\,(n^{-10})$ and so the total contribution from such terms in $O\,((\,\overset{\infty}{\underset{1}{\Sigma}}\,d_k(n)\,n^{-2})^2)$.

Lemma 6. Let now $\underset{T\,\leq\,t\,\leq\,T+\,Y}{\max}\,|\,\zeta(\tfrac{1}{2}+it)\,| = M$. Then there exists a constant C_5 such that in the region $T+(Y/800) \leqslant t \leqslant T+Y-Y/800$, $\sigma \geqslant \tfrac{1}{2}$ we have

$$\max\,|\,\zeta(s)\,| \leqslant C_5\,(M+1).$$

Proof. It suffices to confine to the sub region $\frac{1}{2} \leqslant \sigma \leqslant 2$. Let s_0 be a point at which the maximum is attained. Then we apply maximum modulus principle to the function $\zeta (s) e^{(s-s_0)^{4a+2}}$ (where a is a large positive integer constant depending on C) to the rectangle $T \leqslant t \leqslant T+Y$, $\frac{1}{2} \leqslant \sigma \leqslant 2$. The proof follows.

Lemma 7. Let $\sigma = \frac{1}{2} + (\log k)^{-1}$ and $T + Y/400 \leqslant t \leqslant T + Y - Y/400$. Then either

$$\exp (10^{-10} \, Y) \leqslant (C_6(M+1))^k$$

or

$$\left| \sum_{n=1}^{\infty} d_k(n)n^{-s} \, e^{-n/X} \right| \leqslant (C_7(M+1))^k.$$

Also the error in breaking off the series at $n=[X(\log X)^3]$ is at most 100^k in absolute value.

Proof. We start with

$$\sum_{n=1}^{\infty} d_k(n)n^{-s} \, e^{-n/X} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (\zeta(s+w))^k \, \Gamma (w) X^w \, dw.$$

We first break off the portions Im $(s+w) \leqslant T + Y/600$ and Im $(s+w) \geqslant T + Y - Y/600$ of the integral and move the remaining portion to Re $w = \frac{1}{2} - \sigma$. This proves the lemma.

Lemma 8. Let $\sigma = \frac{1}{2} + (\log k)^{-1}$ and the other parameters as before. Then

$$\int_I \left| \sum_{n < Y^{1/9}} d_k (n)n^{-s} \, e^{-n/x} \right|^2 dt \gg \sum_{n < Y^{1/9}} \frac{(d_k (n))^2}{n^{2\sigma}} \, e^{-2n/X} (Y+O(Y^{1/8}))$$

where I is the interval for t given by Lemma 7.

Proof. Follows from the well known result that for arbitrary $\{a_n\}$ and $T \geqslant 10$, $N \geqslant 2$,

$$\int_0^T \left| \sum_{n \leqslant N} a_n \, n^{-it} \right|^2 dt = \sum_{n \leqslant N} (T+O(N \log N)) \, |a_n|^2.$$

The well known result follows on using

$$|\log (n/n')| \gg \left| \frac{n-n'}{n+n'} \right|$$

for positive integers n, n'.

Lemma 9. Let

$$\sigma = \tfrac{1}{2} + (\log k)^{-1}.$$

Then there exists a point $s = \sigma + it$, with

$$T + Y/400 \leqslant t \leqslant T + Y - Y/400$$

such that

$$\left| \sum_{n=1}^{\infty} d_k(n) n^{-s} e^{-n/X} \right| > e^{C_8 k^2}.$$

Proof. Follows from lemmas 5 and 8.

Lemma 10. Let k be defined as the largest integer satisfying $\exp(Ek^2 \log k) \leqslant Y$ (with a large constant E defined already). Then at least one of the following two inequalities is true:

$$(C_6(M+1))^k \geqslant \exp(10^{-10} Y), \quad e^{C_8 k^2} \leqslant ((M+1)C_9)^k \quad \text{i.e.}$$

$$\log M \gg \min(k, k^{-1} Y).$$

Proof. Follows from lemmas 7 and 9.
       Lemma 10 proves the theorem completely.

3. Final remarks

To prove the first result of remark 3 we use $\sum'_{n \leqslant X} (d_k(n))^2 (nq^k)^{-2\sigma}$ (where n runs through square free integers generated by S, and X defined to be $e^{E'k^2 \log k}$, E' being now a large constant) exceeds $\exp(C'k^2)$. The second result is also proved similarly. We take

$$X = \prod'_{p < k^{1/\sigma}} p$$

and the lower bound $\prod'_{p < k^{1/\sigma}} (1 + k^2 p^{-2\sigma})$ for $f_k(\sigma)$, (here $\prod'$ denotes primes in S).
The last result in remark 3 follows by

$$\prod_{p} \left(1 + \sum_{r=1}^{k} p^{2\sigma r} \left(\frac{k(k-1)\dots(k-r+1)}{r!} \right)^2 \right) >$$

$$\prod_{p < k^{1/\sigma} (\log k)^{-2}} \left\{ p^{-2\sigma r} p \left(\frac{k(k-1)\dots(k-r_p+1)}{r_p!} \right)^2 \right\}$$

where $r_p = [kp^{-\sigma}]$. We define X to be $\prod_{p < k^{1/\sigma} (\log k)^{-2}} (p^r p)$.

We have then to use asymptotic formulae for $n!$. Also for this result we need in place of $|F(s)| \leqslant T^A$ in the region mentioned, the result $|F(s)| \leqslant (\log T)^A$ in the region $T \leqslant t \leqslant T+Y, \sigma \geqslant 1$, we need analytic continuation in $\sigma \geqslant 1$ and no further information for $F(s)$ in $\sigma < 1$ is required. More precise information than the above theorems about $|\zeta(s)|$ is contained (by way of upper bounds for the maximum, for some set of values of T and Y) without any hypothesis, in Ramachandra's (1977) paper.

Appendix

Let $\{a_n\}$ be a sequence of complex numbers such that the dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is convergent for some s. For each positive integer k define

$$(f(s))^k = \sum_{n=1}^{\infty} a_k(n) \, n^{-s}.$$

Suppose $f(s)$ can be continued analytically in $\sigma \geqslant a, T \leqslant t \leqslant T+Y$ where Y is as in theorem 1, and in that region satisfies $|f(s)| \leqslant T^A$.
Put

$$M = \max_{T < t < T+Y} |f(a+it)|.$$

Then arguing on the line $a + (\log Y)^{-1}$ as in the proof of theorem 1 (note that the first conclusion of lemma 7 is unnecessary if we choose $X = Y^{\frac{1}{2}}$) we are led to the following theorem.

Theorem 2. There exist positive constants C_{10} and C_{11} independent of a, k and Y such that

$$M + C_{10} > C_{11}(\phi(a, Y, k) \, (\log Y)^{-2})^{1/2k}$$

for every k, where

$$\phi(a, Y, k) = \sum_{n < Y^{1/10}} |a_k(n)|^2 \, n^{-2a}.$$

Corollary. Let $\psi(s)$ be an analytic function in the region $\sigma \geqslant a, T \leqslant t \leqslant T+Y$ and suppose $\psi(s)$ is bounded both above and below by positive constants. Put

$$M_1 = \max_{T < t < T+Y} \text{Re}(\psi(a+it) \, f(a+it))$$

$$M_2 = M_1 \, (\log Y)^2.$$

Then M_2 also satisfies an inequality similar to the conclusion of theorem 2 provided that in addition to $Y \leqslant T$ we also have $\log Y / \log T \gg 1$.

Remark. Consider the special case $\zeta(s)$. Put

$$\phi_a(X) = \text{maximum of } (\phi(a, X, k))^{1/2k}$$

as k varies over all positive integers. Let a be any constant satisfying $\frac{1}{2} \leqslant a < 1$. Then the best Ω theorem for $|\zeta(a+it)|$ obtainable by our contributions to the fundamental ideas of Titchmarsh is $|\zeta(a+it)| = \Omega(\phi_a(t))$. It is easy to prove that

$$\log \phi_a(X) \gg \frac{(\log X)^{1-a}}{(\log \log X)^{\theta(a)}}$$

where

$$\theta(a) = \tfrac{1}{2} \text{ if } a = \tfrac{1}{2} \text{ and } 1 \text{ if } \tfrac{1}{2} < a < 1.$$

By an ingenious argument Balasubramanian (to appear) has proved that*

$$\log \phi_a(X) \ll \frac{(\log X)^{1-a}}{(\log \log X)^{\theta(a)}} .$$

This shows that either our results are best possible or only slight improvements are possible and ideas entirely different from Titchmarsh's are necessary for such improvements.

Theorem 3. If $\frac{1}{2} < a < 1$, then we have

$$\frac{1}{k} \log \left(\sum_{n < X} \frac{d_k^2(n)}{n^{2\sigma}} \right) = O \left(\frac{(\log X)^{1-\sigma}}{(2\sigma-1) \log \log X} \right), \text{ where the } O\text{-constant is}$$

absolute.

Theorem 4. We have

$$\frac{1}{k} \log \left(\sum_{n < X} \frac{d_k^2(n)}{n} \right) = O \left(\sqrt{\frac{\log X}{\log \log X}} \right) \text{ where the } O\text{-constant is}$$

absolute.

Proof. For the proof of the theorems, we need the following

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*Balasubramanian has simplified his proof very much and the proof as it stands now is not worth publishing separately. So we give it as a continuation of this appendix. (We may also note that in the definition of $\phi_a(X)$, X is arbitrary and should not be confused with the earlier limitation on X.)

Lemma 11. If $\frac{1}{2} < \sigma < 1$, then

$$\log \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} = O\left(\frac{k^{1/\sigma}}{(2\sigma-1)\log k}\right)$$

where the O-constant is absolute.

Proof. We have, by an application of lemmas 1 and 2,

$$\log \sum_{n-1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} = \sum_{p-1}^{\infty} \log A_p$$

$$= \underset{p < k^{1/\sigma}}{\Sigma} \log A_p + \underset{p > k^{1/\sigma}}{\Sigma} \log A_p$$

$$= O\left(k \underset{p < k^{1/\sigma}}{\sum} \frac{1}{p^{\sigma}}\right) + O\left(\underset{p > k^{1/\sigma}}{\sum} \frac{k^2}{p^{2\sigma}}\right)$$

$$= O\left(\frac{k^{1/\sigma}}{\log k}\right) + O\left(\frac{k^{1/\sigma}}{(2\sigma-1)\log k}\right)$$

$$= O\left(\frac{k^{1/\sigma}}{2\sigma-1)\log k}\right).$$

Proof of Theorem 3. If $\log X^{\sigma} > k^{1/\sigma}$, then, using lemma 11,

$$\frac{1}{k}\log \sum_{n < X} \frac{d_k^2(n)}{n^{2\sigma}} = O\left(\frac{1}{k}\log \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}}\right).$$

$$= O\left(\frac{k^{1/\sigma-1}}{(2\sigma-1)\log k}\right)$$

$$= O\left(\frac{(\log X)^{1-\sigma}}{(2\sigma-1)\log\log X}\right).$$

Hence we can assume that $\log X \leqslant k^{1/\sigma}$.

Now put $\delta = \dfrac{k}{(\log X)^{\sigma} \log\log X}$.

Hence $\dfrac{1}{k}\log \sum_{n < X} \dfrac{d_k^2(n)}{n^{2\sigma}} = O\left(\dfrac{1}{k}\log\left(X^{2\delta} \sum_{n=1}^{\infty} \dfrac{d_k^2(n)}{n^{2\sigma+2\delta}}\right)\right)$

which is $O\left(\dfrac{(\log X)^{1-\sigma}}{\log\log X}\right)$ if $\delta > 10$.

If $\delta < 10$, then, using lemma 11,

$$\frac{1}{k} \log \sum_{n < X} \frac{d_k^2(n)}{n^{2\sigma}} = O\left(\frac{1}{k} \log\left(X^{2\delta} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma+2\delta}}\right)\right)$$

$$= O\left(\frac{2\delta}{k} \log X\right) + O\left(\frac{K^{(1/\sigma+\delta)-1}}{(2(\sigma+\delta)-1) \log k}\right).$$

By the choice of δ,

$$\frac{2\delta}{k} \log X = O\left(\frac{(\log X)^{1-\sigma}}{\log \log X}\right).$$

Since $\delta < 10$, and $k \geqslant (\log X)^\sigma$,

$$\frac{k^{(1/\sigma+\delta)-1}}{(2(\sigma+\delta)-1) \log k} = O\left(\frac{k^{1/\sigma-1-\delta/100}}{(2\sigma-1) \log k}\right)$$

$$= O\left(\frac{k^{1/\sigma-1}}{(2\sigma-1) \log k} \cdot \exp\left(-\frac{\delta}{100} \log k\right)\right).$$

$$\exp\left(\frac{\delta}{100} \log k\right) = \exp\left(\frac{k \log k}{100 (\log X)^\sigma \log \log X}\right)$$

$$\geqslant \exp\left(\frac{k}{100 (\log X)^\sigma}\right)$$

$$\gg \left(\frac{k}{(\log X)^\sigma}\right)^{1/\sigma-1} \frac{\log k}{\log \log X},$$

since

$$e^{a/b} \geqslant \left(\frac{a}{b}\right)^\alpha \frac{\log a}{\log b},$$

uniformly in $0 \leqslant a \leqslant 2$ and $b \geqslant a \geqslant 2$, as can be easily verified by distinguishing the cases $a < b^2$ and $a \geqslant b^2$. This completes the proof of theorem 3.

Proof of Theorem 4:   Let $\delta = \dfrac{k}{\sqrt{\log X} \sqrt{\log \log X}}$

Now $\dfrac{1}{k} \log \sum_{n < X} \dfrac{d_k^2(n)}{n} = O\left(\dfrac{1}{k} \log\left(X^{2\delta} \sum_{n=1}^{\infty} \dfrac{d_k^2(n)}{n^{1+2\delta}}\right)\right)$

$$= O\left(\frac{2\delta}{k}\log X\right) + O\left(\frac{1}{k}\log\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{1+2\delta}}\right)$$

$$= O\left(\sqrt{\frac{\log X}{\log\log X}}\right) + O\left(\frac{1}{k}\log\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{1+2\delta}}\right).$$

If $\log X > k^2\log k$, then $\delta = O\left(\dfrac{1}{\log k}\right)$ and consequently, by lemma 4,

$$\frac{1}{k}\log\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{1+2\delta}} = O\left(k\log\left(\frac{e}{\delta\log k}\right)\right)$$

$$= O\left(k\log\left(\frac{e\sqrt{\log X}\sqrt{\log\log X}}{k\log X}\right)\right)$$

$$= O\left(\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\right).$$

as can be easily verified.

If $\log X < k^2\log k$, then using lemma 11, we have

$$\frac{1}{k}\log\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{1+2\delta}} = O\left(\frac{k}{\delta\log k}\right) = O\left(\frac{\sqrt{\log X}\sqrt{\log\log X}}{k\log k}\right).$$

$$= O\left(\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\right).$$

and this completes the proof.

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