Universal Families on Moduli Spaces of Principal Bundles on Curves

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1 Introduction

Let X be a compact connected Riemann surface of genus at least three. Let $\mathcal{M}_X(n, d)$ denote the moduli space of all stable vector bundles over X of rank n and degree d, which is a smooth irreducible quasiprojective variety defined over $\mathbb{C}$. A vector bundle $\mathcal{E}$ over $X \times \mathcal{M}_X(n, d)$ is called *universal* if for every point $m \in \mathcal{M}_X(n, d)$, the restriction of $\mathcal{E}$ to $X \times \{m\}$ is in the isomorphism class of holomorphic vector bundles over X defined by m. A well-known theorem says that there is a universal vector bundle over $X \times \mathcal{M}_X(n, d)$ if and only if d is coprime to n (see [13] for existence in the coprime case, [8] for nonexistence in the noncoprime case, and [7] for a topological version of nonexistence in the case $d = 0$).

Let H be a connected semisimple linear algebraic group defined over the field of complex numbers. Ramanathan extended the notion of (semi-)stability to principal H-bundles and constructed moduli spaces for stable principal H-bundles over X [9–11]. The construction works for any given topological type, yielding a moduli space which is an irreducible quasiprojective variety defined over $\mathbb{C}$. We are concerned here with the case of topologically trivial stable principal H-bundles. Let $\mathcal{M}_X(H)$ denote the moduli space of topologically trivial stable principal H-bundles over X.

Let $Z(H) \subset H$ be the centre. For any H-bundle E_H, the group $Z(H)$ is contained in the automorphism group $\text{Aut}(E_H)$. Let

$$\mathcal{M}'_X(H) \subset \mathcal{M}_X(H) \tag{1.1}$$

be the subvariety consisting of all H-bundles E_H over X with the property $\text{Aut}(E_H) \cong Z(H)$. It is known that $\mathcal{M}'_X(H)$ is a dense Zariski open subset contained in the smooth locus of $\mathcal{M}_X(H)$.

A principal H-bundle E over $X \times \mathcal{M}'_X(H)$ will be called a *universal bundle* if for every point $m \in \mathcal{M}_X(n, d)$, the restriction of $\mathcal{E}$ to $X \times \{m\}$ is in the isomorphism class of stable H-bundles over X defined by the point m of the moduli space.

The following theorem is the main result proved here.

Theorem 1.1. There is a universal H-bundle over $X \times \mathcal{M}'_X(H)$ if and only if $Z(H) = e$.   □

Remark 1.2. Nonexistence for $H = SL(n, \mathbb{C})$ was previously known [8, 7], as also was existence for $H = PGL(n, \mathbb{C})$.

2   Existence of universal bundle

We begin this section by recalling very briefly certain facts from [2]. Unless otherwise specified, all bundles and sections considered will be algebraic.

Remark 2.1. Let E be a principal G-bundle over X, where G is a reductive linear algebraic group defined over $\mathbb{C}$ and $H \subset G$ is a Zariski closed semisimple subgroup. For any variety Y equipped with an action of G, the fibre bundle $(E \times Y)/G$ over X associated to E will be denoted by $E(Y)$.

(1) There is a natural action of the group $\text{Aut}_G E$, defined by all automorphisms of E over the identity map of X that commute with the action of G, on $\Gamma(X, E(G/H))$ (the space of all holomorphic sections of the fibre bundle $E(G/H) = E/H$ over X) and the orbits correspond to the equivalence classes of H-reductions of E with two reductions being equivalent if the corresponding principal H-bundles are isomorphic.

(2) Let $G = GL(n, \mathbb{C})$ and let $\phi : H \hookrightarrow G$ be a faithful representation of the semisimple group H. Let Q denote the open subset of semistable principal G-bundles (or equivalently of topologically trivial semistable vector bundles of rank n) of the usual "Quot scheme," and let $Q(\phi)$ be the "Quot scheme" which parametrises pairs of the form (E', s), where E' is a principal G-bundle and s is a reduction of structure group of E' to H. Then $Q(\phi)$ is in a sense a "relative Quot scheme." As is clear from the definition and the notation, this scheme is dependent on the choice of the inclusion $\phi : H \hookrightarrow G$ (for details, see [2, 10, 11]).

One also has a *tautological* sheaf on $X \times Q$ which in fact is a vector bundle. We denote by $\mathcal{E}$ the *associated tautological* principal G-bundle on $X \times Q$.

Recall that the moduli space of principal G-bundles ($G = GL(n, \mathbb{C})$) is realised as a good quotient of Q by the action of a reductive group $\mathcal{G}$. We may also assume that the group $\mathcal{G}$ is with trivial centre (see, e.g., [4]).

Remark 2.2. It is immediate that the action of $\mathcal{G}$ on Q lifts to an action on $Q(\phi)$, where $\mathcal{G}$, Q, and $Q(\phi)$ are defined in Remark 2.1.

We have a morphism

$$\psi : Q(\phi) \longrightarrow Q, \tag{2.1}$$

which sends any H-bundle E' to the $GL(n, \mathbb{C})$-bundle obtained by extending the structure group of E' using the homomorphism ϕ. In fact, ψ is a $\mathcal{G}$-equivariant *affine morphism* (see [2]).

Continuing with the notation in the above two remarks, consider the $\mathcal{G}$-action on $Q(\phi)$ (defined in Remark 2.1(2)) with the linearisation induced by the affine $\mathcal{G}$-morphism ψ in (2.1). Since a good quotient of Q by $\mathcal{G}$ exists and since ψ is an affine $\mathcal{G}$-equivariant map, a good quotient $Q(\phi)//\mathcal{G}$ exists (see [11, Lemma 5.1]).

Moreover by the universal property of categorical quotients, the canonical morphism

$$\overline{\psi} : Q(\phi)//\mathcal{G} \longrightarrow Q//\mathcal{G} \tag{2.2}$$

given by ψ is also *affine*.

Theorem 2.3. Let $\mathcal{M}_X(H)$ denote the scheme $Q(\phi)//\mathcal{G}$ (see (2.2)). Then this scheme is the coarse moduli scheme of semistable H-bundles. Further, the scheme $\mathcal{M}_X(H)$ is projective, and if $H \hookrightarrow GL(V)$ is a faithful representation, then the canonical morphism

$$\overline{\psi} : \mathcal{M}_X(H) \longrightarrow \mathcal{M}_X\big(GL(V)\big) = Q//\mathcal{G} \tag{2.3}$$

is finite. □

Let $Q(\phi)^s$ be the open subscheme of $Q(\phi)$ consisting of stable H-bundles.

Lemma 2.4. Let $Q'_H \subset Q(\phi)^s$ be the subset parametrising all stable H-bundles whose automorphism group is $Z(H)$. Then the action of $\mathcal{G}$ on the subset Q'_H is free, and furthermore, the quotient morphism $Q'_H \to Q'_H/\mathcal{G}$ is a principal $\mathcal{G}$-bundle. In fact, $Q'_H/\mathcal{G}$ is precisely the Zariski open subset $\mathcal{M}'_X(H)$ (the variety $\mathcal{M}'_X(H)$ is defined in the introduction). □

Proof. For any point $E_H \in Q(\phi)$, the isotropy subgroup of E_H for the action of $\mathcal{G}$ coincides with $\mathrm{Aut}(E_H)/Z(H)$. This can be seen as follows: firstly, the point E_H is a pair (E, s), where $E \in Q$ and s is a reduction of structure group to H of the G-bundle E. It is well known for the action of $\mathcal{G}$ on Q that the isotropy at E is precisely the group $\mathrm{Aut}(E)/Z(G)$. From this, it is easy to see that the isotropy of E_H for the action of $\mathcal{G}$ on $Q(\phi)$ is the group $\mathrm{Aut}(E, s)/Z(H) = \mathrm{Aut}(E_H)/Z(H)$.

Hence it follows that the action of $\mathcal{G}$ on the open subset Q'_H is free, and the proof of the lemma is complete. ∎

We remark that, at least when H is of adjoint type, Q'_H is a nonempty open subset of $Q(\phi)^s$. Openness is easy and can be seen, for example, from [5, Theorem II.6(ii)]. Nonemptiness follows from Proposition 2.6 below.

We prove a proposition on semisimple groups, possibly known to experts but which we could not locate in any standard text.

Proposition 2.5. Let H be a semisimple algebraic group. Then H has a faithful irreducible representation $\phi : H \to GL(V)$ if and only if the centre of H is cyclic. ☐

Proof. One way the implication is easy, namely, suppose that a faithful irreducible representation exists, then the centre $Z(H)$ of H is cyclic. To see this, first note that under the representation ϕ, the centre $Z(H)$ maps to a subgroup which commutes with all elements of $\phi(H)$. Since ϕ is irreducible, this implies by Schur's lemma that $\phi(Z(H)) \subset Z(GL(V))$, where $Z(GL(V))$ is the centre of $GL(V)$. Observe further that since H is semisimple, we have $\phi(H) \subset SL(V)$. Hence, $Z(H) \subset Z(SL(V))$ and is therefore cyclic.

The contrapositive statement is harder to prove. We proceed as follows. Let H' be the simply connected cover of H and let $\overline{H}$ be the associated adjoint group, namely, $H/Z(H)$. Let Λ (resp., Λ_R) be the weight lattice (resp., root lattice). In other words, by the Borel-Weil theorem $\Lambda = \mathfrak{X}(B)$ and $\Lambda_R = \mathfrak{X}(\overline{B})$, where B, $\overline{B}$ are fixed Borel subgroups of H, $\overline{H}$. Then, from the exact sequence

$$e \longrightarrow Z(H) \longrightarrow H \longrightarrow \overline{H} \longrightarrow e \tag{2.4}$$

we see that $\Lambda/\Lambda_R \simeq \mathfrak{X}(Z(H))$. In other words, the quotient group Λ/Λ_R is cyclic of order m.

Let $\overline{\lambda}$ be a generator of the cyclic group Λ/Λ_R. Then, by an action of the Weyl group we may assume that the coset representative $\lambda \in \Lambda$ is actually a dominant weight.

Suppose that the root lattice has the following decomposition (corresponding to the simple components of $\overline{H}$):

$$\Lambda_R = \bigoplus_{i=1}^{\ell} \Lambda^i. \tag{2.5}$$

Then, by possibly adding dominant weights from the Λ^i, we may assume that $(m \cdot \lambda) \in \Lambda_R$ has all its direct sum components $\lambda_i \neq 0$, $i \in [1, \ell]$, where λ is as above.

For this choice of $\lambda \in \Lambda$ let V_λ be the corresponding H-module given by

$$\phi_\lambda : H \longrightarrow GL\left(V_\lambda\right). \tag{2.6}$$

Then one knows that ϕ_λ is an irreducible representation of H.

We claim that the representation ϕ_λ is even *faithful*. Suppose that this is not the case. Let $K_\lambda := \mathrm{kernel}(\phi_\lambda) \neq e$.

Firstly, $K_\lambda \subset Z(H)$. To see this, let K' be the inverse image of K_λ in the simply connected cover H' of H. Then the choice of λ so made that its simple components are nonzero in fact forces the following. Suppose that $H' = H_1 \times \cdots \times H_\ell$ is the decomposition of H' into its almost simple factors. (We recall that a semisimple algebraic group is called almost simple if the quotient of it by its centre is simple.) Then the normal subgroup K' in its decomposition in H' is such that $K_i := K' \bigcap H_i$ are proper normal subgroups of H_i. In particular, $K_i \subset Z(H_i)$ for all $i \in [1, \ell]$. This implies that $K' \subset Z(H')$ and hence $K_\lambda \subset Z(H)$.

Note that the dominant character λ is nontrivial on the generator of the centre $Z(H)$ because $\mathfrak{X}(Z(H)) = \Lambda/\Lambda_R$. Now $K_\lambda \subset Z(H)$ and $Z(H)$ cyclic implies that λ is nontrivial on the generator of K_λ as well. This contradicts the fact that $K_\lambda = \mathrm{kernel}(\phi_\lambda)$. This proves the claim. Therefore, the proof of the proposition is complete. ∎

If H is of adjoint type, then by Proposition 2.5 we can choose the inclusion $\phi : H \hookrightarrow G$ in Remark 2.1(2) to be an irreducible representation. Henceforth, ϕ will be assumed to be irreducible.

Let E be a stable H-bundle of trivial topological type. Recall that one can realise E from a unique, up to an inner conjugation, irreducible representation of $\pi_1(X)$ in a maximal compact subgroup of H (see [9]). For notational convenience, we will always suppress the base point in the notation of fundamental group. Denote by $M(E)$ the Zariski closure of the image of $\pi_1(X)$ in H.

Proposition 2.6. Let H be of adjoint type and let ϕ be a faithful irreducible representation of H in V (see Proposition 2.5). Let $U \subset Q(\phi)^s$ be the subset defined as follows:

$$U = \left\{E \in Q(\phi)^s \mid M(E) = H\right\}. \tag{2.7}$$

Then U is nonempty and is contained in the subset Q_H' of stable H-bundles which have trivial automorphism group. (Since H is of adjoint type, $Z(H) = e$.) □

Proof. Since ϕ is irreducible, using [**12**, Lemma 2.1] we conclude that there is an irreducible representation

$$\rho : \pi_1(X) \longrightarrow H \tag{2.8}$$

which has the property that the composition $\phi \circ \rho : \pi_1(X) \to G := GL(V)$ continues to remain irreducible. This implies that $M(E_\rho) = H$ by the construction in [**12**, Lemma 2.1] and hence $E_\rho \in U$, that is, U is nonempty.

Let $E \in U$. Then by the definition of U, there exists a representation ρ of $\pi_1(X)$ in H such that $E \simeq E_\rho$, where E_ρ is the flat principal H-bundle given by ρ. Observe that E_ρ is a stable H-bundle and the associated G-bundle is also stable. Hence all the automorphisms of this associated G-bundle lie in $Z(G)$. Since H is of adjoint type, it follows that the H-bundle E_ρ has no nontrivial automorphisms. Hence it follows that $U \subset Q'_H$.   ∎

We have the following theorem on existence of universal families.

Theorem 2.7. Let H be a group of adjoint type and $\mathcal{M}'_X(H)$ the Zariski open subset of $\mathcal{M}_X(H)$ defined in the introduction. Then there exists a universal family of principal H-bundles on $X \times \mathcal{M}'_X(H)$.   □

Proof. We first observe that the variety $\mathcal{M}'_X(H)$ is precisely the image of Q'_H under the quotient map for the action of $\mathcal{G}$. We recall that Q'_H is nonempty by Proposition 2.6.

Since $Z(H) = e$, it follows that $H \subset \overline{G} := G/Z(G)$, where $G = GL(V)$ is as in Remark 2.1(2).

Consider the tautological principal G-bundle $\mathcal{E}$ on $X \times Q^s$, and let $\overline{\mathcal{E}}$ be the corresponding $\overline{G}$-bundle obtained by extending the structure group. Then it is well known that the adjoint universal bundle $\overline{\mathcal{E}}$ descends to the quotient $\mathcal{M}_X(G)^s$ (see [**4**]).

We follow the same strategy for $\mathcal{M}'_X(H)$ as well. Consider the pulled-back $\overline{G}$-bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}$ on $X \times Q'_H$, where ψ is the map in (2.1) and $\overline{G} := G/Z(G)$. The action of $\mathcal{G}$ on Q'_H is free by Lemma 2.4. Therefore, the quotient $Q'_H \to \mathcal{M}'_X(H)$ is a principal $\mathcal{G}$-bundle. Further, the action of $\mathcal{G}$ lifts to the tautological bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}$. In particular, the principal $\overline{G}$-bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}$ descends to a principal $\overline{G}$-bundle over $X \times \mathcal{M}'_X(H)$.

Let us denote this descended $\overline{G}$-bundle over $X \times \mathcal{M}'_X(H)$ by $\overline{\mathcal{E}}_0$.

Let $\pi : (\mathrm{Id}_X \times \psi)^* \mathcal{E}(G/H) \to (\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(\overline{G}/H)$ be the natural map induced by the projection $G/H \to \overline{G}/H$, where ψ is the map in (2.1). We note that the universal H-bundle over $X \times U$, where U is defined in Proposition 2.6, is a reduction of structure group of the pulled-back G-bundle $(\mathrm{Id}_X \times \psi)^* \mathcal{E}$. Let

$$\sigma : X \times U \longrightarrow \big(\mathrm{Id}_X \times \psi \big)^* \mathcal{E}(G/H) \tag{2.9}$$

be the section giving this reduction of structure group. Then the composition $\pi \circ \sigma$ is a section of $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(\overline{G}/H)$ over $X \times U$.

Since H is semisimple, a lemma of Chevalley says that there is a $\overline{G}$-module W and an element $w \in W$ such that H is precisely the isotropy subgroup for w (see [3, page 89, Theorem 5.1]). Therefore, $\overline{G}/H$ is identified with the closed $\overline{G}$-orbit in W defined by w. Then we see that $\overline{\mathcal{E}}(\overline{G}/H) \hookrightarrow \overline{\mathcal{E}}(W)$. We may therefore view the section

$$\pi \circ \sigma : X \times U \longrightarrow \left(\mathrm{Id}_X \times \psi \right)^* \overline{\mathcal{E}}\left(\overline{G}/H\right) \tag{2.10}$$

as a section of the vector bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W)$ over $X \times U$.

Since $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}$ descends to the $\overline{G}$-bundle $\overline{\mathcal{E}}_0$ over $X \times \mathcal{M}'_X(H)$, it follows that the associated vector bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W)$ also descends to $X \times \mathcal{M}'_X(H)$. Clearly this vector bundle is nothing but the associated vector bundle $\overline{\mathcal{E}}_0(W)$, associated to the $\overline{G}$-bundle $\overline{\mathcal{E}}_0$ on $X \times \mathcal{M}'_X(H)$ for the $\overline{G}$-module W.

Since each point of $\mathcal{M}'_X(H)$ represents an isomorphism class of stable H-bundles, it follows that, set theoretically, the reduction section $\pi \circ \sigma \in \Gamma((\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W))$ descends to a section on $X \times \mathcal{M}'_X(H)$ of the descended vector bundle $\overline{\mathcal{E}}_0(W)$.

We now appeal to [6, Proposition 4.1], which implies that the section $\pi \circ \sigma$ in fact descends to give a holomorphic section of $\overline{\mathcal{E}}_0(W)$ over $X \times \mathcal{M}'_X(H)$.

Again set theoretically, the image of this section of $\overline{\mathcal{E}}_0(W)$ lies in $\overline{\mathcal{E}}_0(\overline{G}/H) \subset \overline{\mathcal{E}}_0(W)$. As before, from [6, Proposition 4.1] it follows that $\pi \circ \sigma$ in (2.10) gives a reduction of structure group to H of the descended $\overline{G}$-bundle $\overline{\mathcal{E}}_0$. The H-bundle over $X \times \mathcal{M}'_X(H)$ obtained this way is the required universal H-bundle. This completes the proof of the theorem. ∎

3 Nonexistence of universal bundle

Let H be a complex semisimple linear algebraic group and let $K \subset H$ be a maximal compact subgroup. The Lie algebra of K will be denoted by $\mathfrak{k}$. A homomorphism

$$\rho : \pi_1(X) \longrightarrow K \tag{3.1}$$

is called *irreducible* if no nonzero vector in $\mathfrak{k}$ is fixed by the adjoint action of the subgroup $\rho(\pi_1(X)) \subset K$ on $\mathfrak{k}$. Let $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ denote the space of all irreducible homomorphisms such that the corresponding K-bundle is topologically trivial. So any homomorphism in $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ is induced by a homomorphism from $\pi_1(X)$ to the universal cover of K.

Let g denote the genus of X. Assume that $g \geq 3$. If we choose a basis

$$\{a_1, \ldots, a_g, b_1, \ldots, b_g\} \subset \pi_1(X) \tag{3.2}$$

such that

$$\pi_1(X) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \right\rangle, \tag{3.3}$$

then $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ gets identified with a real analytic subspace of K^{2g}.

For any $\rho \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$, the principal H-bundle obtained by extending the structure group of the principal K-bundle given by ρ is stable. A theorem of Ramanathan says that all topologically trivial stable principal H-bundles arise in this way, that is, the space of all equivalence classes of irreducible homomorphisms of $\pi_1(X)$ to K is in bijective correspondence with the space of all stable H-bundles over X [9, Theorem 7.1]. More precisely (see the proof of [9, Theorem 7.1]), the real analytic space underlying the moduli space $\mathcal{M}_X(H)$ parametrising topologically trivial stable H-bundles is analytically isomorphic to the quotient space

$$R\big(\pi_1(X), H\big) := \mathrm{Hom}^{\mathrm{irr}}\big(\pi_1(X), K\big)/K, \tag{3.4}$$

for the action constructed using the conjugation action of K on itself. Let

$$q : \mathrm{Hom}^{\mathrm{irr}}\big(\pi_1(X), K\big) \longrightarrow R\big(\pi_1(X), H\big) \cong \mathcal{M}_X(H) \tag{3.5}$$

be the quotient map. The open subset $\mathcal{M}'_X(H) \subset \mathcal{M}_X(H)$ is a smooth submanifold and the restriction of the map in (3.5),

$$q\,|_{q^{-1}(\mathcal{M}'_X(H))} : q^{-1}\big(\mathcal{M}'_X(H)\big) \longrightarrow \mathcal{M}'_X(H), \tag{3.6}$$

is a smooth principal K/Z-bundle, where Z is the centre of K. Note that Z coincides with the centre of H (as H is semisimple, its centre is a finite group).

If $\mathcal{U}_H$ is a universal H-bundle over $X \times \mathcal{M}'_X(H)$, then we have a reduction of structure group of $\mathcal{U}_H$ to K which is constructed using the correspondence established in [9] between stable H-bundles and irreducible flat K-bundles. Let

$$\mathcal{U}_K \longrightarrow X \times \mathcal{M}'_X(H) \tag{3.7}$$

be the smooth principal K-bundle obtained from $\mathcal{U}_H$ this way. Consider the principal K/Z-bundle over $X \times \mathcal{M}'_X(H)$, where $Z \subset K$ is the centre, obtained by extending the structure group of $\mathcal{U}_K$ using the natural projection of K to K/Z. The restriction of this K/Z-bundle to $p \times \mathcal{M}'_X(H)$, where p is the base point in X used for defining $\pi_1(X)$, is identified with the K/Z-bundle in (3.6). To see this first note that the universal K-bundle over $X \times \mathrm{Hom}(\pi_1(X), K)$ is obtained as a quotient by the action of $\pi_1(X)$ on $\widetilde{X} \times \mathrm{Hom}(\pi_1(X), K) \times K$, where $\widetilde{X}$ is the pointed universal cover of X for the base point p; the action of $z \in \pi_1(X)$ sends any (α, β, γ) to $(\alpha, z, \beta, \beta(z)^{-1}\gamma)$. From this it follows that the restriction of this universal bundle to $p \times \mathrm{Hom}(\pi_1(X), K)$ is canonically trivialized. The above-mentioned identification is constructed using this trivialization.

Our aim is to show that no K-bundle over $\mathcal{M}'_X(H)$ exists that produces the K/Z-bundle in (3.6) by extension of structure group, provided the centre Z is nontrivial.

Let F_3 denote the free group on three generators. Fix a surjective homomorphism

$$f : \pi_1(X) \longrightarrow F_3 \tag{3.8}$$

that sends a_i, $1 \leq i \leq 3$, to the ith generator of F_3 and sends a_i, $4 \leq i \leq g$, and b_i, $1 \leq i \leq g$, to the identity element, where a_j, b_j are as in (3.3) (recall that $g \geq 3$).

Set

$$R(F_3, H) := \mathrm{Hom}^{\mathrm{irr}}(F_3, K)/K \tag{3.9}$$

to be the equivalence classes of irreducible representations. Note that $\mathrm{Hom}(F_3, K) \simeq K^3$, and under this identification the action

$$\mu : \mathrm{Hom}(F_3, K) \times K \longrightarrow \mathrm{Hom}(F_3, K) \tag{3.10}$$

given by $\mu(\rho, A) = A^{-1}\rho A$ corresponds to the simultaneous diagonal conjugation action of K on the three factors.

Since F_3 is a free group and K is connected, any homomorphism ρ from F_3 to K can be deformed to the trivial homomorphism. This implies that the principal H-bundle corresponding to the homomorphism

$$\rho \circ f : \pi_1(X) \longrightarrow K \tag{3.11}$$

is topologically trivial, where f is defined in (3.8). Therefore, we have an embedding

$$R(F_3, H) \longrightarrow R(\pi_1(X), H) \cong \mathcal{M}_X(H) \tag{3.12}$$

that sends any ρ to $\rho \circ f$. Let $R'(F_3, H) \subset R(F_3, H)$ be the inverse image of the open subset $\mathcal{M}'_X(H)$ under the above map. By Proposition 2.6 it follows that $R'(F_3, H)$ is a *nonempty* open subset of $R(F_3, H)$.

Fix a point $p \in X$, which will also be the base point for the fundamental group. Consider the restriction of the principal K-bundle $\mathcal{U}_K$ in (3.7) to $p \times \mathcal{M}'_X(H) \hookrightarrow X \times \mathcal{M}'_X(H)$ and denote it by $\mathcal{U}_{K,p}$.

Let

$$\gamma : p \times R'(F_3, H) \longrightarrow p \times \mathcal{M}'_X(H) \tag{3.13}$$

be the map given by the embedding $R'(F_3, H) \to \mathcal{M}'_X(H)$ constructed in (3.12). Taking the pullback of the above-defined principal K-bundle $\mathcal{U}_{K,p}$ over $p \times \mathcal{M}'_X(H)$ under the morphism γ in (3.13)

$$
\begin{array}{ccc}
\widetilde{M} & & \mathcal{U}_{K,p} \\
\downarrow & & \downarrow \\
p \times R'(F_3, H) & \xrightarrow{\ \gamma\ } & p \times \mathcal{M}'_X(H)
\end{array}
\tag{3.14}
$$

we obtain a principal K-bundle $\widetilde{M} := \gamma^* \mathcal{U}_{K,p}$ on $R'(F_3, H)$ whose associated K/Z-bundle (as before, Z is the centre of K) is precisely the K/Z-bundle $q^{-1}(R'(F_3, H)) \to R'(F_3, H)$ constructed in (3.6). In particular, $q^{-1}(R'(F_3, H)) \simeq \widetilde{M}/Z$.

Let $\mathrm{Hom}^{\mathrm{irr}}(F_3, K)' \subset \mathrm{Hom}^{\mathrm{irr}}(F_3, K)$ be the inverse image of $R'(F_3, H)$ under the quotient map in (3.9). Let

$$q_0 : \mathrm{Hom}^{\mathrm{irr}}(F_3, K)' \longrightarrow R'(F_3, H) \tag{3.15}$$

be the restriction of the quotient map in (3.9). So q_0 is the quotient for the conjugation action of K. This map q_0 defines a principal K/Z-bundle over $R'(F_3, H)$ which is evidently identified with the K/Z-bundle $\widetilde{M}/Z \to R'(F_3, H)$ obtained from (3.14).

We will prove (by contradiction) that such a K-bundle $\widetilde{M}$ does not exist. In other words, we will show that there is no principal K-bundle over $R'(F_3, H)$ whose extension of structure group is the K/Z-bundle in (3.15).

Assume the contrary, then we get the following diagram of topological spaces and morphisms:

$$
\begin{array}{ccccc}
Z & \longrightarrow & K & \longrightarrow & K/Z \\
\Big\| & & \Big\downarrow & & \Big\downarrow \\
Z & \longrightarrow & \widetilde{M} & \longrightarrow & q_0^{-1}\big(R'(F_3, H)\big) \\
& & \Big\downarrow & & \Big\downarrow \\
& & R'(F_3, H) & =\!=\!=\!= & R'(F_3, H)
\end{array}
\tag{3.16}
$$

We will need a couple of results. The proof of the nonexistence of the bundle $\widetilde{M}$ will be completed after establishing Proposition 3.2.

Lemma 3.1. There exist finitely many compact differentiable manifolds N_i and differentiable maps

$$
f_i : N_i \longrightarrow K^3,
\tag{3.17}
$$

$1 \le i \le d$, such that if $N := \bigcup_{i=1}^d f_i(N_i)$, then
  (1) the complement $K^3 \setminus N$ is contained in $q_0^{-1}(R'(F_3, H))$, where q_0 is the projection in (3.15),
  (2) $\dim K^3 - \dim N_i \ge 4$ for all $i \in [1, d]$.   □

Proof. Take any homomorphism $\rho : \pi_1(X) \to K$. Let E_ρ denote the corresponding polystable principal H-bundle over X [9]. The automorphism group of the polystable H-bundle E_ρ coincides with the centraliser of $\rho(\pi_1(X))$ in H. Hence if $\rho(\pi_1(X))$ is dense in K, then the automorphism group of E_ρ is the centre $Z(H) \subset H$. Therefore, if the topological closure $\overline{\rho(\pi_1(X))}$ of $\rho(\pi_1(X))$ is K and E_ρ is topologically trivial, then $E_\rho \in \mathcal{M}'_X(H)$.

Now assume that the centraliser $C(\rho) \subset H$ of $\rho(\pi_1(X))$ in H properly contains $Z(H)$. Since the complexification of $\overline{\rho(\pi_1(X))}$ is reductive and the centraliser of a reductive group is reductive, we conclude that $C(\rho)$ is reductive. Take a semisimple element $z \in (H \setminus Z(H)) \bigcap C(\rho)$ (since $C(\rho)$ is reductive and larger than $Z(H)$, such an element exists). Let $C_z \subset K$ be the centraliser of z in K. We have $\overline{\rho(\pi_1(X))} \subset C_z$ and C_z is a proper subgroup of K as $z \notin Z(H)$. Also, C_z contains a maximal torus of K.

Fix a maximal torus $T \subset K$. (Since any two maximal tori are conjugate, any subgroup C_z of the above type would contain T after an inner conjugation.) Consider all proper Lie subgroups M of K satisfying the following two conditions:
  (i) $T \subseteq M$,

(ii) there exists a semisimple element $z \in H \setminus Z(H)$ such that M is the centraliser of z in K.

Let $\mathbb{S}$ denote this collection of Lie subgroups of K.

The connected ones among $\mathbb{S}$ are precisely the maximal compact subgroups of Levi subgroups of proper parabolic subgroups of H containing T. Note that if P is a proper parabolic subgroup of H, then $P \cap K$ is a maximal compact subgroup of a Levi subgroup of P. All the connected ones among $\mathbb{S}$ arise as $P \cap K$ for some proper parabolic subgroup $P \subset H$ containing T.

This collection $\mathbb{S}$ is a finite set. To see this, we first note that there are only finitely many parabolic subgroups of H that contain T. For a proper parabolic subgroup $P \subset H$ containing T, there are only finitely many Lie subgroups of K that have $P \cap K$ as the connected component containing the identity element. Thus $\mathbb{S}$ is a finite set.

Let $M_1, \ldots, M_d$ be the subgroups of K that occur in $\mathbb{S}$.

We will show that the codimension of each M_i in K is at least two. It suffices to show that for a maximal proper parabolic subgroup P of G containing T, the codimension of $M := P \cap K$ in K is at least two.

To prove that the codimension of $M = P \cap K$ in K is at least two, let $\mathfrak{k}$ be the Lie algebra of K and let $\mathfrak{h}$ be the Lie algebra of T (the Cartan subalgebra). Let

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{k}^{\alpha} \tag{3.18}$$

be the root space decomposition of $\mathfrak{k}$. Let $\mathfrak{m}$ be a Lie algebra of M which is a Lie subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}$. Then we have the decomposition

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{m}^{\alpha} \tag{3.19}$$

for $\mathfrak{m}$ where each $\mathfrak{m}^{\alpha}$ is irreducible for $\mathfrak{h}$, and hence has to coincide with one of the $\mathfrak{k}^{\alpha}$. Then $\operatorname{codim}_{\mathfrak{k}}(\mathfrak{m}) \geq 2$ since if α is a root for the subalgebra $\mathfrak{m}$, then so is $-\alpha$ (see [1, page 83, Corollary 4.15]).

Consequently, the codimension of M in K is at least two. Thus the codimension of each M_i, $i \in [1, d]$, in K is at least two.

For each $i \in [1, d]$, let

$$\overline{f}_i : K \times M_i^3 \longrightarrow K^3 \tag{3.20}$$

be the map defined by $(x, (y_1, y_2, y_3)) \mapsto (xy_1x^{-1}, xy_wx^{-1}, xy_3x^{-1})$. Consider the free action of M_i on $K \times M_i^3$ defined by

$$z \cdot \left(x, \left(y_1, y_2, y_3\right)\right) = \left(xz^{-1}, \left(zy_1z^{-1}, zy_2z^{-1}, zy_3z^{-1}\right)\right), \tag{3.21}$$

where $x \in K$ and $z, y_1, y_2, y_3 \in M_i$. The map f_i in (3.20) clearly factors through the quotient

$$\frac{K \times M_i{}^t}{M_i} \tag{3.22}$$

for the above action. Therefore, we have

$$f_i : N_i := \frac{K \times M_i{}^t}{M_i} \longrightarrow K^t \tag{3.23}$$

induced by f_i.

To prove part (1) of the lemma, we recall the earlier remark that for any homomorphism $\rho' \in q_0^{-1}(R'(F_3, H))$ (the map q_0 is defined in (3.15)), the automorphism group of the principal H-bundle corresponding to $\rho' \circ f$ (the homomorphism f is defined in (3.8)) coincides with $Z(H)$ if the image of ρ' is dense in K. From the properties of the collection $\{M_1, \ldots, M_d\}$ we conclude that

$$N := \bigcup_{i=1}^{d} f_i\left(N_i\right) \supseteq q_0^{-1}\left(R'\left(F_3, H\right)\right)^c, \tag{3.24}$$

where f_i are defined in (3.23) and

$$q_0^{-1}\left(R'\left(F_3, H\right)\right)^c \subset K^3 \tag{3.25}$$

is the complement of $q_0^{-1}(R'(F_3, H))$ in K^3.

Therefore, proof of part (1) is complete. To prove part (2), we note that

$$\dim N_i = \dim K + 2 \cdot \dim M_i \tag{3.26}$$

for all $i \in [1, d]$. It was shown earlier that $\dim M_i \leq \dim K - 2$. Therefore,

$$\dim K^3 - \dim N_i = 3 \cdot \dim K - \dim N_i \geq 3 \cdot \dim K - 3 \cdot \dim K + 4 = 4. \tag{3.27}$$

This completes the proof of the lemma. ∎

Proposition 3.2. Consider the K/Z-principal bundle $q_0^{-1}(R'(F_3, H)) \to R'(F_3, H)$, where q_0 is the projection in (3.15). The induced homomorphism on fundamental groups

$$\pi_1(K/Z) \longrightarrow \pi_1\left(q^{-1}\left(R'\left(F_3, H\right)\right)\right) \tag{3.28}$$

obtained from the homotopy exact sequence is trivial. □

Proof. Let $x_1, x_2, x_3 \in K$ be regular elements; we recall that $x \in K$ is called regular if the centralizer $C(x) = \{y \in K \mid yx = xy\}$ is a maximal torus in K. Since the set of regular elements is dense in K, and $q_0^{-1}(R'(F_3, H)) \subset K^3$ is a nonempty open dense subset (this follows from Lemma 3.1), we may choose these x_i, $i = 1, 2, 3$, to lie in $q_0^{-1}(R'(F_3, H))$.

Consider the orbit $\text{Orb}_{K^3}(x_1, x_2, x_3)$ of $(x_1, x_2, x_3) \in K^3$ for the adjoint action of the group K^3 on itself. Clearly we have the following identification of this orbit:

$$\text{Orb}_{K^3}(x_1, x_2, x_3) = \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)} \tag{3.29}$$

with $C(x_i) \subset K$ being the centralizer of x_i. The conjugation action of K on $\text{Hom}(F_3, K)$ coincides with the restriction of the above action of K^3 to the image of the diagonal map $K \hookrightarrow K^3$. Therefore, the fibre K/Z through the point $(x_1, x_2, x_3) \in q_0^{-1}(R'(F_3, H))$ of the K/Z-bundle

$$q_0^{-1}\left(R'\left(F_3, H\right)\right) \longrightarrow R'\left(F_3, H\right) \tag{3.30}$$

is contained in the orbit

$$\frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}. \tag{3.31}$$

Now if we choose a point $(x_1, x_2, x_3) \in K^3$ which is general enough, then by the definition of the inverse image $q_0^{-1}(R'(F_3, H))$ and Lemma 3.1(2) it follows immediately that the complement of the open dense subset

$$q_0^{-1}(R'(F_3, H)) \bigcap \left(\frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}\right) \subset \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)} \tag{3.32}$$

is of codimension at least four.

Since the image of K/Z in $q_0^{-1}(R'(F_3, H))$ lies in $(q^{-1}(R'(F_3, H))) \bigcap \text{Orb}_{K^3}(x_1, x_2, x_3)$, whose complement is of codimension at least four in

$$\text{Orb}_{K^3}(x_1, x_2, x_3) = \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}, \tag{3.33}$$

it follows that the homomorphism $\pi_1(K/Z) \to \pi_1(q_0^{-1}(R'(F_3, H)))$ in the proposition factors through

$$
\begin{aligned}
\pi_1\big(&(q_0^{-1}(R'(F_3, H))) \cap \mathrm{Orb}_{K^3}(x_1, x_2, x_3)\big) \\
&= \pi_1\left(\frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}\right) = \pi_1(K/T)^3,
\end{aligned}
\tag{3.34}
$$

where T is a maximal torus of K.

For any maximal torus $T \subset K$, the quotient K/T is diffeomorphic to H/B, where B is a Borel subgroup of H. Since H/B is simply connected, we conclude that $(K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3))$ is simply connected. This completes the proof of the proposition. ∎

We now complete the proof of the nonexistence of the covering $\widetilde{M} \to q^{-1}(R'(F_3, H))$ *as in (3.16)*.

Since the homomorphism of fundamental groups $\pi_1(K/Z) \to \pi_1(q_0^{-1}(R'(F_3, H)))$ is trivial (see **Proposition 3.2**), the induced covering $K \to K/Z$ is trivial (see the diagram (3.16)). But this is a contradiction to the fact that K is connected.

Therefore, we have proved the following theorem.

Theorem 3.3. If the centre $Z(H)$ is nontrivial, then there is no universal bundle over $X \times \mathcal{M}'_X(H)$. □

Acknowledgments

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