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Limit theorems for semi-Markov processes

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A new construction of regeneration times is exploited to prove ergodic and renewal theorems for semi-Markov processes on general state spaces. This work extends results of the authors in *Ann. Probability* (6 (1978), 788-797).

1. Introduction

This note is a continuation of [2] and [3], where we introduced a construction of regeneration times to show that recurrent Markov chains on general state spaces act as if they had a single recurrence point which is visited infinitely often. This device was used to give renewal theoretic proofs of ergodic and renewal theorems for Markov and semi-Markov chains. The semi-Markov results were proved under a strong aperiodicity condition on the underlying Markov chain, and a further strong restriction on the distributions of sojourn times in a state. Our objective here is to remove some of these restrictions.

Consider a space S with a σ-algebra of subsets $\mathcal{S}$. Let $\{X_n;\ n = 0, 1, \ldots\}$ be a Markov chain on $(S, \mathcal{S})$ with homogeneous transition function $P(s, E)$, $s \in S$, $E \in \mathcal{S}$, and let $\{G_{xy}(\cdot);\ x, y \in S\}$ be a family of distribution functions on $R^+ = [0, \infty)$. Given a realization $\{X_n = x_n;\ n = 0, 1, \ldots\}$ of the chain, generate independent random variables $\{L_n;\ n = 0, 1, \ldots\}$ such that

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$$(1.1)\qquad P\{L_i \le t \mid X_n = x_n,\ n = 0, 1, \dots\} = G_{x_i, x_{i+1}}(t)\ ,\quad t \ge 0\ .$$

Set

$$W(t) = \big(Z(t), A(t)\big) = \begin{cases} (X_0,\ t) & \text{when} \quad 0 \le t < L_0 \\ (X_1,\ t-L_0) & \text{when} \quad L_0 \le t < L_0+L_1 \\ \vdots & \quad\vdots \qquad\qquad \vdots \\ \left(X_k,\ t - \sum_0^{k-1} L_i\right) & \text{when} \quad \sum_0^{k-1} L_i \le t < \sum_0^{k} L_i \\ \vdots & \end{cases}$$

The process $\{W(t);\ 0 \le t\}$ is a Markov process. $\{Z(t);\ 0 \le t\}$ is called a *semi-Markov process*, and $\{A(t);\ 0 \le t\}$ is its associated *age process*. Also $\{X_n\}$ is referred to as the state process, $\{L_n\}$ as its "sojourn" times, and $\{X_n, L_{n-1}\}$ as a *semi-Markov chain*. (Note that this pair chain is actually a Markov chain; the joint distribution of (X_{n+1}, L_n) depends only on X_n .) The objective is to determine the limiting behaviour of $W(t)$; more specifically of functions of the form $Ef\big(W(t)\big)$ and $E\left(\sum_{n=0}^{\infty} g\left(X_n,\ t - \sum_{i=0}^{n} L_i\right)\right)$, for reasonable classes of functions f and g .

The key hypothesis is designed to guarantee the required recurrence structure.

HYPOTHESIS H_k . *(i) There exists a set* $A \in S$ *such that for some integer* k ,

$$(1.2)\qquad P_x\{X_{nk} \in A \ \text{ for some } \ n \ge 1\} = 1\ ,\quad x \in S\ .$$

(ii) There exists a probability measure φ *on* $(S \cap A,\ S \cap A)$, *a family of probability measures* $\mu(x, \cdot)$ *on* $\mathcal{B}(R^+)$ *for* $x \in A$, *and a number* $\lambda \in (0, 1)$, *such that*

$$(1.3)\qquad P_x\left(X_k \in E,\ \sum_{i=0}^{k-1} L_i \in D\right) \ge \lambda\varphi(E \cap A)\mu(x, D)$$

for all $x \in A$, $D \in \mathcal{B}(R^+)$.

We then have the following

THEOREM. *Assume* H_k. *Then*

(a) *there exists a* σ*-finite, invariant measure* ν *for* P *which is unique up to multiplicative constants;*

(b) *if furthermore*

(i) $m \equiv \int m(x)\nu(dx) < \infty$, *where* $m(x) = E_x L_0$,

(ii) $P\left\{\sum_0^{k-1} L_i \le u\right\} = F_k(u)$ *is non-lattice, and*

(iii) $f : S \times R^+ \to R^+$ *is bounded, measurable, and* $\nu\{x : f(x, t)$ *is discontinuous for some* $t\} = 0$,

then

$$(1.4)\quad \lim_{t\to\infty} Ef(W(t)) = \left\{\int_S \left\{\int_0^\infty f(x, u)P_x(L_0 > u)du\right\}\nu(dx)\right\} / \left\{\int_S \left\{\int_0^\infty P_x(L_0 > u)du\right\}\nu(dx)\right\} ;$$

(c) *if (b) (i) and (ii) hold, and* $g(x, t) : S \times R^+ \to R$ *is bounded, measurable, continuous in* t , *and satisfies*

$$(1.5)\qquad g(x, t) \to 0 \text{ almost surely } (\nu) \text{ as } t \to \infty ,$$

$$(1.6)\qquad \int_S \sum_{n=0}^{\infty} \sup_{nh\le t<(n+1)h} g(x, t)\nu(dx) < \infty ,$$

and

$$(1.7)\qquad \int \{\sup_t |g(x, t)|\}\nu(dx) < \infty ,$$

then as $t \to \infty$,

$$(1.8)\qquad E_\varphi \sum_{n=0}^{\infty} g\left\{X_n, t - \sum_{i=0}^{n-1} L_i\right\} \to \frac{1}{m}\int_0^\infty \int_S g(x, t)\nu(dx)dt .$$

REMARKS. 1. The theorem extends results of [2] in two ways. First, the distributions $G_{xy}(\cdot)$ were allowed to depend only on x (not y) in the earlier work; that is, there the sojourn times depend only on the state the chain is "coming from". Secondly, only the case $k = 1$ is admitted in [2].

2. The awkward conditions (1.5)-(1.7) are designed to assure the direct Riemann integrability of certain functions related to g . These conditions can be eliminated at the expense of strengthening the smoothness hypothesis on F_k in *(b) (ii)*. (see the related discussion in Arjas, Nummelin, and Tweedie [1], and Athreya and Ney [4].)

3. Nummelin [7], [8], has proved many results related to this subject, including one like the above theorem under apparently slightly stronger hypotheses. He also uses regeneration methods, but treats the semi-Markov process as a Markov chain on the enlarged state space $S \times R$, and applies discrete renewal theory to this chain. We work instead with the continuous time process $\big(Z(t), A(t)\big)$ and use renewal theory on R to draw our conclusions. This approach seems to us to yield somewhat more transparent proofs.

4. Nummelin has also shown (see [7]) that a condition very close to H_k is always satisfied (for some k), provided the semi-Markov chain satisfies a weak irreducibility condition.

5. That (1.8) is in fact a "renewal theorem", can be seen by taking $g(x, t)$ of the form $\chi_A(x)\chi_I(t)$ for $A \in S$, and $I \subset R^+$ an interval. As usual in such results, if $F_k(\cdot)$ is lattice, then there is a lattice version of (1.8).

6. Similarly (1.4) can be seen to imply the convergence of $P\{Z(t) \in E\}$, $E \in S$.

7. Since *(a)* asserts the uniqueness of the invariant measure, one can use any measure $\pi(\cdot)$ that is invariant with respect to P in place of ν .

8. If $k = 1$ and G_{xy} depends only on x , then H_1 automatically holds.

9. The result *(c)* of the theorem carries over without difficulty to the "two sided" case when the $G_{xy}(\cdot)$ are distributions on $(-\infty, \infty)$, along the lines of Theorem 4.1 of [2].

10. Renewal theorems of the type in *(c)* of the theorem under different hypotheses have been proved by Jacod [5] and Kesten [6].

2. Proof of the theorem

Hypothesis H_k *(ii)* applied to $D = R^+$ implies that the Markov chain $\{X_{nk};\ n = 0, 1, \ldots\}$ is $(A, \lambda, \varphi, 1)$-recurrent in the sense of Definition (2.2) of [3]. Hence, by Theorem (6.1) of that paper, there exists a unique (up to multiplicative constants) invariant measure, say ν_k, for P. It is now easy to verify that

$$\nu = \nu_k + \nu_k P + \ldots + \nu_k P^{k-1} \tag{2.1}$$

is a (necessarily unique) invariant measure for P.

As in the earlier work [2], [3], the basic idea behind the proof is an appropriate

REGENERATION LEMMA. *If* H_k *holds for some* $k \geq 1$, *then there exists a random time* N *such that* $P_x(N < \infty) = 1$ *for all* $x \in S$, *and*

$$P_x\left\{X_{Nk} \in E,\ \sum_{(N-1)k}^{Nk-1} L_i \in D \mid X_{jk},\ \sum_{i=0}^{jk} L_i,\ j = 0, 1, \ldots, N-1\right\} = \varphi(E \cap A)\mu\left(X_{(N-1)k}, D\right), \tag{2.2}$$

almost surely.

Proof. Let $U_j = L_{jk} + \ldots + L_{(j+1)k-1}$, and consider the "skeleton" semi-Markov chain $\{(X_{jk}, U_{j-1}),\ j = 1, 2, \ldots\}$ (k is fixed throughout). Whenever $X_{jk} \in A$ for some j (say $X_{jk} = x \in A$), randomize the next transition of the chain as follows:

(i) with probability p $(0 < p < \lambda =$ the constant in (1.3)) distribute $(X_{(j+1)k}, U_j)$ over $A \times R^+$ *independently* with distributions $\varphi(\cdot)$ and $\mu(x, \cdot)$ respectively;

(ii) with probability $(1-p)$ distribute $(X_{(j+1)k}, U_j)$ over the entire state space $S \times R^+$ according to a transition function $Q(x, \cdot)$, chosen so that the overall transition probabilitities of the chain

$\{(X_{jk}, U_{j-1})\}$ remain unchanged. This is accomplished by defining Q so that

$$P^k(x, E\times D) = p\varphi(E \cap B)\mu(x, D) + (1-p)Q(x, E\times D) .$$

That this is in fact possible, follows from (1.3). Now since A is visited by $\{X_{nk}\}$ infinitely often, and each time, with probability $p > 0$, the next transition is distributed independently according to (φ, μ), this event will ultimately occur at some time $N < \infty$ almost surely. This proves the lemma. (The reader wishing to see a more detailed argument is referred to the proof of (3.1) in [3], which contains a careful proof of a special case of the above lemma.)

COROLLARY 1. *There exists a sequence of random times* $N_1, N_2, \ldots$ *for which* (2.1) *holds.*

COROLLARY 2. *Let* $T = \sum_0^{Nk-1} L_i$ *and* $Z = X_{Nk}$. *Then* Z *and* T *are independent random variables.*

Proof. Let f and g be bounded, measurable functions on S and R^+, respectively. Then

$$\begin{aligned} Ef(Z)g(T) &= E\{E[f(Z)g(T) \mid (X_{jk}, U_{j-1}), j = 0, \ldots, N-1]\} \\ &= E\left\{E\left[f(X_{Nk})g\left(\sum_0^{N-2} U_i+U_{N-1}\right) \mid (X_{jk}, U_{j-1}), j = 0, \ldots, N-1\right]\right\} \\ &= E\left\{\int_A f(y)\varphi(dy) \int_R g\left(\sum_0^{N-2} U_i+u\right)\mu(X_{(N-1)k}, du)\right\} . \end{aligned}$$

Taking the first integral outside the expectation, this equals $Ef(Z)Eg(T)$, proving the corollary. Before turning to the main part of the proof of the theorem, we prepare one more

PROPOSITION. *Let* $h : S \times S \to R$ *be bounded and measurable, and* $\tilde{h}(x) = E_x h(X_0, X_1)$. *Then*

$$E_\varphi \sum_{n=0}^{Nk-1} h(X_n, X_{n+1}) = \int_S \tilde{h}(x)\nu(dx) \tag{2.3}$$

where $\nu(\cdot)$ *is the stationary measure whose existence was asserted in part* *(a) of the theorem.*

Proof.

$$E_\varphi \sum_{n=0}^{Nk-1} h(X_n, X_{n+1}) = E_\varphi \sum_{j=0}^{N-1} \sum_{n=jk}^{(j+1)k-1} h(X_n, X_{n+1})$$
$$= E_\varphi \sum_{j=0}^{\infty} \sum_{n=jk}^{(j+1)k-1} h(X_n, X_{n+1})\chi_{(N>j)} .$$

Now extend the chain $\{X_n\}$ to a chain $\{X_n, \delta_n\}$, $n \ge 0$, where $\{\delta_n; n \ge 0\}$ is a sequence of independent "coin tossing" variables with $P(\text{Head}) = P(\delta_n = 1) = 1 - P(\delta_0 = 0) = p$. The evolution of $\{X_n\}$ is independent of $\{\delta_n\}$ except when $X_{nk} \in A$, at which time the randomization described in the proof of the regeneration lemma takes place. Thus the random variable $\chi_{(N>j)}$ is completely determined by the history $\{(X_i, \delta_i), i = 1, \ldots, jk\}$, and hence conditioned on this history (say F_{jk}) $(X_{jk}, \ldots, X_{(j+1)k})$, and $\chi_{(N>j)}$ are trivially independent. Thus for any bounded, measurable $\psi : S^k \to R$, we have

$$E\psi(X_{jk}, \ldots, X_{(j+1)k})\chi_{(N>j)} = E\{E[\psi\chi | F_{jk}]\} = E\{\chi E[\psi | F_{jk}]\}$$
$$= E\{\chi_{N>j} E[\psi(X_{jk}, \ldots, X_{(j+1)k}) \mid X_{jk}]\}$$
$$\equiv E\{\tilde{\psi}(X_{jk})\chi_{(N>j)}\} ,$$

where we have let

$$\tilde{\psi}(x) = E\{\psi(X_0, \ldots, X_k) \mid X_0 = x\} .$$

Also let

$$H(x) = E_x \sum_{n=0}^{k-1} h(X_n, X_{n+1}) = \sum_{n=0}^{k-1} E_x\tilde{h}(X_n)$$
$$\left(E_x h(X_n, X_{n+1}) = E_x E_{X_n} h(X_n, X_{n+1}) = E_x(\tilde{h}(X_n))\right) .$$

Then

$$(2.4)\qquad E_\varphi \sum_{n=0}^{Nk-1} h(X_n, X_{n+1}) = E_\varphi \sum_{j=0}^{N-1} \sum_{n=jk}^{(j+1)k-1} h(X_n, X_{n+1})$$
$$= E_\varphi \sum_{j=0}^{N-1} H(X_{jk}) = E_\varphi \sum_{j=0}^{\infty} H(X_{jk})\chi_{N>j} .$$

By Theorem (6.1) of [3] we know that an invariant measure for P^k is given by $\nu_k(E) = E_\varphi \sum_{j=0}^{N-1} \chi_E(X_{jk})$, and hence (2.4) equals

$$\int_S H(x)\nu_k(dx) = \int_S \tilde{h}(y) \sum_{n=0}^{k-1} \int_S P^n(x, dy)\nu_k(dy)$$
$$= \int_S \tilde{h}(y)\nu(dy) \quad \text{by (2.1).}$$

This, with (2.4), is (2.3), proving the proposition.

We now turn to the main part of the proof of part *(b)* of the theorem. Since $P_x(T < \infty) = 1$ for all $x \in S$, it is sufficient to prove this result for the case when the initial state X_0 is distributed according to φ . Let $m(t) = Ef(W(t))$, and $a(t) = E\{f(W(t)); T > t\}$, where f satisfies the conditions of part *(b)* of the theorem, and T is as in Corollary 2. Due to the independence assertion of that corollary, T is a "regeneration time", at which the chain $\{X_n\}$ undergoes a transition and is distributed over A according to φ , independent of the history of the process up to that time. Hence $m(\cdot)$ satisfies the one-dimensional renewal equation

$$m(t) = a(t) + \int_0^t m(t-u)dF(u) ,$$

where

$$F(u) = P_\varphi(T \le u) .$$

The direct Riemann integrability of $a(\cdot)$ under the hypotheses on $f(\cdot)$ now follows exactly as in the proof of Theorem (3.1) of [2]. Hence, since $F(\cdot)$ is non-lattice (note that this is a little weaker than the hypothesis F_k non-lattice), we have by the renewal theorem that

$$(2.5) \qquad m(t) \to \left(E_\varphi T\right)^{-1} \int_0^\infty a(y)dy \ , \text{ as } \ t \to \infty \ .$$

It thus remains only to identify the limit. To this end we write

$$(2.6) \qquad \begin{aligned} \int_0^\infty a(t)dt &= E_\varphi \int_0^T f\big(W(u)\big)du \\ &= E_\varphi \sum_{n=0}^{Nk-1} \int_0^{L_n} f\big(X_n, u\big)du \\ &= E_\varphi\left\{E_\varphi\left[\sum_{n=0}^{Nk-1} \int_0^{L_n} f\big(X_n, u\big)du \mid X_0, \ldots, X_{N-1}, N\right]\right\} \\ &= E_\varphi \sum_{n=0}^{Nk-1} h\big(X_n, X_{n-1}\big) \ , \end{aligned}$$

where

$$\begin{aligned} h(x, y) &= E\left[\int_0^{L_0} f\big(X_0, u\big)du \mid X_0 = x, X_1 = y\right] \\ &= \int_0^\infty f(x, u)\left[1-G_{xy}(u)\right]du \ . \end{aligned}$$

Applying (2.3) of the proposition, we see that

$$\int_0^\infty a(t)dt = \int_S \tilde{h}(x)\nu(dx) \ ,$$

where

$$\begin{aligned} \tilde{h}(x) &= E_x Eh\big(X_0, X_1\big) \\ &= \int_S \int_0^\infty f(x, u)\left[1-G_{xy}(u)\right]du P(x, dy) \\ &= \int_0^\infty f(x, u) P_x\big(L_0 > u\big)du \ . \end{aligned}$$

Thus $\int a(t)dt$ equals the numerator in (1.4). For the denominator, we observe from (2.6) that $E_\varphi T = \int_0^\infty a_1(t)dt$, where $a_1(\cdot)$ is $a(\cdot)$ for the special case when $f(x, t) \equiv 1$. This implies part *(b)* of the theorem.

Finally, turning to part *(c)*, we again observe that we need only consider the case when X_0 has distribution φ . Let

$$K(t) \equiv E_\varphi \sum_{n=0}^{Nk-1} g\left(X_n,\ t - \sum_{i=0}^{n-1} L_i\right) \quad \text{and} \quad T_j = \sum_{N_jk}^{N_{j+1}k-1} L_i\ , \text{ where } \{N_j;\ j \geq 0\}$$

are the successive regeneration times for $\{X_{nk}\}$. Then $\{T_j;\ j \geq 0\}$ are independent identically distributed as T in Corollary 2, and are regeneration times, in the previously described sense, for the continuous time process. Hence

$$E_\varphi \sum_{n=N_jk}^{N_{j+1}k-1} g\left(X_n,\ t - \sum_{i=0}^{n-1} L_i\right) = E_\varphi \sum g\left(X_n,\ t - \sum_{i=0}^{j-1} T_i - \sum_{i=N_jk}^{n-1} L_i\right)$$
$$= E_\varphi K\left(t - \sum_{i=0}^{j-1} T_i\right) .$$

Thus

$$M(t) \equiv E_\varphi \sum_{n=0}^{\infty} g\left(X_n,\ t - \sum_{i=0}^{n-1} L_i\right) = E_\varphi \sum_{j=0}^{\infty} \sum_{n=N_jk}^{N_{j+1}k-1} g\left(X_n,\ t - \sum_{i=0}^{n} L_i\right)$$
$$= E_\varphi \sum_{j=0}^{\infty} g\left(X_n,\ t - \sum_{i=0}^{j-1} T_i\right) .$$

But now we are ready to apply the one dimensional renewal theorem (exactly as in Section 4 of [2]), to conclude that if $K(\cdot)$ is direct Riemann integrable, then

$$M(t) \to \left(E_\varphi T\right)^{-1} \int_0^\infty K(u)du \quad \text{as} \quad t \to \infty . \tag{2.7}$$

The hypotheses on g are exactly as those in [2], and are designed to assure this direct Riemann integrability.

To identify the limit in (1.8) we write

$$\int_0^\infty K(t)dt = E_\varphi \sum_{n=0}^{Nk-1} \int_0^\infty g\left(X_k,\ t - \sum_{i=0}^{n-1} L_i\right)dt$$
$$= E_\varphi \sum_{n=0}^{Nk-1} \int_0^\infty g\left(X_k,\ t\right)dt \ ,$$

(we may define $g(x,\ t) = 0$ for $t < 0$)

$$= \int_S \int_0^\infty g(x,\ t)\nu(dx)dt \ .$$

We have already seen that $E_\varphi(T) = m$ (*(b) (i)*). This completes the proof of the theorem.

We again ask the reader to observe that the above proof of *(c)* in no way depends on the non-negativity of the L_i's .

References

[1] Elja Arjas, Esa Nummelin, Richard L. Tweedie, "Uniform limit theorems for non-singular renewal and Markov renewal processes", *J. Appl. Probability* 15 (1978), 112-125.

[2] K.B. Athreya, D. McDonald and P.E. Ney, "Limit theorems for semi-Markov processes and renewal theory for Markov chains", *Ann. Probability* 6 (1978), 788-797.

[3] K.B. Athreya and P.E. Ney, "A new approach to the limit theory of recurrent Markov chains", *Trans. Amer. Math. Soc.* (to appear).

[4] K.B. Athreya and P.E. Ney, "Regeneration methods for Markov chains and related processes", under preparation.

[5] J. Jacod, "Théorème de renouvellement et classification pour les chaînes semi-markoviennes", *Ann. Inst. H. Poincaré Sect. B (N.S.)* 7 (1971), 83-129.

[6] Harry Kesten, "Renewal theory for functionals of a Markov chain with general state space", *Ann. Probability* 2 (1974), 355-386.

[7] Esa Nummelin, "A splitting technique for σ-recurrent Markov chains" (Report-HTKK-MAT-A80, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 1976).

[8] Esa Nummelin, "Uniform and ratio limit theorems for Markov renewal and semi-regenerative processes on a general state space" (Report HTKK-MAT-A98, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 1977).

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