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On the zeros of a class of generalised Dirichlet series–XIV

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Dedicated to the memory of Professor K G Ramanathan

Abstract. We prove a general theorem on the zeros of a class of generalised Dirichlet series. We quote the following results as samples.

Theorem A. *Let $0<\theta<\frac{1}{2}$ and let $\{a_n\}$ be a sequence of complex numbers satisfying the inequality $\left|\sum_{m=1}^{N} a_m - N\right| \leqslant (\frac{1}{2}-\theta)^{-1}$ for $N=1,2,3,\ldots$, also for $n=1,2,3,\ldots$ let α_n be real and $|\alpha_n|\leqslant C(\theta)$ where $C(\theta)>0$ is a certain (small) constant depending only on θ. Then the number of zeros of the function*

$$\sum_{n=1}^{N} a_n(n+\alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n(n+\alpha_n)^{-s} - n^{-s})$$

in the rectangle $(\frac{1}{2}-\delta\leqslant\sigma\leqslant\frac{1}{2}+\delta, T\leqslant t\leqslant 2T)$ (where $0<\delta<\frac{1}{2}$) is $\geqslant C(\theta,\delta)\, T\log T$ where $C(\theta,\delta)$ is a positive constant independent of T provided $T\geqslant T_0(\theta,\delta)$ a large positive constant.

Theorem B. *In the above theorem we can relax the condition on a_n to $\left|\sum_{m=1}^{N} a_m - N\right| \leqslant (\frac{1}{2}-\theta)^{-1}N^{\theta}$ and $|a_N|\leqslant(\frac{1}{2}-\theta)^{-1}$. Then the lower bound for the number of zeros in $(\sigma\geqslant\frac{1}{2}-\delta, T\leqslant t\leqslant 2T)$ is $>C(\theta,\delta)\, T\log T(\log\log T)^{-1}$. The upper bound for the number of zeros in $(\sigma\geqslant\frac{1}{2}+\delta$, $T\leqslant t\leqslant 2T)$ is $O(T)$ provided $\sum_{n\leqslant x} a_n = x + O_\varepsilon(x^\varepsilon)$ for every $\varepsilon>0$.*

Keywords. Generalised Dirichlet series; distribution of zeros; neighbourhood of the critical line.

1. Introduction

This paper ought to have been paper XII of this series. But elsewhere [5] the second author has referred to this paper as paper XIV, because there are two new additions to this series namely, On the zeros of $\zeta'(s)-a$, (on the zeros of a class of generalised Dirichlet series-XII) and On the zeros of $\zeta(s)-a$, (on the zeros of a class of generalized Dirichlet series-XIII) both of which will appear in *Acta Arithmetica* with the short titles only. The addition elsewhere of the title in the brackets have been made only for some technical convenience. In the present paper we continue the investigations of the papers III [1], IV [2], V [4], and VI [3]. Just as VI [3], was in the nature of an addendum to the earlier papers, this note is a modest progress beyond the paper

VI [3], and the previous papers. Apart from an innovation, the main change consists in replacing the old kernel $\exp(w^{4k+2})$ by the function $R(w)=\exp((\sin w/100)^2)$. Thus in place of $\Delta(\chi)$, our new function $\Delta_1(\chi)$ will be defined for all $\chi>0$ by

$$\Delta_1(\chi)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\chi^w R(w)\frac{dw}{w}.$$

Also by moving the line of integration from $\mathrm{Re}\,w=2$ to $\mathrm{Re}\,w=-2$, we see that $\Delta_1(\chi)=O(\chi^2)$ and also $\Delta_1(\chi)=1+O(\chi^{-2})$ where the O-constants are absolute. As a special case of a more general theorem we prove the following two theorems.

Theorem 1. *Let $0<\theta<\frac{1}{2}$ and let $\{a_n\}$ be a sequence of complex numbers satisfying the inequality*

$$\left|\sum_{m=1}^{N}a_m-N\right|\leqslant\left(\frac{1}{2}-\theta\right)^{-1}$$

for $N=1,2,3,\ldots$. Also for $n=1,2,3,\ldots$ let α_n be real with $|\alpha_n|\leqslant C(\theta)$ where $C(\theta)>0$ is a certain (small) constant depending only on θ. Then the number of zeros of the function

$$\sum_{n=1}^{\infty}a_n(n+\alpha_n)^{-s}=\zeta(s)+\sum_{n=1}^{\infty}(a_n(n+\alpha_n)^{-s}-n^{-s})$$

in the rectangle $(\sigma\geqslant\frac{1}{2}-\delta,\ T\leqslant t\leqslant 2T)$ (where $0<\delta<\frac{1}{2}$) is $\geqslant C(\theta,\delta)\,T\log T$, where $C(\theta,\delta)$ is a positive constant independent of T provided $T\geqslant T_0(\theta,\delta)$, a large constant.

Theorem 2. *Let $0<\theta<\frac{1}{2}$ and $\{a_n\}$ a sequence of complex numbers satisfying the inequalities*

$$|a_N|\leqslant\left(\frac{1}{2}-\theta\right)^{-1}\quad\text{and}\quad\left|\sum_{m=1}^{N}a_m-N\right|\leqslant\left(\frac{1}{2}-\theta\right)^{-1}N^{\theta}$$

for $N=1,2,3,\ldots$. Let α_n be as before. Then the number of zeros of the function

$$\sum_{n=1}^{\infty}a_n(n+\alpha_n)^{-s}=\zeta(s)+\sum_{n=1}^{\infty}(a_n(n+\alpha_n)^{-s}-n^{-s})$$

in the rectangle $(\sigma\geqslant\frac{1}{2}-\delta, T\leqslant t\leqslant 2T)$ (where $0<\delta<\frac{1}{2}-\theta$) is $>C(\theta,\delta)\,T\log T\,(\log\log T)^{-1}$, where $C(\theta,\delta)$ is a positive constant independent of T provided $T\geqslant T_0(\theta,\delta)$, a large positive constant.

Remark 1. In Theorem 1 the number of zeros of the function in question in $(\sigma\geqslant\frac{1}{2}+\delta, T\leqslant t\leqslant 2T)$ is $O(T)$. But in Theorem 2 to prove a similar result, we require $\sum_{n\leqslant x}a_n=x+O_\varepsilon(x^\varepsilon)$ for every $\varepsilon>0$. To prove these we have to prove that the mean square of the absolute value of the function in question in $(T,2T)$ is $O_\varepsilon(T^\varepsilon)$ (for every $\varepsilon>0$) on the line $\sigma=\frac{1}{2}$. We have then to use an idea of J E Littlewood (see Theorem 9.15 (A) on page 230 of [7]).

Remark 2. Let $\{x_n)$ and $\{y_n\}$ be any two sequences of complex numbers and let $0<\lambda_1<\lambda_2<\ldots$ and further let $\lambda_{n+1}-\lambda_n (n=1,2,\ldots)$ lie between two positive constants. Then

$$\frac{1}{T}\int_0^T\left(\sum_{n=1}^{\infty} x_n\lambda_n^{it}\right)\left(\sum_{n=1}^{\infty} \bar{y}_n\lambda_n^{-it}\right)dt=\sum_{n=1}^{\infty} x_n\bar{y}_n+O\left(T^{-1}\left(\sum_{n=1}^{\infty} n|x_n|^2\right)^{1/2}\times\left(\sum_{n=1}^{\infty} n|y_n|^2\right)^{1/2}\right)$$

where the O-constants depend only on the constants appearing in the conditions for the sequence $\{\lambda_n\}$. Also $\bar{y}_n$ denotes the complex conjugate of y_n. This fundamental result is due to H L Montgomery and R C Vaughan (see [6] for a simple proof due to the second of us). It will be very much useful for our work.

2. Notation

From now on we adopt the following notation. The symbol $\Delta_1(\chi)$ is already explained. We begin by explaining two Dirichlet series

$$\sum_{n=1}^{\infty} a_n b_n \lambda_n^{-s} \text{ and } F(s)=\sum_{n=1}^{\infty} a_n b_n \mu_n^{-s}$$

satisfying the conditions (i) to (ix) below. (We nearly borrow from VI [3]. Note the following typographical corrections. In place of $g(x)g'(x)$ in the condition (iii) on page 247 of VI [3] there should be $g(x)g''(x)$. Again in VI [3] page 248 line 7 from the top, x should be X and there should be extra term $X^{1-2\sigma}(f(X))^2$ in the bracket and in line 8 from the top "X". should read X and σ if $\sigma\geqslant 0$. Also in V [4] on page 304 line 11 from the bottom $F(s)$ should be ψ). Throughout we assume $a_n=O(1)$.

(i) $0<\lambda_1<\lambda_2<\ldots$ and $\lambda_{n+1}-\lambda_n (n=1,2,3,\ldots)$ should lie between two positive constants. The sequence $\{\lambda_n\}$ is further restricted by the condition (vii) or (viii) as the case may be.
Let $f(x)$ and $g(x)$ be two positive real valued functions defined in $x\geqslant 0$ satisfying.

(ii) $f(x)x^{\eta}$ is monotonic increasing and $f(x)x^{-\eta}$ is monotonic decreasing for every fixed $\eta>0$ and all $x\geqslant x_0(\eta)$.

(iii) $\lim_{x\to\infty}(g(x)x^{-1})=1$.

(iv) For all $x\geqslant 0, 0<a\leqslant g'(x)\leqslant b$ and $0<a\leqslant (g'(x))^2-g(x)g''(x)\leqslant b$ where a and b are constants.
Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers having the following properties.

(v) $|b_n|$ lies between $a f(n)$ and $b f(n)$ for all n.

(vi) For all $X\geqslant 1$, $\sum_{X\leqslant n\leqslant 2X}|b_{n+1}-b_n|\ll f(X)$.

We next assume that $\{a_n\}$ and $\{b_n\}$ satisfy at least one of the following two conditions (vii) and (viii).

(vii) Monotonicity condition. There exists an arithmetic progression $\mathscr{A}$ such that

$$\lim_{x\to\infty}\left(x^{-1}\sum_{n\leqslant x}{}' a_n\right)=h,\ (h\neq 0),$$

where the accent denotes the restriction of n to $\mathscr{A}$. Also $|b_n|\lambda_n^{-1/400}$ is monotonic decreasing as n varies over $\mathscr{A}$.

(viii) Real part condition. There exists an arithmetic progression $\mathscr{A}$ of integers such that

$$\liminf_{x\to\infty}\left(\frac{1}{x}\sum_{x\leqslant \lambda_n\leqslant 2x,\mathrm{Re}\, a_n>0}{}' \mathrm{Re}\, a_n\right)>0$$

and

$$\lim_{x\to\infty}\left(x^{-1}\sum_{x\leqslant \lambda_n\leqslant 2x,\mathrm{Re}\, a_n<0}{}' \mathrm{Re}\, a_n\right)=0$$

where the accent denotes the restriction of n to $\mathscr{A}$. (We can manage with $\mathrm{Im}\, a_n$; but this is included in the condition stated since we can change b_n in φ on p. 304 of V [4] to ib_n in fact to $\pm b_n$ or $\pm ib_n$).

Note that any of (vii) or (viii) implies that

$$\left|\sum_{n\leqslant x}{}' a_n|b_n|^2\lambda_n^{-2\sigma}\right|\gg\frac{x^{1-2\sigma}(f(x))^2}{1-2\sigma},$$

for $\sigma<\frac{1}{2}$ and σ close to $\frac{1}{2}$, where the constant implied by $\gg$ is independent of σ.

(ix) Finally let $\beta(>0)$ be a constant. We write $\lambda_n=\beta g(n)$ for n in $\mathscr{A}$. Otherwise λ_n are arbitrary but the sequence $\{\lambda_n\}$ is subject to the condition (i), mentioned above. Next we write $\mu_n=\lambda_n+\alpha_n$ where $\{\alpha_n\}$ is any sequence of real numbers subject to $|\alpha_n|\leqslant C_1$, C_1 being a positive constant which is small enough. How small should C_1 be will be stated later. (C_1 will be independent of the constant δ which appears from Theorem 7 onwards).

Remark 1. The earlier results were proved with the condition $\lambda_n=g(n)+u_n+v_n$ (for all n) where $\{u_n\}$ and $\{v_n\}$ denoted two arbitrary monotonic bounded sequences of real numbers. Since bounded monotonic sequences of real numbers are convergent (say $u_n+v_n\to l$ as $n\to\infty$) and in place of $g(x)$, $g(x)+l$ satisfies the conditions satisfied by $g(x)$, the results of the present paper are more general. However we use the results of the earlier papers III [1], IV [2], V [4] and VI [3].

Remark 2. Our new results are Theorems 7, 8 and 9 and their Corollaries.

3. Some preparations

We begin by stating

Theorem 3. *Let* $F_1(s)=\sum_{n=1}^{\infty}(a_nb_n\Delta_1(T/\lambda_n)\lambda_n^{-s})$. *Then for* $0<\sigma<\frac{1}{2}$ *and* $T\geqslant 10$ *we* *have,*

$$\frac{1}{T}\int_T^{2T}|F_1(\sigma+it)|\mathrm{d}t > C_2\, T^{(1/2)-\sigma}f(T),$$

where $C_2>0$ is independent of T.

Also for $1\leqslant X\leqslant T$, we have,

$$\frac{1}{T}\int_T^{2T}\left|\sum_{n=1}^{\infty}\left(a_nb_n\Delta_1\left(\frac{X}{\lambda_n}\right)\lambda_n^{-\sigma-it}\right)\right|^2\mathrm{d}t < C_3X^{1-2\sigma}(f(X))^2,$$

where $0<\sigma<\frac{1}{2}$ and $C_3>0$ is independent of T and X.

We make two remarks by way of proof.

Remark 1. The first part of the theorem is nearly explained in V [4]. The role of $F_3(s)$ in Lemma 7 (Here $\max_{t \text{ in } I}|F_3(s)|>0$ should read $\max_{t \text{ in } I}|F_3(s)|\leqslant D$) of § 2 of paper VI [3] is played by $F_5(s)=\sum'_{\lambda_n\leqslant T} b_n\lambda_n^{-s}$ where the accent denotes the restriction of the sum to the integers $\mathscr{A}$ occurring in the condition (vii) or (viii) as the case may be. Then the function $F_5(s)$ possesses a gth power mean with $g=g(\sigma)>2$ if $\sigma<\frac{1}{2}$ in the sense

$$\frac{1}{T}\int_T^{2T}|F_5(\sigma+it)|^g\mathrm{d}t = O((T^{(1/2)-\sigma}f(T))^g).$$

This gth power moment is easily deducible from Lemma 6 of paper IV [2] which is quoted as Theorem 4 in paper V [4]. The rest of the proof follows V [4] except that $\exp(-(\lambda_n/T))$ is replaced by $\Delta_1(T/\lambda_n)$. The first part of the Theorem 3 is first proved for σ close to $\frac{1}{2}$ by the above method and then extended by convexity for all $\sigma(0<\sigma<\frac{1}{2})$.

Remark 2. Let $\sigma>0$. Then by using the theorem of Montgomery and Vaughan [6] quoted already we see that the LHS of the second inequality of Theorem 3 is

$$\leqslant C_4\left(\sum_{\lambda_n\leqslant X}(f(n))^2n^{-2\sigma}+X^2\sum_{\lambda_n\geqslant X}(f(n))^2n^{-2\sigma-2}+\frac{X^2}{T}\sum_{\lambda_n\geqslant X}(f(n))^2n^{-2\sigma-1}\right).$$

Using the fact that $f(n)n^{\eta}$ is monotonic increasing and $f(n)n^{-\eta}$ is monotonic decreasing for all fixed $\eta>0$ and all $n\geqslant n_0(\eta)$, we see that the theorem is proved.

Note that if $0<\mu<\frac{1}{2}-\sigma$ we have

$$X^{1-2\sigma}(f(X))^2 = X^{1-2\sigma-2\mu}(f(X)X^{\mu})^2 \leqslant X^{1-2\sigma-2\mu}(f(T)T^{\mu})^2$$
$$\leqslant\left(\frac{X}{T}\right)^{1-2\sigma-2\mu}T^{1-2\sigma}(f(T))^2$$

and so the RHS of the second inequality of Theorem 3 is

$$O\left(\left(\frac{X}{T}\right)^{1-2\sigma-2\mu}T^{1-2\sigma}(f(T))^2\right)$$

for all $X \geqslant X_0(\mu)$ and $T \geqslant T_0(\mu)$. We can fix $\mu = 1/4 - \sigma/2$ and $X = TD$ where $D(0 < D < 1$: not to be confused with D occurring in Remark 1 below Theorem 3) is a small constant. In that case this expression is $O(D^{(1/2)-\sigma}(f(T)\, T^{(1/2)-\sigma})^2)$, where the O-constant depends only on σ. Note also that the second part of Theorem 3 uses only the properties $0 < \lambda_1 < \lambda_2 < \ldots$ and $1 \ll \lambda_{n+1} - \lambda_n \ll 1$ of $\{\lambda_n\}$. We now state a Corollary to Theorem 3.

Theorem 4. *Let*

$$F_2(s) = \sum_{n=1}^{\infty} \left(a_n b_n \left(\Delta_1 \left(\frac{T}{\lambda_n} \right) - \Delta_1 \left(\frac{DT}{\lambda_n} \right) \right) \lambda_n^{-s} \right)$$

where $D(0 < D < 1)$ is a sufficiently small positive constant. Then if $0 < \sigma < \frac{1}{2}$, we have,

$$\frac{1}{T} \int_T^{2T} |F_2(\sigma + it)| \mathrm{d}t > C_5\, T^{1/2 - \sigma} f(T)$$

and

$$\frac{1}{T} \int_T^{2T} |F_2(\sigma + it)|^2 \mathrm{d}t < C_6\, T^{1 - 2\sigma} (f(T))^2,$$

where $C_5(>0)$ and $C_6(>0)$ are independent of T.

4. Main results

We now proceed to prove the analogue of Theorem 4 where $\{\lambda_n\}$ is replaced by $\{\mu_n\}$.

Theorem 5. *Let*

$$F_3(s) = \sum_{n=1}^{\infty} \left(a_n b_n \left(\Delta_1 \left(\frac{T}{\mu_n} \right) - \Delta_1 \left(\frac{DT}{\mu_n} \right) \right) \mu_n^{-s} \right),$$

where D is the positive constant occurring in $F_2(s)$. Then if $0 < \sigma < \frac{1}{2}$, we have,

$$\frac{1}{T} \int_T^{2T} |F_3(\sigma + it)| \mathrm{d}t > C_7\, T^{1/2 - \sigma} f(T)$$

and

$$\frac{1}{T} \int_T^{2T} |F_3(\sigma + it)|^2 \mathrm{d}t < C_8 (T^{1/2 - \sigma} f(T))^2,$$

where $C_7(>0)$ and $C_8(>0)$ are independent of T, provided $C_1(>0)$ of condition (ix) *in* § 2 *is sufficiently small.*

Remark. Our proof gives this theorem where the constants depend on σ but uniformly in a certain range for σ in $\sigma < \frac{1}{2}$. By convexity, the theorem can be upheld for all $\sigma < \frac{1}{2}$ uniformly in $\sigma \leqslant \frac{1}{2} - C_9$ where $C_9(>0)$ is any constant less than $\frac{1}{2}$. We can even secure C_1 to be independent of C_9, but C_7 and C_8 depend on C_9.

Proof. The second inequality follows by the well-known Montgomery-Vaughan theorem (see [6]) as in Remark 2 below Theorem 3. The first part can be deduced from that of Theorem 4 as follows. Put

$$\phi(u)=\left(\Delta_1\left(\frac{T}{\lambda_n+u}\right)-\Delta_1\left(\frac{DT}{\lambda_n+u}\right)\right)(\lambda_n+u)^{-s}.$$

Then

$$\Delta_1\left(\frac{T}{\mu_n}\right)-\Delta_1\left(\frac{DT}{\mu_n}\right)\mu_n^{-s}-\phi(0)=\int_0^{\alpha_n}\phi'(u)\mathrm{d}u$$

$$=\int_{-C_1}^{C_1} Ch(u,\alpha_n)\phi'(u)\mathrm{d}u,$$

where we define $Ch(u,\alpha_n)$ to be 1 if u lies in $(0,\alpha_n)$ if $\alpha_n>0$ or $(\alpha_n,0)$ if $\alpha_n<0$. Otherwise we define $Ch(u,\alpha_n)=0$. Note that

$$\phi'(u)=-\frac{s(\lambda_n+u)^{-s-1}}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left\{\left(\frac{T}{\lambda_n+u}\right)^w-\left(\frac{DT}{\lambda_n+u}\right)^w\right\}R(w)\frac{\mathrm{d}w}{w}$$

$$-\frac{(\lambda_n+u)^{-s}}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left\{\frac{T^w}{(\lambda_n+u)^{w+1}}-\frac{(DT)^w}{(\lambda_n+u)^{w+1}}\right\}R(w)\mathrm{d}w.$$

Now

$$F_3(s)-F_2(s)=\int_{-C_1}^{C_1}\left(\sum_{n=1}^{\infty}a_nb_nCh(u,\alpha_n)\phi'(u)\right)\mathrm{d}u$$

and so

$$\frac{1}{T}\int_T^{2T}|F_3(s)-F_2(s)|\mathrm{d}t\leqslant\int_{-C_1}^{C_1}\left(\frac{1}{T}\int_T^{2T}\left|\sum_{n=1}^{\infty}(a_nb_nCh(u,\alpha_n)\phi'(u))\right|^2\mathrm{d}t\right)^{1/2}\mathrm{d}u.$$

We write $\phi'(u)=(\phi_1(u)+\phi_2(u))(\lambda_n+u)^{-s}$ with an obvious meaning for $\phi_1(u)$ and $\phi_2(u)$. We have $\phi_1(u)=O(T/\lambda_n\min((T/\lambda_n)^2,(T/\lambda_n)^{-2}))$ and $\phi_2(u)=O(\min(T^2/\lambda_n^3, T^{-2}/\lambda_n^{-1}))$, by moving the line of integration to Re $w=2$ and Re $w=-2$. We now prove that, for $|u|\leqslant C_1$ there holds uniformly in u,

$$\frac{1}{T}\int_T^{2T}\left|\sum_{n=1}^{\infty}a_nb_n\beta_n(\lambda_n+u)^{-s}\right|^2\mathrm{d}t\ll T^{1-2\sigma}(f(T))^2$$

where β_n depends only on n, T and u and further $\beta_n=O(\min(T^3/\lambda_n^3,\lambda_n/T))$ and $\beta_n=O(\min(T^2/\lambda_n^3,\lambda_n/T^2))$. Clearly the second estimate is smaller by a factor $O(1/T)$ and hence it suffices to ignore it. By the well-known Montgomery-Vaughan theorem (see [6]) we see that LHS is

$$O\left(\sum_{n\leqslant T}(f(n))^2n^{-2\sigma}\frac{n}{T}+\sum_{n\geqslant T}(f(n))^2n^{-2\sigma}\frac{T^5}{n^5}\right)=O(T^{1-2\sigma}(f(T))^2).$$

Here the O-constant is independent of C_1 if C_1 is chosen to be smaller than a constant $C^*(>0)$. This completes the proof that

$$\frac{1}{T}\int_T^{2T}|F_3(s)-F_2(s)|dt=O(C_1\,T^{(1/2)-\sigma}f(T))$$

where the O-constant is independent of C_1. This completes the proof of Theorem 5.

Theorem 6. *There are $\gg T$ distinct integers M in $(T, 2T)$ for each of which there holds*

$$\int_M^{M+1} |F_3(\sigma + it)|\,dt \gg T^{(1/2)-\sigma} f(T)$$

provided $0 < \sigma < \frac{1}{2}$.

Proof. Divide $[T, 2T]$ into intervals G of unit length ignoring a bit at one end. Put $\Lambda(G) = \int_G |F_3(\sigma + it)|\,dt$ and $Q = T^{(1/2)-\sigma} f(T)$. Theorem 5 gives

$$\sum_G \Lambda(G) \gg TQ \text{ and } \sum_G (\Lambda(G))^2 \ll TQ^2.$$

This leads to Theorem 6.

Theorem 7. *Suppose that $F(s)$ defined in $\sigma > 1$ by*

$$F(s) = \sum_{n=1}^{\infty} a_n b_n \mu_n^{-s}$$

can be continued analytically in $(\sigma \geqslant \frac{1}{2} - 2\delta, T \leqslant t \leqslant 2T)$ and there $\max|F(s)| \leqslant T^B$ *where $B(>0)$ is a constant. Then there are $\gg T(\text{loglog } T)^{-1}$ distinct integers M in $(T, 2T)$ for each of which there holds*

$$\int_M^{M+1} |F(\sigma + it)|\,dt \gg T^{(1/2)-\sigma} f(T)$$

where $\sigma = \frac{1}{2} - \delta$.

Using Theorem 3 of paper III [1] we obtain the following Corollary.

COROLLARY

$F(s)$ has $\gg T\log T(\text{loglog } T)^{-1}$ zeros in $(\sigma \geqslant \frac{1}{2} - 2\delta, T \leqslant t \leqslant 2T)$.

Remark. It is not hard to prove that in many cases (for example $\sum_{n \geqslant x} a_n = x + O_\varepsilon(x^\varepsilon)$ for every $\varepsilon > 0$) that

$$\frac{1}{T}\int_T^{2T} \left|F\left(\frac{1}{2} + it\right)\right|^2 dt \ll_\varepsilon T^\varepsilon$$

(for every $\varepsilon > 0$) and in this case it follows that the number of zeros of $F(s)$ in $(\sigma \geqslant \frac{1}{2} + \delta, T \leqslant t \leqslant 2T)$ is $O(T)$.

Proof. (Of Theorem 7). We have for $s = \sigma + it$

$$F_3(s) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s+w)(T^w - (DT)^w)R(w)\frac{dw}{w}.$$

We deform the contour $(2 - i\infty, 2 + i\infty)$ to $(2 - i\infty, 2 - iC_{10}\text{loglog } T, -iC_{10}\text{loglog } T,$

$iC_{10}\log\log T$, $2+iC_{10}\log\log T$, $2+i\infty$), use $|F_3(s)| \leqslant \int |\,|\cdots|\mathrm{d}w/w|$ (over the new contour) and integrate with respect to t from M to $M+1$ of Theorem 6. We obtain the theorem by slight work. (Here $C_{10}(>10)$ is a large constant).

The remark below the Corollary to Theorem 7 follows from an idea of J E Littlewood (see Theorem 9.15(A) on page 230 of [7]).

Theorem 8. *We have, for* $\sigma < \frac{1}{2}$,

$$\frac{1}{T}\int_T^{2T} |F(\sigma+it)|\,\mathrm{d}t \gg T^{(1/2)-\sigma} f(T).$$

Proof. Let $T+(T/10) \leqslant t \leqslant 2T-(T/10)$ and $\sigma < \frac{1}{2}$. We start with the formula for $F_3(s)$ as in the proof of Theorem 7 above and deform the contour exactly as before. It follows that

$$\frac{1}{T}\int_{T-C_{10}\log\log T+(T/10)}^{2T+C_{10}\log\log T-(T/10)} |F(\sigma+\mathrm{i}t)|\,\mathrm{d}t \gg T^{(1/2)-\sigma} f(T)$$

on using the first part of Theorem 5. For this we need

$$\frac{1}{T}\int_{T+T/10}^{2T-T/10} |F_3(\sigma+it)|\,\mathrm{d}t \gg T^{(1/2)-\sigma} f(T).$$

But this can be proved just as we proved the first part of Theorem 7. This completes the proof of Theorem 8.

Theorem 9. *Let* $\sum_{n \leqslant x} a_n = O(1)$. *Then for* $0 < \sigma < \frac{1}{2}$, *we have,*

$$\frac{1}{T}\int_T^{2T} |F(\sigma+it)|^2\,\mathrm{d}t \ll T^{1-2\sigma}(f(T))^2$$

and

$$\frac{1}{T}\int_T^{2T} \left|F\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \ll_\varepsilon T^\varepsilon$$

for every $\varepsilon > 0$.

COROLLARY

Let $0 < \sigma < \frac{1}{2}$. *Then there are* $\gg T$ *distinct integers* M *in* $(T, 2T)$ *for each of which there holds*

$$\int_M^{M+1} |F(\sigma+\mathrm{i}t)|\,\mathrm{d}t \gg T^{(1/2)-\sigma} f(T).$$

Hence as before $F(s)$ *has* $\gg T\log T$ *zeros in* $(\sigma \geqslant \frac{1}{2}-\delta,\ T \leqslant t \leqslant 2T)$ *and only* $O(T)$ *zeros in* $(\sigma \geqslant \frac{1}{2}+\delta,\ T \leqslant t \leqslant 2T)$.

We remark finally that Theorems 7, 8 and 9 are valid even if we omit N terms (other than the first term) in $F(s)$ where $N = O_\varepsilon(T^\varepsilon)$ for every $\varepsilon > 0$.

P.S. In a forthcoming paper (On the zeros of a class of generalised Dirichlet series-XV) we consider zeros of functions like $\sum_{n=1}^{\infty} d(n)(n+\alpha_n)^{-s}$ and $\sum_{n=1}^{\infty} d_3(n)(n+\alpha_n)^{-s}$ and prove some interesting lower bounds for the number of zeros in $(\sigma \geqslant \frac{1}{2}-\delta, T \leqslant t \leqslant 2T)$ like $\gg T \log T$. Also in paper XVI with the same title K Ramachandra and A Sankaranarayanan have proved the upper bound $\ll T$ in $(\sigma \geqslant \frac{1}{2}+\delta, T \leqslant t \leqslant 2T)$ for the functions such as those mentioned above.

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