Semistable Principal Bundles-II (in positive characteristics)

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1 Introduction

Let H be a semisimple algebraic group and let X be a smooth projective curve defined over an algebraically closed field k.

One of the important problems in the theory of principal H-bundles on X is the construction of the moduli spaces of semistable H-bundles when the characteristic of k is positive. Over fields of characteristic 0 this work was done by A.Ramanathan (cf.[R1]). For principal $GL(n)$-bundles this is classical, over fields of any characteristic (cf.[Ses]).

The purpose of this paper is to prove the existence and the projectivity of the moduli spaces of semistable principal H-bundles on X for fields k of characteristic $p > 0$ with precise bounds on the prime p, the restrictions being imposed by the representation theory of H.

It might seem, by the general method of reduction modulo p, that the existence of the moduli space in char.0 implies its existence for large primes. To the best of our knowledge this is not the case. (cf Remark 4.10). The representation theoretic considerations involving heights are essential to the proving of the existence of the moduli.

The broad strategy of this paper is along the same lines as in the precursor to this paper ([BS]) where a different approach for the construction

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and projectivity of these moduli spaces (in characteristic zero) was given. However, its implementation involves several new inputs. The key input for the *existence* of the moduli comes from the paper of Ilangovan-Mehta-Parameswaran ([IMP]) which establishes in positive characteristics the links between the semistability of principal bundles and the concept of a *low height representation*. In proving the *projectivity* of the moduli space, the key ideas come from a natural interplay of the recent results of Serre on the representation theory in positive characteristics ([S2], [S3]), and ideas inspired by the papers of Ramanan-Ramanathan and Rousseau ([RR], [Rou]). The principal difficulty is to replace *the tensor product theorem of semistable bundles* and *unitary representations of fundamental groups* which are central to the characteristic 0 theory. The notions of height and saturated groups provide just the right replacements.

Let H be a semisimple algebraic group (as coming by reduction from a Chevalley group scheme defined over $\mathbf{Z}$), and fix a faithful representation $H \hookrightarrow G = SL(n)$ arising as reduction modulo p of a representation defined over $\mathbf{Z}$. Let us denote by $ht_H(G)$ the height of G as an H-representation. (cf. Definition 3.1). We say a representation $H \hookrightarrow G$ is of *low height* if $char(k) = p > ht_H(G)$. Then we have the following:

Theorem 4.6 Let $H \hookrightarrow G$ be a faithful low height representation. Then there exists a coarse moduli scheme $M_X(H)$, for semistable principal H-bundles. Further, the moduli space $M_X(H)$ is quasi-projective and the canonical morphism $\overline{\mu} : M_X(H) \longrightarrow M_X(G)$ is *affine*.

The proof of the projectivity of the moduli spaces requires more refined prime bounds. Towards this we introduce a new index which we term the *separable index* associated to a G-module W (cf. Definition 5.2). We denote this by $\psi_G(W)$ and we say a G-module W is of *low separable index* if $char(k) = p > \psi_G(W)$. We fix throughout, a finite dimensional G-module W such that the subgroup H is realised as the isotropy of a closed orbit, hence giving rise to a closed embedding $G/H \subset W$. We term these G-modules for convenience as *affine* (G, H)*-modules* (cf. Def 3.12). Let A be a complete discrete valuation ring and let K be its quotient field and k its residue field. Then we have the following theorem:

Theorem 11.1 (*Semistable reduction*) Let W be a finite dimensional *affine* (G, H)*-module* associated to H and G and let $p > \psi_G(W)$. Let H_K denote the group scheme $H \times \operatorname{Spec} K$, and P_K be a semistable H_K-bundle on

X_K. Then there exists a finite extension L/K, with B as the integral closure of A in L such that the bundle P_K, after base change to $\operatorname{Spec} B$, extends to a semistable H_B-bundle P_B on X_B.

This in particular implies that the moduli spaces $M_X(H)$ are projective over fields k with $char(k) = p > \psi_G(W)$. Together with Theorem 4.6 we can conclude that the canonical morphism $\overline{\mu} : M_X(H) \longrightarrow M_X(G)$ is *finite*. As a corollary we also obtain the irreducibility of the moduli spaces when H is semisimple and simply connected.(Cor 11.10). A large part of this paper is devoted to proving Theorem 11.1.

The crucial difference between the present approach and the classical proof of Langton for the properness of the moduli space of semistable vector bundles can be briefly described as follows. Langton first extends the family of semistable vector bundles (or equivalently principal $GL(n)$-bundles) to a $GL(n)$-bundle in the limit although non-semistable. In other words, the structure group of the limiting bundle remains $GL(n)$. Then by a sequence of *Hecke modifications* the semistable limit is attained without changing the isomorphism class of the bundle over the generic fibre.

Instead, we extend the family of semistable H_K-bundles to an H'_A-bundle with the limiting bundle remaining semistable, but the structure group scheme H'_A, is non-reductive in the limit. In other words one loses the reductivity of the structure group scheme. Then, by using Bruhat-Tits theory (cf §10), we relate the group scheme H'_A to the reductive group scheme H_A without changing the isomorphism class of the bundle over the generic fibre as well as the semistability of the limiting bundle.

We note that the boundedness of semistable principal bundles over curves in positive characteristics is proved in the preprint ([HN]).[1]

Throughout the paper, we make an effort to specify carefully the bounds on the characteristic of k that are forced on us. We believe that our methods can probably be stretched to include more primes and we indicate at every stage the possible difficulties. The representation theoretic indices that we have developed here may possibly be of independent interest.

Before we proceed to describe the contents we pause to remark that there is some overlap between the present paper and [BS].

[1]The problem of the construction of the moduli is being considered independently by V.B.Mehta and S.Subramaniam.

The layout of the paper is as follows. In §3 we recall low height representations and some results from [S2] which we need in later sections. Here we also define the basic functors for semistable principal H- and G-bundles and we prove a technical lemma involving the choice of a "base point on the curve" which, in some sense gives the motivation for the rest of the work. In this paper we work with more than one base point so as to achieve better height bounds.

In §4 we give a simple construction of the moduli space of H-bundles under the right characteristic bounds. The idea of the proof comes from [BS] and the ingredients involving heights from §3.

The rest of the paper is devoted to proving the semistable reduction theorem. In §5 some new representation theoretic indices are introduced and these give the bounds that we need to impose on the characteristic p in what follows. Here the main point is to give a criterion for the *strong separability* of a linear action of a reductive group. In sections §6 and §7 we construct and study the flat closure of H'_K in G_A and realise it as *isotropy group schemes* along the lines of the classical theorem on semi-invariants (cf. [B]).

In §8 we prove the key lemmas on the relationship between polystable bundles and semistable sections inspired mainly by the papers of Ramanan-Ramanathan ([RR]) and Rousseau ([Rou]). More precisely, we obtain a notion paralleling that of *monodromy* subgroup of a polystable G-bundle which is realised as a *saturated subgroup* of G. This enables us to prove a *local constancy* for polystable bundles in char.p. In §9 we prove that the family of bundles extends to a semistable bundle with structure group as a non-reductive group scheme H'_A with generic fibre H_K. In §10 using Bruhat-Tits theory we relate the non-reductive group scheme H'_A with the reductive group scheme H_A. In §11 we complete the proof of the semistable reduction theorem.

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Contents

2 Notations and Conventions

Throughout this paper, unless otherwise stated, we have the following notations and assumptions:

(i) We work over an algebraically closed field k of characteristic $p > 0$.

(ii) H is a *semisimple* algebraic group, and G, unless otherwise stated will always stand for the special linear group $SL(n)$. Their representations are finite dimensional and rational.

(iii) A is a discrete valuation ring (which could be assumed to be complete) with residue field k, and quotient field K.

(iv) We recall that $\pi : E \longrightarrow X$ is a principal bundle with structure group H, or a principal H-bundle for short if H acts on E on the right and π is H-invariant and isotrivial, i.e, locally trivial in the étale topology.

(v) Let E be a principal G-bundle on $X \times T$ where T is $\operatorname{Spec} A$. Let $x \in X$ be a closed point which we fix throughout and we shall denote by $E_{x,A}$ or $E_{x,T}$ (resp $E_{x,K}$) the restriction of E to the subscheme $x \times \operatorname{Spec} A$ or $x \times T$ (resp $x \times \operatorname{Spec} K$). Similarly, $l \in T$ will denote the closed point of T and the restriction of E to $X \times l$ will be denoted by E_l.

(vi) We shall denote $T - l$ by T^* throughout this paper.

(vii) In the case where the structure group is $GL(n)$, when we speak of a principal $GL(n)$-bundle we identify it often with the associated vector bundle (and can therefore talk of the degree of the principal $GL(n)$-bundle).

(viii) We denote by E_K (resp E_A) the principal bundle E on $X \times \operatorname{Spec} K$ (resp $X \times \operatorname{Spec} A$) when viewed as a principal H_K-bundle (resp H_A-bundle). Here H_K and G_K (resp H_A and G_A) are the product group scheme $H \times \operatorname{Spec} K$ and $G \times \operatorname{Spec} K$ (resp $H \times \operatorname{Spec} A$ and $G \times \operatorname{Spec} A$).

(ix) If H_A is an A-group scheme, then by $H_A(A)$ (resp $H_K(K)$) we mean its A (resp K)-valued points. When $H_A = H \times \operatorname{Spec} A$, then we simply write $H(A)$ for its A-valued points. We denote the closed fibre of the group scheme by H_k.

(x) Let Y be any G-scheme and let E be a G-principal bundle. For example Y could be a G-module. Then we denote by $E(Y)$ the associated bundle with fibre type Y which is the following object: $E(Y) = (E \times Y)/G$ for the twisted action of G on $E \times Y$ given by $g.(e, y) = (e.g, g^{-1}.y)$.

(xi) If we have a group scheme H_A (resp H_K) over $\operatorname{Spec} A$ (resp $\operatorname{Spec} K$) an H_A-module Y_A and a principal H_A-bundle E_A, then we shall denote the associated bundle with fibre type Y_A by $E_A(Y_A)$.

(xii) By a family of H bundles on X parametrised by T we mean a principal H-bundle on $X \times T$, which we also denote by $\{E_t\}_{t \in T}$.

3 Low height representations and some consequences

Let k be an algebraically closed field of characteristic $p > 0$. Let H be a connected reductive algebraic group over k. Let T be a maximal torus of H, $X(T) := Hom(T, \mathbf{G}_m)$ be the character group of T and $Y(T) := Hom(\mathbf{G}_m, T)$ be the 1-parameter subgroups of T. Let $R \subset X(T)$ be the root system of H with respect to T. Let $\mathcal{W}$ be the Weyl group of the root system R. Let $(\ , \ )$ denote the $\mathcal{W}$-invariant inner product on $X(T) \otimes \mathbf{R}$. For $\alpha \in R$, the corresponding co-root $\alpha^\vee$ is $2\alpha/(\alpha, \alpha)$. Let $R^\vee \subset X(T) \otimes \mathbf{R}$ be the set of all co-roots. Let $B \subset H$ be a Borel subgroup containing T. This choice defines a base Δ^+ of R called the *simple roots*. Let $\Delta^- = -\Delta^+$. A root in R is said to be *positive* if it is a non-negative linear combination of simple roots. We take the roots of B to be positive by convention. Let $\Delta^\vee \subset R^\vee$ be the basis for the corresponding dual root system. Then we can define the Bruhat ordering on $\mathcal{W}$. The longest element with respect to this ordering of $\mathcal{W}$ is denoted by w_0. A reductive group is classified by these *root-data*, namely the character group, 1-parameter subgroups, the root system, co-roots and the $\mathcal{W}$-invariant pairing.

Let V be a H-module, i.e., V is a k-vector space together with a linear representation of H in Aut (V). Then V can be written as direct sum of eigenspaces for T. On each eigenspace T acts by a character. These are called the *weights* of the representation. A weight λ is called *dominant* if $(\lambda \ , \ \alpha_i^\vee) \geq 0$ for all simple roots $\alpha_i \in \Delta^+$. A weight λ is said to be $\geq$ another weight μ if the difference $\lambda - \mu$ is a non-negative integral linear combination of simple roots, where the difference is taken with respect to the natural abelian group structure of $X(T)$. The *fundamental weights* ω_i are uniquely defined by the criterion $(\omega_i \ , \ \alpha_j^\vee) = \delta_{ij}$. The element ρ of $X(T) \otimes \mathbf{R}$ is defined to be half the sum of positive roots. It can also seen to be equal to the sum of fundamental weights. The *height* (cf. [H], Section 10.1) of a root is defined to be the sum of the coefficients in the expression $\alpha = \Sigma k_i \alpha_i$. We extend this notion of height linearly to the weight space and denote this function by $ht(\ )$. Note that ht is defined for all weights but need not be an integer even for dominant weights. We extend this notion of height to representations as follows:

Definition 3.1.

(i) Given a linear representation V of H, we define the **height** of the representation $ht_H(V)$ (also denoted by $ht(V)$ if H is understood in the given context) to be the maximum of $2ht(\lambda)$, where λ runs over dominant weights occurring in V.

(ii) A linear representation V of H is said to be a **low height** representation if $ht_H(V) < p$, and a weight λ is of low height if $2ht(\lambda) < p$.

Then we have the following theorem (cf. [IMP], [S2])

Theorem 3.2. Let V be a linear representation of H of low height. Then V is semisimple.

Corollary 3.3. Let V be a low height representation of H and $v \in V$ an element such that the H-orbit of v in V is closed. Then V is a semisimple representation for the reduced stabiliser $H_{v,red}$ of v.

Proposition 3.4. Let H be as above and let V be a low-height representation of H. Then we have the following vanishing of group cohomology:

$$H^i(H, V) = 0$$

for all $i \geq 1$.

Proof. We now recall from ([S2] (pp 25,26))the following general result on low height modules of connected reductive groups: Let V be a low height module of H. Let λ be a dominant weight which occurs in V. Then, if $V(\lambda) = H^0(\lambda)$ is the dual of the Weyl module associated to λ, by the definition of height and the low height property of V, it follows that $V(\lambda)$ are also low-height H-modules. In particular, it follows that $V(\lambda)$ are also *irreducible* and they coincide with their socle $L(\lambda)$. Therefore, by the semisimplicity of low height modules, one has $V = \overset{\lambda}{\oplus} V(\lambda)$.

Therefore by the Vanishing Theorem of Cline-Parshall-Scott-van der Kallen (cf. [J] pp 237) we have the required cohomology vanishing since

$$H^i(H, V) = \bigoplus^{\lambda} H^i(H, V(\lambda)) = 0$$

for all $i \geq 1$. Q.E.D.

3.1 *Height and semistability*

Let F be a G-variety. Then a section $s : X \longrightarrow E(F)$ can be described as a morphism from $\psi : E \longrightarrow F$ such that $\psi(e.g) = g^{-1}.s(e)$. In particular, if $H \subset G$ and $F = G/H$ then a section of $E(G/H)$ gives a reduction of structure group of E to H.

We now recall the definitions of semistable, polystable and stable principal bundles. Note that these definitions make sense for reductive groups as well.

Definition 3.5. (A. Ramanathan) E is *semistable* (resp. *stable*) if for every parabolic subgroup P of H, and for every reduction of structure group $\sigma_P : X \longrightarrow E(H/P)$ to P and for any dominant character χ of P, the bundle $\sigma_P^*(L_\chi))$ has degree ≤ 0 (resp.< 0).(cf.[R1]).

Definition 3.6. A reduction of structure group of E to a parabolic subgroup P is called *admissible* if for any character χ on P which is trivial on the center of H, the line bundle associated to the reduced P-bundle E_P has degree zero.

Definition 3.7. An H-bundle E is said to be *polystable* if it has a reduction of structure group to a Levi R of a parabolic P such that the reduced R-bundle E_R is stable and the extended P bundle $E_R(P)$ is an admissible reduction of structure group for E.

Remark 3.8. We note that there is a natural action of the group $\mathrm{Aut}_G E$, of automorphisms of the principal G-bundle E, on $\Gamma(X, E(G/H))$ and the orbits correspond to the H-reductions which are isomorphic as principal H-bundles.

Remark 3.9. Let E_R be a stable R-bundle. Then E_R has no further reduction of structure group to a Levi subgroup L of a parabolic subgroup in R.

Proposition 3.10. Let E be a principal H-bundle on X. Let $H \hookrightarrow SL(V)$ be a *low height* faithful representation. Then the following are equivalent:

(a) The induced bundle $E(V) = E \times^H V$ is semistable.

(b) The bundle E is semistable as a principal H-bundle.

Proof. (b) $\Rightarrow$ (a) follows by ([IMP] Theorem 3.1).

For (a) $\Rightarrow$ (b), we need to proceed as follows. By the Main Theorem of [S2], any low height representation is actually semi-simple. Further, if $V = \bigoplus V_i$ is the decomposition into irreducible H-modules, then the associated bundle $E(V)$ decomposes as $\bigoplus E(V_i)$ and the direct sum is of bundles of degree zero. Therefore it is clear that to prove the converse, we may as well assume that the representation ρ is an irreducible representation of H and also of low height. So since we are assuming that E is non-semistable, there exists a maximal parabolic subgroup $P \subset H$ and a dominant character λ such that the pull back of L_λ has degree $deg(L_\lambda) > 0$. Now it is not very hard to see that there exists a parabolic P_1 in $SL(V)$ such that $P = P_1 \cap H$ (cf [IMP, Lemma 3.5]). Thus we see that, the reduction of structure group of the vector bundle $E(V)$ to P_1 is given by a section $\sigma \in \Gamma(E(SL(V)/P_1))$ and the line bundle L_λ is the restriction of an ample line bundle $L_{\lambda'}$ obtained by a dominant character λ' of P_1. It is clear that $deg(\sigma^*(L_{\lambda'})) > 0$ since $degL_\lambda = deg(\sigma^*(L_{\lambda'}))$. This implies that $E(V)$ is also non-semistable, and we are done.

Remark 3.11. This theorem is strict in the sense that given an almost simple group H and a representation $H \longrightarrow SL(V)$ which is not of low height, there exists a curve X and a semistable H-bundle E on X such that $E(V)$ is not semistable. (The converse works for all but small primes. For more precise details see [IMP])

3.2 *Functorial properness of the evaluation map*

The aim of this section is to define the basic functors and prove some technical lemmas. Let G be $SL(n)$ and let H be a semisimple algebraic group, $H \subset G$. For our convenience we make the following definition:

Definition 3.12. Define an *affine* (G, H)*-module* W associated to (H, G) to be a finite dimensional G-module, such that $G/H \overset{i}{\hookrightarrow} W$ is realised as a closed orbit of a vector $w \in W$. Observe that since G/H is affine, such a W always exists. *We work with this canonical* W *whenever we refer to the affine* (G, H)*-module*. This is a classical result (cf. for example [DM1], p 40 or [B]; also cf. Lemma 7.1 below).

Let

$$F_G : (Schemes) \longrightarrow (Sets)$$

be the functor given by

$$F_G(T) = \left\{ \begin{array}{l} \text{isomorphism classes of semistable } G\text{-bundles of degree 0} \\ \text{on } X \text{ parametrised by } T \end{array} \right.$$

One may similarly define the functor F_H (note that since H is semisimple, for a principal H-bundle the associated vector bundles have degree zero).

Let $x \in X$ be a marked point and let $F_{H,G,x}$ be the functor

$$F_{H,G,x}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (E, \sigma_x),\ E = \{E_t\}_{t \in T} \\ \text{a family of semistable principal } G\text{-bundles of degree 0} \\ \text{and } \sigma_x : T \longrightarrow E(G/H)_x \text{ a section} \end{array} \right\}$$

(Recall that $E(G/H)_x$ denotes the restriction of $E(G/H)$ to $x \times T \approx T$).

Notice that the functor F_H is in fact realisable as the following functor (by Remark 3.8) .

$$F_{H,G}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (E, s),\ E = \{E_t\}_{t \in T} \\ \text{a family of semistable } G\text{-bundles of degree 0 and} \\ s = \{s_t\}_{t \in T} \text{ a section of E(G/H) on } X \times T \\ \text{or what we may call a family of sections of } \{E(G/H)_t\}_{t \in T}. \end{array} \right\}$$

In what follows, we shall identify the functors F_H with $F_{H,G}$. With these definitions we have the following:

Proposition 3.13. Let α_x be the morphism induced by "evaluation of section" at x:

$$\alpha_x : F_H \longrightarrow F_{H,G,x}.$$

Then α_x is a *proper morphism of functors.*(cf. [DM]).

Proof. Let T be an affine smooth curve and let $l \in T$. Let us write $T^* = T - l$. Then by the valuation criterion for properness, we need to show the following:

If E is a family of semistable principal G-bundles on $X \times T$ together with a section $\sigma_x : T \longrightarrow E(G/H)_x$; such that for $t \in T^*$, we are given a family of H-reductions, i.e. a family of sections $s_{T^*} = \{s_t\}_{t \in T^*}$, where, $s_t : X \longrightarrow E(G/H)_t$, with the property that, at x, $s_t(x) = \sigma_x(t)\ \forall\ t \in T^*$; then we need to extend the family s_{T^*} to a section s_T of $E(G/H)$ on $X \times T$ such that $s_l(x) = \sigma_x(l)$ as well.

Let W be an affine (G, H)-module associated to (H, G).(see Def 3.12). Thus we get a closed embedding

$$E(G/H) \hookrightarrow E(W)$$

and a family of vector bundles $\{E(W)_t\}_{t \in T}$ together with a family of sections s_{T-l} and evaluations $\{\sigma_x(t)\}_{t \in T}$ such that $s_t(x) = \sigma_x(t)$, $t \neq p$.

For the section s_{T-l}, viewed as a section of $E(W)_{T-l}$ we have two possibilities:

(a) it extends as a regular section s_T.

(b) it has a pole along $X \times l$.

Observe that if (a) holds, then we have

$$s_T(X \times (T - l)) \subset E(G/H) \subset E(W),$$

since $E(G/H)$ is closed in $E(W)$, it follows that $s_T(X \times l) \subset E(G/H)$. Thus $s_l(X) \subset E(G/H)_l$. Further by continuity, $s_l(x) = \sigma_x(l)$ as well and this proves the proposition.

If (b) holds the reduction section exists over an open $U_T \subset X_T$ which contains all the primes of height 1 in X_T; or equivalently, the H-bundle exists over U_T. We can now appeal to a theorem of Colliot-Thélène and Sansuc ([CS] Theorem 6.13 pp 128) which enables us to extend the principal bundle to X_T. In other words (b) cannot occur. Finally observe that the limiting H-bundle is semistable since it arises as a reduction of structure group of a semistable G-bundle and $H \subset G$ is low height. Q.E.D.

Remark 3.14. A different proof involving the semistability of $E(W)$ is given in [BS]. Here we avoid it so as to improve the prime bounds arising out of height considerations involved in the construction of the moduli spaces.

4 Construction of the moduli space

The aim of this section is to give a construction of the moduli space of H-bundles. This section is somewhat different from the corresponding one in[BS] to enable us to provide the best prime bounds.

Recall that $G = SL(n)$ and $H \subset G$ a semisimple subgroup.

We recall very briefly the Grothendieck Quot scheme used in the construction of the moduli space of vector bundles (cf. [Ses]).

Let $\mathcal{F}$ be a coherent sheaf on X and let $\mathcal{F}(m)$ be $\mathcal{F} \otimes \mathcal{O}_X(m)$ (following the usual notations). Choose an integer $m_0 = m_0(n, d)$ (n = rk, d = deg) such that for any $m \geq m_0$ and any semistable bundle V of rank n and deg d on X we have $h^i(V(m)) = 0$ and $V(m)$ is generated by its global sections.

Let $\chi = h^0(V(m))$ and consider the Quot scheme Q consisting of coherent sheaves $\mathcal{F}$ on X which are quotients of $k^\chi \otimes_k \mathcal{O}_X$ with a fixed Hilbert polynomial P. The group $\mathcal{G} = GL(\chi)$ canonically acts on Q and hence on $X \times Q$ (trivial action on X) and lifts to an action on the universal sheaf $\mathcal{E}$ on $X \times Q$.

Let R denote the $\mathcal{G}$-invariant open subset of Q defined by

$$R = \left\{ q \in Q \mid \begin{array}{l} \mathcal{E}_q = \mathcal{E} \mid_{X \times q} \text{ is locally free and the canonical map} \\ k^\chi \longrightarrow H^0(\mathcal{E}_q) \text{ is an isomorphism}, \det \mathcal{E}_q \simeq \mathcal{O}_X \end{array} \right\}$$

We denote by Q^{ss} the $\mathcal{G}$-invariant open subset of R consisting of semistable bundles and let $\mathcal{E}$ continue to denote the restriction of $\mathcal{E}$ to $X \times Q^{ss}$.

Let $q'' : (\text{Sch}) \longrightarrow (\text{Sets})$ be the following functor:

$$q''(T) = \left\{ (V_t, s_t) \mid \begin{array}{l} \{V_t\} \text{ is a family of semistable principal } G\text{-bundles} \\ \text{parameterised by } T \text{ and } s_t \in \Gamma(X, V(G/H)_t) \ \ \forall\, t \in T \end{array} \right\}.$$

i.e. $q''(T)$ consists pairs of rank n vector bundles (or equivalently principal G-bundles) together with a reduction of structure group to H.

By appealing to the general theory of Hilbert schemes, one can show without much difficulty (cf. [R1, Lemma 3.8.1]) that q'' is representable by a Q^{ss}-scheme, which we denote by Q''.

Let W be an affine (G, H)-module associated to (H, G) (see Def 3.12).

Remark 4.1. Let $E \in Q^{ss}$ and consider $E(W)$ the associated vector bundle. Then, by the boundedness of the family $E \in Q^{ss}$ it follows that there exists an m_0 (*independent of* E) such that if s is any section of $E(W)$, then $\#(zeroes(s)) \leq m_0$. Fix a subset $J \subset X$ such that $\#J > m_0$.

The universal sheaf $\mathcal{E}$ on $X \times Q^{ss}$ is in fact a vector bundle. Let $\mathcal{E}_G$ denote the associated principal G-bundle, set

$$Q' = (\mathcal{E}_G/H)_J = (\mathcal{E}_G/H)_{x_j} \times_{Q^{ss}} (\mathcal{E}_G/H)_{x'_j} \cdots$$

the fibre product being taken over all $j \in J$. Then in our notation $Q' = \mathcal{E}_G(G/H)_J$ i.e. we take the bundle over $X \times Q^{ss}$ associated to $\mathcal{E}_G$ with fibre G/H and take its restriction to $x_j \times Q^{ss} \approx Q^{ss}$ and take the product over Q^{ss}. Let $f : Q' \longrightarrow Q^{ss}$ be the natural map. Then, since H is reductive, f is an *affine morphism.*

Observe that Q' parametrises semistable vector bundles together with "initial values of reductions to H ".

Define the "evaluation map" of Q^{ss}-schemes as follows:

$$\phi_J : Q'' \longrightarrow Q'$$

$$(V, s) \longmapsto \{(V, s(x)) | x \in J\}.$$

Lemma 4.2. The evaluation map $\phi_J : Q'' \longrightarrow Q'$ is *proper* and *injective.*

Proof. Let $G/H \hookrightarrow W$ be as in Definition 3.12 and let (E, s) and $(E', s') \in Q''$ such that $\phi_J(E, s) = \phi_J(E', s')$ in Q'. i.e. $(E, s(x)) = (E', s'(x)) \forall x \in J$. So we may assume that $E \simeq E'$ and that s and s' are two different sections of $E(G/H)$ with $s(x) = s'(x) \forall x \in J$.

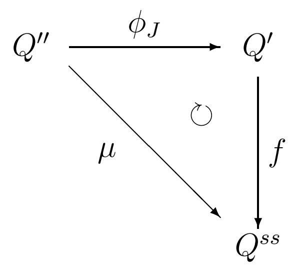
Using $G/H \hookrightarrow W$, we may consider s and s' as sections in $\Gamma(X, E(W))$ and further, as sections of $E(W)$ one has $s(x) = s'(x) \forall x \in J$. By Remark 4.1 this implies $s = s'$. This proves that ϕ_J is *injective* (since $E(G/H) \hookrightarrow E(W)$ is a closed embedding).

The properness of the map follows easily, the proof being as in Proposition 3.13. Thus ϕ_J being proper and injective is *affine.* Q.E.D.

Remark 4.3. In [BS] a single base point served the purpose. Here we employ the standard trick of increasing the number of base points to achieve injectivity without the semistability of $E(W)$. This enables us to improve the prime bounds.

Remark 4.4. It is immediate that the $\mathcal{G}$-action on Q^{ss} lifts to an action on Q''.

Recall the commutative diagram



By Lemma 4.2, ϕ_J is a proper injection and hence affine. One knows that f is affine (with fibres $(G/H)^{\#J}$). Hence μ is a $\mathcal{G}$-equivariant affine morphism.

Lemma 4.5. (cum remark) Let (E, s) and (E', s') be in the same $\mathcal{G}$-orbit of Q''. Then we have $E \simeq E'$. Identifying E' with E, we see that s and s' lie in the same orbit of $\mathrm{Aut}_G E$ on $\Gamma(X, E(G/H))$. Then using Remark 3.8, we see that the reductions s and s' give isomorphic H-bundles.

Conversely, if (E, s) and (E', s') such that $E \simeq E'$ and the reductions s, s' give isomorphic H-bundles, using again Remark 3.8, we see that (E, s) and (E', s') lie in the same $\mathcal{G}$-orbit.

Consider the $\mathcal{G}$-action on Q'' with the linearisation induced by the *affine $\mathcal{G}$-morphism* $\mu : Q'' \longrightarrow Q^{ss}$. It is seen without much difficulty that, since a good quotient of Q^{ss} by $\mathcal{G}$ exists and since $Q'' \longrightarrow Q^{ss}$ is an affine $\mathcal{G}$-equivariant map, a good quotient $Q''/\mathcal{G}$ exists (cf. [R1, Lemma 4.1]).

Moreover by the universal property of categorical quotients, the canonical morphism

$$\overline{\mu} : Q''//\mathcal{G} \longrightarrow Q^{ss}//\mathcal{G}$$

is also *affine*.

Let $M_X(H)$ denote the scheme $Q''//\mathcal{G}$. then we have proved the following theorem.

Theorem 4.6. Let $H \hookrightarrow G$ be a faithful low height representation. Then there exists a coarse moduli scheme $M_X(H)$, for semistable principal H-bundles. Further, the moduli space $M_X(H)$ is quasi-projective and the canonical morphism $\overline{\mu} : M_X(H) \longrightarrow M_X(G)$ is *affine*.

4.1 *Points of the moduli*

In this subsection we will recall briefly the description of the k-valued points of the moduli space $M_X(H)$. The general functorial description of $M_X(H)$ as a coarse moduli scheme follows by the usual process.

Proposition 4.7. The "points " of $M_X(H)$ are given by polystable principal H-bundles.

We firstly remark that since the quotient $q : Q'' \longrightarrow M_X(H)$ obtained above is a good quotient, it follows that each fibre $q^{-1}(E)$ for $E \in M_X(H)$ has a unique closed $\mathcal{G}$-orbit. Let us denote an orbit $\mathcal{G} \cdot E$ by $O(E)$. The proposition will follow from the following:

Lemma 4.8. If $O(E)$ is closed then E is polystable.

Proof. Recall the definition of a polystable bundle Def 3.7 and the definition of *admissible reductions* Def 3.6. If E has no admissible reduction of structure group to a parabolic subgroup then it is polystable and there is nothing to prove.

Suppose then that E has an admissible reduction E_P, to $P \subset H$. Recall by the general theory of parabolic subgroups that there exists a 1-PS $\xi : \mathbf{G}_m \longrightarrow H$ such that $P = P(\xi)$. Let $L(\xi)$ and $U(\xi)$ be its canonical Levi subgroup and unipotent subgroup respectively. The Levi subgroup will be the centraliser of this 1-PS ξ and one knows $P(\xi) = L(\xi) \cdot U(\xi) = U(\xi) \cdot L(\xi)$. In particular, if $h \in P$ then $lim \xi(t) \cdot h \cdot \xi(t)^{-1}$ exists. From these considerations one can show that there is a morphism

$$f : P(\xi) \times \mathbf{A}^1 \longrightarrow P(\xi)$$

such that $f(h, 0) = m \cdot u$, where $h \in P$ and $h = m \cdot u$, $m \in L$ and $u \in U$. (see Lemma 3.5.12 [R1])

Consider the P-bundle E_P. Then, using the natural projection $P \longrightarrow L$ where $L = L(\xi)$, we get an L-bundle $E_P(L)$. Again, using the inclusion $L \hookrightarrow P \hookrightarrow H$, we get a new H-bundle $E_P(L)(H)$. Let us denote this H-bundle by $E_P(L, H)$. It follows from the definition of admissible reductions and polystability that $E_P(L, H)$ is *polystable*.

Further, from the family of maps f defined above, and composing with the inclusion $P(\xi) \hookrightarrow H$ we obtain a family of H-bundles $E_P(f_t)$ for $t \neq 0$ and all these bundle are isomorphic to the given bundle E. Following ([R1] Prop.3.5 pp 313), one can prove that the bundle $E_P(L, H)$ is the limit of $E_P(f_t)$. It follows that $E_P(L, H)$ is in the $\mathcal{G}$-orbit $O(E)$ because $O(E)$ is closed. Now by Lemma 4.5, $E \simeq E_P(L, H)$, implying that E is polystable. Q.E.D.

Remark 4.9. In the above Proposition we have only stated that there is a surjective map from the set of isomorphism classes of polystable H-bundles to the points of the moduli space. We believe that this correspondence is a bijection but one possibly needs to discard a few more primes.

A few remarks are in order regarding the existence and properness of the moduli spaces of principal bundles for "large" primes.

Remark 4.10. " A general principle is that if a statement is true in characteristic zero then it is also true for *large p*" (cf. ([S2])). One might therefore think that this would imply the existence and projectivity of the moduli spaces of semistable H-bundles for large primes, since one already knows this in char 0 (cf. for example [R1], [F] or [BS]).

We observe that this principle would indeed hold if one could show that the subset corresponding to the semistable bundles in a family of H-bundles is open over (**Z** or large p); for the moduli spaces of H-semistable bundles is realised as a GIT-quotient of a quasi-projective scheme and the required results would follow by "reduction modulo p" for large p. To the best of our knowledge the required "openness result" does not follow by any general principle.

A key point of this paper is that even for the existence of the moduli spaces of semistable H-bundles as a quasi-projective scheme for large p, one requires *height* considerations. Moreover we give explicit bounds for p.

Once the moduli space exists as a quasi-projective scheme for large p, its projectivity follows for an unspecified *larger* p. One of the hard parts of this paper is to give specific representation theoretic bounds for p for the projectivity of the moduli spaces.

5 Separable index and slice theorem

Let T be a torus and W be a finite dimensional T-module. Further, let $X(T)$ be the free abelian group of characters of T and $\mathcal{S}$ be the set of distinct characters that occur in W.

For every subset $S \subset \mathcal{S}$ we have the following map:

$$\nu_S : \mathbf{Z}^{|S|} \longrightarrow X(T)$$

given by $e_s \longrightarrow \chi_s$.

Let g_S be the g.c.d of the maximal minors of the map ν_S written under the fixed basis. For any vector $w \in W$, consider the subset $S_w \subset \mathcal{S}$, consisting of characters that occur in w with nonzero coefficients. i.e., if $w = \sum a_\chi(w)e_\chi$, then

$$S_w = \{\chi \in \mathcal{S} | a_\chi(w) \neq 0\}$$

Then we have the following:

Lemma 5.1. The characteristic of the field, p does not divide g_{S_w} if and only if the action of T on the vector w is separable.

Proof. Let $T \cdot w$ denote the orbit of w under the T action. Let T_w be the stabilis er. Then T/T_w is a torus and the character group $X(T_w)$ of the stabiliser is the quotient of the character groups $X(T)/X(T/T_w)$. Moreover the image of the dual map of the quotient map $T \to T/T_w$ is canonically identified with the image of ν_{S_w}. Hence the group T_w is reduced if and only if this cokernel, identified with the cokernel of ν_{S_w}, has no p-torsion. But this cokernel has p-torsion if and only if the rank of ν_{S_w} drops mod p, which in turn happens if and only if p divides all the maximal minors. Hence the lemma.

Notation

$$\mathfrak{p}_T(W) = \{\text{largest prime which divides } g_S | \forall S \subset \mathcal{S}\}$$

Definition 5.2. Let $H \longrightarrow SL(W)$ be a finite dimensional representation of H. Define the *separable index*, $\psi_H(W)$ of the representation as follows:

$$\psi_H(W) = max\{ht_H(W),\ \mathfrak{p}_T(W)\}$$

Remark 5.3. When T is a maximal torus of a semisimple group H and the T-module W is actually an H-module, then the set of characters that occur on W can be written down explicitly using Standard Monomial Theory. From this very explicit form, this separable index is computable, though it could be tedious or may need a computer. The few cases where we made some computations indicated that this index is possibly bounded above by the dimension of W. One can easily observe that the absolute value of each minor of the map ν_S is bounded above by $l! \cdot h^l$, where $l = rank(G)$ and $h = ht_G(W)$. Hence the separable index has a weak upper bound given by $l! \cdot h^l$.

Definition 5.4. A representation $H \longrightarrow SL(W)$ is said to be with **low separable index** if $p > \psi_H(W)$.

Theorem 5.5. If W is a low separable index H-module then the action of H on W is *strongly separable* i.e., the stabilizer at any point is absolutely reduced.

Proof. Since the representation is low height, every nilpotent in the Lie algebra of the H is integrated in $SL(W)$ and hence the nilpotent part of the Lie algebra of the stabiliser at any $w \in W$ will actually lie in the Lie algebra of the reduced stabiliser. Thus by the Jordan decomposition of the stabiliser, it is enough to ensure separability of the action of any maximal torus. Separability index assures that the given maximal torus T acts separably at all points in W. This implies that every maximal torus acts separably at all points as all maximal tori are conjugates. Hence the action of H is strongly separable. Q.E.D.

Remark 5.6. We recall briefly the notions of *saturated subgroups* of $GL(V)$. For details cf.pp 524-526 [S3].

We first define a one parameter subgroup defined by an element of order p. Let V be a finite dimensional k-vector space, and let $s \in GL(V)$ be an element such that $s^p = 1$. One has $s = 1 + u$ where $u^p = 0$. If $t \in k$, we can define an element $s^t \in GL(V)$ by the truncated binomial formula:

$$s^t = 1 + tu + \frac{t(t-1)u^2}{2} + ...$$

summed upto u^i with $i < p$. The map $t \longrightarrow s^t$ defines a homomorphism of algebraic groups:

$$\phi_s : \mathbf{G}_a \longrightarrow GL(V)$$

where $\mathbf{G}_a$ is the additive group. This homomorphism has two characterising properties:

- $\phi_s(1) = s$
- The map $t \longrightarrow \phi_s(t)$ is a polynomial of degree $< p$.

Let $H \subset GL(V)$ be a subgroup. We say that H is *saturated* if every unipotent element $s \in H$ has the following properties:

- $s^p = 1$
- $s^t \in H$ for every $t \in k$

One can see that given any subgroup H there is a smallest saturated subgroup which contains H called the *saturation of* H.

A property of saturated groups which we need is that if H is saturated and H^0 is the connected component of identity of H then the index $[H : H^0]$ is coprime to p. (cf.pp 524-526 [S3]).

One can again generalize all these notions for an arbitrary reductive algebraic group G instead of $GL(V)$. Among the elementary examples of saturated subgroups are parabolic subgroups, centralizers of any subgroup, and Levi subgroups (since they can be realised as the centralizer of a torus). We can isolate a couple of key properties in the theory of low height representations:

(i) If $G \longrightarrow GL(V)$ is a low height representation of G then the isotropy subgroups of closed orbits in V are saturated.

(ii) If S is a reductive and saturated subgroup of G and if $G \longrightarrow GL(V)$ is a low height representation of G then V is a low height module for S as well.(cf. [S2] p.25)

Proposition 5.7. (A version of Luna's étale slice theorem in char.p) Let W be a low separable index G-module. Let F be a fibre of the good quotient $q : W \longrightarrow W//G$, and let F^{cl} be the unique closed orbit contained in F. Then there exists a G-map

$$F \longrightarrow F^{cl}.$$

Proof. Since $\psi_G(W) = max\{ht_G(W),\ \mathfrak{p}_T(W)\}$ the assumption $p > \psi_G(W)$ on the separable index implies the following:

(i) Every stabiliser subgroup for the G-action on W is reduced, the action being *strongly separable* (by Th 5.5).

(ii) It is *saturated*, the representation $G \longrightarrow GL(W)$ being low height.

(iii) When w is a quasistable point in W or equivalently, the orbit $G \cdot w$ is closed, then W is a semisimple representation of the stabiliser G_w. This is a consequence of the main theorem of [S3], namely that low height representations are semi-simple.

For more on this (cf. [S2] pp 20-25); it may be kept in mind that the height of the representation, $ht_G(W)$, coincides with Serre's index $n_G(W)$.

A close examination of Luna's proof shows that the key point is the complete reducibility of the tangent space $T_w(W)$, of the affine G-module. This is used then to get a splitting of the canonical injection of the tangent space of the closed G-orbit in $T_w(W)$. Once this is achieved the slice can be constructed. The above proposition is then a corollary to the main slice theorem applied to a single orbit. (For details cf. [BR] Prop 8.5 p 312). Q.E.D.

6 Towards the flat closure

Fix as in §3.2 a faithful *low height* representation $H \hookrightarrow G$ defined over k as well as an *affine* (G, H)*-module* associated to the pair (H, G).(cf.Def 3.12).

Consider the extension of structure group of the bundle P_K via the induced K-inclusion $H_K \hookrightarrow G_K$. We denote the associated G_K-bundle $P_K(G)$ by E_K.

Then, since $G = SL(n)$, by the projectivity of the moduli space of semistable vector bundles, there exists a *semistable extension* of $P_K(G) = E_K$ to a G_A-bundle on $X \times \operatorname{Spec} A$, which we denote by E_A. Call the restriction of E_A to $X \times l$ (identified with X) the *limiting bundle* of E_A and denote it by E_l (as in §1). One has in fact slightly more, which is what we need.

Lemma 6.1. Let E_K denote a family of semistable G_K-bundles on $X \times \operatorname{Spec} K$ (or equivalently a family of semistable vector bundles of rank n and trivial determinant on $X \times T^*$). Then (by going to a finite cover S of T if need be) the principal bundle E_K extends to E_A with the property that the limiting bundle E_l is in fact *polystable* i.e, a direct sum of stable bundles.

Proof. The proof of this Lemma is possibly well known but for the sake of completeness we give it here. Recall notations as in §4 regarding Quot schemes etc.

Note that the moduli space in question, namely of semistable principal G-bundles, is a GIT quotient $Q^{ss} \longrightarrow M$ by $\mathcal{G}$, and the family $E_A(G)$ is given by a morphism $T \longrightarrow M$. Lift the K-valued point, namely, r_K, given by the family E_K, to Q^{ss} and consider the $\mathcal{G}$-orbit R_0 of r_K in Q^{ss}. Let $\overline{R}_0$ be its closure in Q^{ss}. Since the K-valued point r_K is in fact an A-valued point of M, the GIT quotient of $\overline{R}_0$ is indeed the curve T. Also, observe that the closure intersects the closed fibre. Consider the morphism $\psi : \overline{R}_0 \longrightarrow T$. Since the base is a curve T, one has a *multi-section* for the morphism ψ, and one obtains the curve S. The general fibre has been modified only in the orbit, therefore the isomorphism class of the bundles remains unchanged. Q.E.D.

Remark 6.2. It is to be noted that the definition of polystability given here coincides with that in Def 3.7, in the sense that a closed orbit in the Quot scheme corresponds to a polystable vector bundle.

We observe the following:

- Note that giving the H_K-bundle P_K is giving a reduction of structure group of the G_K-bundle E_K which is equivalent to giving a section s_K of $E_K(G_K/H_K)$ over X_K .
- We fix a base point $x \in X$ and denote by $x_A = x \times \operatorname{Spec} A$, the induced section of the family (which we call the *base section*):

$$X_A \longrightarrow \operatorname{Spec} A$$

- Let $E_{x,A}$ (resp $E_{x,K}$) be as in §1, the restriction of E_A to x_A (resp x_K). Thus, $s_K(x)$ is a section of $E_K(G_K/H_K)_x$ which we denote by $E_x(G_K/H_K)$.
- Since $E_{x,A}$ is a principal G-bundle on $\operatorname{Spec} A$ and therefore trivial, it can be identified with the group scheme G_A itself. *For the rest of the article we fix one such identification, namely:*

$$\xi_A : E_{x,A} \longrightarrow G_A.$$

- Since we have fixed ξ_A we have a canonical identification

$$E_x(G_K/H_K) \simeq G_K/H_K$$

  which therefore carries a natural *identity section* e_K (i.e the coset $id.H_K$). Using this identification we can view $s_K(x)$ as an element in the homogeneous space G_K/H_K.

- Let $\theta_K \in G(K)$ be such that $\theta_K^{-1} \cdot s_K(x) = e_K$. Then we observe that, the isotropy subgroup scheme in G_K of the section $s_K(x)$ is $\theta_K.H_K.\theta_K^{-1}$.

- On the other hand one can realise $s_K(x)$ as the identity coset of $\theta_K.H_K.\theta_K^{-1}$ by using the following identification:

$$G_K/\theta_K.H_K.\theta_K^{-1} \xrightarrow{\sim} G_K/H_K.$$

$$g_K(\theta_K.H_K.\theta_K^{-1}) \longmapsto g_K\theta_K.H_K.$$

Definition 6.3. Let H'_K be the subgroup scheme of G_K defined as:

$$H'_K := \theta_K.H_K.\theta_K^{-1}.$$

Using ξ_A we can have a canonical identification:

$$E_x(G_K/H'_K) \simeq G_K/H'_K.$$

Then we observe that, using the above identification we get a section s'_K of $E_K(G_K/H'_K)$, with the property that, $s'_K(x)$ is the *identity section* and moreover, since we have conjugated by an element $\theta_K \in G_A(K)(= G(K))$, the isomorphism class of the H_K-bundle P_K given by s_K does not change by going to s'_K.

Thus, in conclusion, the G_A-bundle E_A has a reduction to H'_K given by a section s'_K of $E_K(G_K/H'_K)$, with the property that, at the given base section $x_A = x \times \operatorname{Spec} A$, we have an equality $s'_K(x_A) = e'_K$, with the *identity element* of G_K/H'_K (namely the coset $id.H'_K$).

Definition 6.4. The *flat closure* of the reduced group scheme H'_K in G_A is defined to be the schematic closure of H'_K in G_A with the reduced scheme structure. Let H'_A denote the flat-closure of H'_K in G_A.(cf. Lemma 7.1)

We then have a canonical identification via ξ_A:

$$E_x(G_A/H'_A) \simeq G_A/H'_A.$$

One can easily check that H'_A is indeed a subgroup scheme of G_A since it contains the identity section of G_A, and moreover, it is faithfully flat over A.

Notice however that H'_A *need not* be a *reductive* group scheme; that is, the special fibre H_k over the closed point need not be reductive.

Observe further that $s'_K(x)$ extends in a trivial fashion to a section $s'_A(x)$, namely the *identity coset section* e'_A of $E_x(G_A/H'_A)$ identified with G_A/H'_A .

Remark 6.5. If H'_A is *reductive* then the semistable reduction theorem (Theorem 11.1) follows quite easily. Indeed, firstly by the *rigidity* of reductive group schemes over Spec A (SGA 3, Expose III Cor 2.6 pp 117), by going to a finite cover, we may assume that $H'_A = H \times \operatorname{Spec} A$. Secondly, in this case one can realise H'_A, as the isotropy subgroup scheme for a closed orbit $w_A \in W_A$. Then we have a *closed* G*-immersion* of G/H in a G-module W, and one may view s_K as a section of $E_K(W_K)$. Note that $E_K(G_K/H'_K) \subset E_K(W_K)$.

By choice, along x_A, the section $s_K(x)$ extends regularly to a section of $E_A(G_A/H'_A) \subset E_A(W_A)$. Hence by Proposition 3.13, s_K extends to a section s_A which gives the required reduction over $X \times \operatorname{Spec} A$.

7 Affine embedding of G_A/H'_A

As we have noted, H'_A need not be reductive and the rest of the proof is to get around this difficulty. Our first aim is to prove that the structure group of the bundle $E_A(G_A)$ can be reduced to H'_A which is the statement of Proposition 9.1.

We need to prove the following generalisation of a well-known result (cf. for example [B]).

Lemma 7.1. There exists a finite dimensional G_A-module W_A such that $G_A/H'_A \hookrightarrow W_A$ is a G_A-immersion.

Proof. We follow the standard proof. Let I_K be the ideal defining the subgroup scheme H'_K in $K(G)$ (note that G_A (resp G_K) is an affine group scheme and we denote by $A(G)$ (resp $K(G)$) its coordinate ring).

Set $I_A = I_K \cap A(G)$. Then it is easy to see that since we are over a discrete valuation ring, I_A is in fact the ideal in $A(G)$ defining the flat closure H'_A. Observe also that I_A is a *primitive* A submodule of $A(G)$, that is, $A(G)/I_A$

is torsion free; further, $I_A \otimes k = I_k$ is the defining ideal in $k(G)$ of H'_k in G_k and $I_A \otimes K$ is I_K.

Since $A(G)$ and the other modules involved are free over the discrete valuation ring A, a set generates $I_A \otimes k = I_k$ if and only if it generates I_A. Thus we may now choose a finite generating set $\{f_i\}$ of I_A, such that their images $f_{i,k}$ generate I_k.

As in the classical proof, one has a finite dimensional G_K-submodule, V_K, containing the $\{f_i\}$. Now set $V_A = V_K \cap A(G)$ and $M = V_A \cap I_A$. Observe that I_A, V_A and hence M are all G_A-submodules of $A(G)$. This can be seen by keeping track of the co-module operations. Then clearly V_A is primitive in $A(G)$ and M is also primitive in $A(G)$ and in particular, primitive in V_A.

If we set

$$M_k = M \otimes k \text{ and } V_k = V_A \otimes k$$

we see that the inclusion $M \hookrightarrow V_A$ induces an inclusion $M_k \hookrightarrow V_k$. Observe that

$$f_i \ \in M \ , f_{i,k} \ \in \ M_k \text{ and } M \ \subset \ I_A$$

$$M_k \ \subset \ I_k \text{ and } M_k \ = \ V_k \ \cap \ I_k$$

We claim that, for $g \in G_A(k)$, one has

$$g \cdot M_k \subset M_k \iff g \in H'_k$$

Obviously, if $g \in H'_k$, then $g \cdot M_k \subset M_k$, since V_k is G-stable and I_k is H'_k-stable. Thus, it suffices to show that

$$f_{i,k}(g) = 0 \text{ for all } i$$

that is,

$$f_{i,k} \text{ vanish on } \ g$$

Since $f_{i,k} \in M_k$, it suffices to show that

$$\phi(g) = 0 \text{ for } \phi \ \in \ M_k$$

But $\phi(g) = (g^{-1} \cdot \phi)(id)$, where $g^{-1} \cdot \phi$ is the action of G on functions on G. Now, by hypothesis, $(g^{-1} \cdot \phi) \in M_k$. Since $M_k \subset I_k$, and $id \in H'_k$, we see that $(g^{-1} \cdot \phi)(id) = 0$. This proves the above claim.

Similarly, if we set

$$M_F = M \otimes_A F \text{ and } V_F = V_A \otimes_A F$$

where F is any field containing A, we see that for $g \in G(F)$

$$g \cdot M_F \subset M_F \iff g \in H'_A(F)$$

Let L denote the primitive rank one A-submodule $\wedge^d M \hookrightarrow \wedge^d V = W_A$, and $[L]$ the A-valued point of $\mathbf{P}(W_A)$ defined by L. Here, $\mathbf{P}(W_A)$ is defined by the functor associated to rank one direct summands of W_A. Then, the above discussion means that, we can recover H'_A as the isotropy subgroup scheme at $[L]$ for the G_A-action on $\mathbf{P}(W_A)$.

Recall that, for any field F, the isotropy subgroup of $G_A(F)$, at the point of $\mathbf{P}(W_A(F))$ represented by the base change of L by F, is $H'_A(F)$.

Fix a generator $l \in L$ so that l is a primitive element in W_A and consider the isotropy subgroup scheme H''_A at l for the G_A-action on W_A. We claim that, H''_A coincides with H'_A. To see this, observe that, H''_A is the subgroup scheme of G_A which leaves the closed subscheme (identified with $Spec(A)$) determined by l invariant (with the corresponding automorphism on this subscheme being identity). We see then that, H''_A is a *closed* subgroup scheme of G_A. Further, we see that since H''_A is the isotropy subgroup of the vector $l \in L$ and H'_A that of the line $[L]$ we have $H''_A \hookrightarrow H'_A$. Since H'_K is semi-simple, it has no characters and therefore, the isotropy subgroup scheme at $(l \otimes K) \in (W_A \otimes K)$ is precisely H'_K. This means that, $H''_K = H'_K$. Now, H'_K is open (dense) in H'_A (since H'_A is the flat closure of H'_K in G_A) so that, H'_K is also dense in H''_A. This implies that, H'_A and H''_A coincide set theoretically. Observe also that H'_A is *reduced* by the definition of flat closure. Thus, it follows that $H'_A = H''_A$. This implies that, $G_A/H'_A \hookrightarrow W_A$ is a G_A-*immersion* and the above lemma follows. Q.E.D.

Remark 7.2. Regarding the Lemma 7.1 proved above, we note that usually the subgroup scheme H'_A can be realised only as the isotropy subgroup scheme of a line in a G_A-module. But here, since the generic fibre of H'_A is semisimple, one is able to realise H'_A as the isotropy subgroup scheme of a primitive element in a G_A-module and the limiting group scheme also as an isotropy subgroup scheme for an element in a G_k-module. We note here that last part of the above proof is seen easily by observing that a non-trivial character of H'_A by definition is a non-trivial character of H'_K and hence $H''_A = H'_A$.

Remark 7.3. We make the following key observations about the group scheme H'_A. The flat group scheme $H'_A = Stab(w_A)$, is the isotropy subgroup scheme of G_A at an A-valued point $w_A \in W_A$, where W_A can be realised as $W \otimes A$ (after going to a finite cover of A if need be) and W is the affine (G, H)-module such that $G/H \subset W$.

Moreover, it is also shown as a part of the proof that the closed fibre $H'_k = Stab(w_k)$, is the isotropy subgroup scheme of G_k for a vector $w_k \in W$.

Thus if we assume that $p > \psi_G(W)$, it follows by Theorem 5.5 that H'_k is **reduced**.

8 Semistable bundles, semistable sections and saturated groups

The aim of this section is to prove some general lemmas on polystable bundles and semistable sections. We assume that $p > \psi_G(W)$, notations as in §5.

Definition 8.1. (following Bogomolov) Let E be a principal G-bundle and let $G \longrightarrow GL(V)$ be a representation of G. Let s be a section of the associated bundle $E(V)$. Then we call the section s stable (resp semistable, unstable) relative to G if at one point $x \in X$ (and hence at every point on X) the value of the section $s(x)$ is stable (resp semistable, unstable).

(It is easy to see the non-dependence of the definition on the point $x \in X$. Consider the inclusion $k[V]^G \hookrightarrow k[V]$ and the induced morphism $V \longrightarrow V/G$. This induces a morphism $E(V) \longrightarrow E(V/G)$. Observe that V/G is a trivial G-module. Thus we have the following diagram:

$$s : X \longrightarrow E(V) \longrightarrow E(V/G) \simeq X \times V/G$$

Composing with the second projection we get a morphism $X \longrightarrow V/G$ which is constant by the projectivity of X. Hence the value of the section is determined by one point in its G-orbit.) (cf. [Rou] 1.10)

Lemma 8.2. Let $E(W)$ be a semistable vector bundle of *degree zero* and let R be a *saturated reductive* subgroup of $GL(W)$. Suppose that $E(W)$ has a reduction of structure group to E_R, a stable R-bundle, and further suppose that we have a non-zero section $s : X \longrightarrow E_R(W) = E(W)$. Then s is a semistable section in the sense of Def 8.1.

Proof. Suppose that this is not the case. Then as observed in the definition, if $s(x)$ unstable for a single $x \in X$ implies it is unstable for all $x \in X$. In particular for the generic point $x_0 \in X$. (cf. [Rou] Prop 1.5)

Since s is a non-zero section of $E_R(W)$ and $E_R(W)$ is semistable of degree zero, it is nowhere zero. This section gives a reduction of structure group of $E_R(W)$ to a maximal parabolic subgroup P_s, given by the extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E_R(W) \longrightarrow V \longrightarrow 0$$

for some degree zero vector bundle V and where the first inclusion is given by the section s.

Notice that $SL(W)/P_s = \mathbf{P}(W)$. In the language of [RR], the section s can be thought of as taking values in the cone W and $deg(s) = 0$.

We now claim that w.l.o.g we may assume that the representation W is an irreducible R-module.

Since W is a low height R-module it is completely reducible, i.e it can be expressed as

$$W = \bigoplus W^\alpha$$

where W^α are irreducible R-module. Any element $w \in W$ can be expressed as $w = \oplus w^\alpha$ with $w^\alpha \in W^\alpha$. It is easy to see that if, w is R-unstable and if λ is a Kempf 1-PS in R which drives w to 0 then λ drives all the w^α's to 0 as well. Further, as bundles

$$E_R(W) = \bigoplus E_R(W^\alpha)$$

and since $E_R(W)$ is semistable of degree 0 all the $E_R(W^\alpha)$ are semistable of degree 0 being direct summands of $E_R(W)$. The given section also breaks up as $s = \oplus s^\alpha$ to give non-zero (and hence nowhere zero!) sections of $E_R(W^\alpha)$ (since $s(x) = w = \oplus w^\alpha$, here of course, not all α's may be involved!).

Again by Def 8.1, the new sections s^α continue to remain unstable since instability is determined at a point $x \in X$. This proves the claim.

Once W is irreducible as an R-module by Schur Lemma the connected component $Z^0(R)$ of center of R acts as scalars on W and hence *trivially* on $\mathbf{P}(W)$ and as scalars on the ample line bundle L on it.

Since $m = s(x_0)$ is unstable we have a Kempf instability flag $P(m)$ and the corresponding 1-PS μ, are also defined over the field $K(X)$. This follows by the low separable index assumption, namely $p > \psi_G(W)$, which in particular implies W is a low height module for G and hence for the saturated subgroup R (cf. [RR, Prop 3.13] and [IMP, Theorem 3.1]).

The parabolic subgroup being defined over $K(X)$ gives a reduction of structure group of E_R to a parabolic P of R. Let $W = \bigoplus W_i$ be the weight space decomposition of W with respect to μ. Let $m = m_0 + m_1$, with m_0 of weight $j > 0$ and m_1 the sum of terms of higher weights. In other words, in the projective space $\mathbf{P}(W)$ we see that $\mu(t) \cdot m \longrightarrow m_0$. It is not too hard to see that we have an identification of the Kempf parabolic subgroups associated to the points m and m_0. i.e $P(m) = P(m_0)$.(cf. [RR, Proposition 1.9]).

In the generic fibre $E_R(W)_{x_0}$ we have the projection

$$\bigoplus_{i \geq j} W_i \longrightarrow W_j$$

which takes m to m_0. This gives a line sub-bundle L_0 of degree zero of $E_R(W)$ corresponding to m_0. It then follows that m_0 is in fact semistable for the action of P/U, the Levi of P, for a suitable choice of linearisation obtained by twisting the action by a dominant character χ of P. (This is essentially the content of [RR, Prop.1.12] and we can apply it since we work in the degree 0.)

The semistability of the point m_0 with this new linearisation the forces the degree inequality:

$$deg(L_0 \otimes L(\chi)^{-1}) \leq 0$$

But since $deg(L_0) = 0$, this implies $deg(L(\chi)) \geq 0$. This contradicts the stability of E_R. Q.E.D.

Remark 8.3. We note that the condition of semistability of the vector bundle $E_R(W)$ is assumed here since [IMP] proves it only for semisimple groups. But in the situation in which we need (cf. Prop 9.2) this condition automatically holds since we have the following inclusion

$$R \hookrightarrow G \hookrightarrow GL(W)$$

and therefore $E_R(W) = E(W)$ and $E(W)$ is semistable since W is a low height representation of G.

Lemma 8.4. Let E_R be a stable R-bundle as above and let I be a saturated reductive (possibly non-connected) subgroup of R such that E_R has a reduction of structure group to I. Then the reduced I-bundle is also stable.

Proof. We first *claim* that I is *irreducible* in R: if not, then by the low height property there exists a parabolic P and a Levi L in it such that $I \subset L$ and this is irreducible. This gives a reduction of structure group of E_R to L and this again contradicts the stability of E_R, by Remark 3.9.

Now to prove the Lemma, suppose that the reduced I-bundle E_I is *not stable*. Then, E_I has an reduction of structure group σ, to a maximal parabolic $P \subset I$. Observe that any parabolic subgroup of a reductive algebraic group looks like $P(\lambda)$ for a 1-PS $\lambda : \mathbf{G}_m \longrightarrow I$. Now consider $P_R(\lambda)$ the induced parabolic in R. Then, it is clear that $P_R(\lambda)$ gives a reduction of structure group for E_R.

Notice that $P_R(\lambda)$ in R may not be a maximal parabolic, but there exists a maximal parabolic Q containing it. Now note that by the irreducibility of $I \subset R$ seen above, $Q \cap I$ is a proper parabolic in I and contains $P_I(\lambda)$. Therefore by the maximality of P_I it follows that $Q \cap I = P_I$.

Let χ be a dominant character of $P(\lambda)$ and let the induced line bundle be L_χ such that $deg(\sigma^*(L_\chi)) \geq 0$. Then, since Q is a maximal parabolic a multiple of χ extends to a dominant character of Q and the induced line bundle L_χ on I/P is the restriction of the line bundle from R/Q. Therefore, the degrees of the pull backs to X remain the same. This contradicts the stability of E_R. Q.E.D.

Proposition 8.5. Let E_R be a stable R-bundle and s be a non-zero section of $E_R(W)$ as in Lemma 8.2. Let $s(x) = w$. Then the R-orbit of w is closed and s takes its image in the closed orbit.

Proof. By Lemma 8.2, since E_R is stable, $w \in W^{ss}$. Therefore the section s which can be thought of as a map

$$s : E_R \longrightarrow W^{ss}$$

which further takes its values in a fibre F of the GIT quotient:

$$W^{ss} \longrightarrow W^{ss}//R$$

Thus the section s gives the following map:

$$s : E_R \longrightarrow F$$

and F contains the vector w.

We need to show that the orbit $R{\cdot}w$ is closed. We prove this by contradiction.

Suppose then that orbit of $R \cdot w$, is *not* closed. Let I be the isotropy at a point $f \in F$ such that $R \cdot f$ is closed. Note that the identity component I^o is reductive and saturated and I is also reduced.

Then by Proposition 5.7 we have an I-invariant "slice", $S \subset F$ and an R-isomorphism

$$\theta : R \times^I S \simeq F$$

$$\theta([r, s]) = r \cdot s$$

This gives a R-equivariant morphism

$$l : F \simeq R \times^I S \longrightarrow R/I \simeq F^{cl}.$$

The composition $l \circ s = s_1$ of the maps s and l gives a reduction of structure group, $E_I \subset E_R$ to the isotropy $I = Stab_R(f)$ of a point $f \in F^{cl}$. By Lemma 8.2 the I-bundle E_I is stable.

Consider the given section s of $E_R(W)$ as obtained via the reduction of structure group to I. This is given as follows:

$$s_1 : E_I \longrightarrow F \hookrightarrow W$$

which is I-equivariant. Observe that without loss of generality (by taking a conjugate of the isotropy I) we may assume that $w \in Im(s_1)$.

(This is easy to see. Indeed, starting with a pair (I, S) namely a slice and an isotropy subgroup at $f \in S$, the given point $w \in F$ can be expressed as an equivalence class $w = [r, s_0]$. Then by translating the slice S by the element $r \in R$ we get a new slice $r \cdot S = S'$ and a new pair (I', S') where $I' = r \cdot I \cdot r^{-1}$. It is clear that we have an isomorphism

$$F \simeq R \times^I S \simeq R \times^{I'} S'$$

and under this identification we get a reduction of structure group to I' with the property that the image of the section contains the given vector w.)

Further, by assumption $w \in W - W^I$.

Moreover, the I-orbit closure of w contains $f \in W^I$. Therefore, if $\overline{w}$ is the image of w in the quotient space W/W^I, then clearly $\overline{w}$ is an I-*unstable* vector in W/W^I.

Observe also that since I is saturated, by [S2], W is I-cr and hence $W/W^I \hookrightarrow W$ obtained as an I-splitting. Note that $E_I(W) = E_R(W) = E(W)$ is semistable of degree 0 and since W/W^I is an I-direct summand of W the associated bundle $E_I(W/W^I)$ is a direct summand of the degree 0 semistable vector bundle $E_I(W)$.

This implies that $E_I(W/W^I)$ is also semistable of degree 0.

Composing the section s_1 and the I-map $W \longrightarrow W/W^I$ we have:

$$\overline{s}_1 : E_I \longrightarrow W \longrightarrow W/W^I$$

and $\overline{w} \in Im(\overline{s}_1)$. This gives a non-zero *unstable* section of $E_I(W/W^I)$ which contradicts the stability of the bundle E_I by Lemma 8.2.

This contradicts the assumption that the orbit $R \cdot w$ is *not* closed and completes the proof of the Proposition. Q.E.D.

Remark 8.6. The theme in this section fits in with the general theme of Kempf-Luna in the char.0 case. In char.0 the polystable bundle E comes from an representation of $\pi_1(X) \longrightarrow G$. Let R be the Levi of an admissible parabolic and E_R be as in §9. Then E_R is stable. So the representation $\pi_1(X) \longrightarrow G$ which factors via R is irreducible . Let M be the Zariski closure of the image. Then the inclusion $M \hookrightarrow R$ is irreducible in the following natural sense of [S2] and [S3]: namely, there exists no parabolic subgroup $P \subset R$ such that $M \hookrightarrow P$.

In this case the proof of Proposition 8.5 now follows easily by results of Kempf. We need to check that the orbit $R \cdot w$ is closed. Now M is a reductive subgroup of R which fixes w since $\pi_1(X)$ fixes w (by classical local constancy). If $R{\cdot}w$ is *not closed* then R possesses a non-trivial one-parameter subgroup and since M fixes w there exists a Kempf parabolic P such that $M \hookrightarrow P \hookrightarrow R$ contradicting irreducibility of $M \subset R$. (cf. [K, Cor 4.4,4.5])

Remark 8.7. The Proposition 8.5 appears in [RR] but only in char.0. In [RR] there is an error in the proof of the second half of their theorem. Here we give a different proof of this and this works in the situation when the action is separable which in particular takes care of char.0 as well.

9 Extension to the flat closure

Recall that the section $s'_K(x)$ extends along the base section x_A, to give $s'_A(x) = w_A$. The aim of this section is to prove the following key theorem.

Theorem 9.1. The section s'_K, extends to a section s'_A of $E_A(G_A/H'_A)$. In other words, the structure group of E_A can be reduced to H'_A; in particular, if H'_k denotes the closed fibre of H'_A, then the structure group of E_k can be reduced to H'_k.

9.1 *Saturated monodromy groups and Local constancy*

Proposition 9.2. Let E be a *polystable* principal G-bundle on X. Let W be a G-module of low separable index, $w \in W$ and $H' = Stab(w)$. Let $Y = G/H'$ the G-subscheme of W defined by the reduced subgroup $H' \subset G$. If s is a section of $E(W)$ such that for some $x \in X$, the evaluation at x, namely $s(x) = w$ is in $E(Y)_x$, then the entire image of s lies in $E(Y)$. In fact we have a reduction of structure group to a reductive **saturated** subgroup R_w of H' and in particular, the reduced R_w-bundle is stable.

Proof. Since the G-bundle E is assumed polystable, by Def 3.7, there is an admissible reduction to a parabolic subgroup $Q \subset G$ and a further reduction of structure group E_R, to a Levi subgroup $R \subset Q$ with E_R actually **stable**.

Note that since R is a Levi of a parabolic in G, the maximal torus of G and R are the same.

Further, being a Levi of a parabolic R is a saturated subgroup of G. Since the height of the representation $G \longrightarrow SL(W)$ is low, it follows that W as an R-module is also of low height (cf. [S2] pp 22).

Thus, we can conclude that W as an R-module is also of *low separable index*.

Consider the R-bundle E_R and the R-module W. We are given a section $s : X \longrightarrow E(W) = E_R(W)$ such that at $x \in X$ $s(x) = w$ is the given vector in W with $Stab_G(w) = H'$.

By Proposition 8.5, since E_R is stable, the orbit $R \cdot w = F^{cl}$ is a closed orbit. Since the action of R on W is separable the isotropy, $R_w = Stab_R(w)$ is reduced, and we have an isomorphism $R.w \simeq R/R_w$. Note further that R_w is *saturated* and reductive.

As one has observed in the previous proof the section s takes its values in the fibre F and since $w \in F^{cl}$ we have the following:

$$s : E_R \longrightarrow F^{cl} \simeq R/R_w.$$

This gives a reduction of structure group of E_R to R_w. We thus have the following inclusion of bundles:

$$E_{R_w} \hookrightarrow E_R \hookrightarrow E$$

Note that $R_w = Stab_R(w) \subset Stab_G(w) = H'$. This inclusion gives the required reduction of structure group of E to H' which indeed comes as an extension of structure group from E_{R_w}. Furthermore, R_w is saturated and reductive. This complete the proof of the Proposition. Q.E.D.

9.2 *Completion of proof of Theorem 9.1*

By Lemma 7.1 ,we have

$$E_A(G_A/H'_A) \hookrightarrow E_A(W_A).$$

The given section s'_K of $E_K(G_K/H'_K)$ therefore gives a section u_K of $E(W_K)$. Further, $u_K(x)$, the restriction of u_K to $x \times T^*$, extends to give a section $u_A(x)$ of $E_x(W_A)$ (restriction of $E_A(W_A)$ to $x \times T$).Thus, by Proposition 3.13, and by the semistability of $E_l(W_A)$, the section u_K extends to give a section u_A of $E(W_A)$ over $X \times T$.

Now, to prove the Theorem 9.1 , we need to make sure that:

$$\text{The image of this extended section } u_A \text{ actually lands in } E_A(G_A/H'_A)\ . \tag{$*$}$$

This would then define s'_A.

To prove $(*)$, it suffices to show that $u_A(X \times l)$ lies in $E_A(G_A/H'_A)_l$ (the restriction of $E_A(G_A/H'_A)$ to $X \times l$).

Observe that, $u_A(x \times l)$ lies in $E_A(G_A/H'_A)_l$ since $u_A(x) = s'_A(x) = w_A$.

Observe further that, if E_l denotes the principal G-bundle on X, which is the restriction of the G_A-bundle E_A on $X \times T$ to $X \times l$, then $E_A(W_A)_l = E_A(W_A)|X \times l$, and we also have

$$\begin{array}{ccc} E_A(G_A/H'_A)_l & \xrightarrow{\simeq} & E_l(G_k/H'_k) \\ \downarrow & & \downarrow \\ E_A(W_A)_l & \xrightarrow{\simeq} & E_l(W) \end{array}$$

and the vertical maps are inclusions:

$$E_A(G_A/H'_A)_l \hookrightarrow E_A(W_A)_l, E_l(G_k/H'_k) \hookrightarrow E_l(W)$$

where $E_l(W) = E_l \times^{H'_k} W$ with fibre as the G-module $W = W_A \otimes k$. Note that G/H'_k is a G-subscheme Y of W.

Recall that E_l is polystable of degree zero. Then, from the foregoing discussion, the assertion that $u_A(X \times l)$ lies in $E_A(G_A/H'_A)$, is a consequence of Proposition 9.2 applied to E_l. (Note that the group $H'_k = Stab_{G_k}(w_k)$ satisfies the hypothesis of Proposition 9.2).

Thus we get a section s'_A of $E_A(G_A/H'_A)$ on $X \times T$, which extends the section s'_K of $E_A(G_A/H'_A)$ on $X \times T^*$. This gives a reduction of structure group of the G_A-bundle E_A on $X \times T$ to the subgroup scheme H'_A and this extends the given bundle E_K to the subgroup scheme H'_A.

In summary, we have extended the original H_K-bundle upto isomorphism to a H'_A-bundle. The extended H'_A-bundle has the further property that the limiting bundle E'_l which is an H'_k-bundle comes with a reduction of structure group to a reductive and *saturated* subgroup R_w of H'_k. Q.E.D

Remark 9.3. The proof of Theorem 9.1 is not as simple as in the proof of Proposition 3.13, since

$$E_A(G_A/H'_A) \hookrightarrow E_A(W_A)$$

is *not* a closed immersion. The group scheme H'_A is not reductive and therefore, we are *not* given a *closed* G-embedding of G_A/H'_A in G_A-module W_A (cf. Remark 6.5).

Remark 9.4. The reductive saturated subgroup R_w plays the role of "monodromy" subgroup of the polystable G-bundle E. (cf. [BS])

10 Potential good reduction

Recall that by virtue of the separability of the action the group scheme H'_A is *smooth.*

To complete the proof of the Theorem 11.1, we need to extend the H_K-bundle to an H_A-bundle where H_A denotes the reductive group scheme $H \times \operatorname{Spec} A$ over A.

Proposition 10.1. There exists a finite extension L/K with the following property: If B is the integral closure of A in L, and if H'_B are the pull-back group schemes, then we have a morphism of B-group schemes

$$\chi_B : H'_B \longrightarrow H_B$$

which extends the isomorphism $\chi_L : H'_L \cong H_L$.

Proof. Observe first that the *lattice* $H'_A(A)$ is a *bounded subgroup* of $H_A(K)$, in the sense of the Bruhat-Tits theory [BT]. Here, we make the identifications:

$$H'_K \cong H_K \text{ as K-group schemes}$$

Hence,

$$H'_A(A) \subset H'_K(K) \cong H_K(K) = H_A(K)$$

Then we use the following crucial fact:

$$\left\{ \begin{array}{l} \text{There exists a finite extension } L/K \text{ and an element } g \in H'_A(L) \text{ such that} \\ g.H'_A(A).g^{-1} \hookrightarrow H_A(B). \end{array} \right\} \quad (*)$$

This assertion is a consequence of the following result from, ([S1] Prop 8, p 546) (cf. also [Gi] Lemma I.1.3.2, or [La] Lemma 2.4).

(Serre) There exists a totally ramified extension L/K having the following property: For every bounded subgroup M of $H(K)$, there exists $g \in H(K)$ such that $g.M.g^{-1}$ has *good reduction* in $H(L)$ (i.e $h.M.h^{-1} \subset H(B)$, where B is the integral closure of A in L).

For the sake of clarity we gather all the identifications of the subgroups under consideration:

$$H'_A(K) = H'_K(K) \text{ and } H'_A(L) = H'_B(L) = H'_L(L)$$

$$H'_A(A) \subset H'_B(B)$$

$$H_A(B) = H_B(B)$$

Thus, we see that the isomorphism $\chi_L : H'_L \longrightarrow H_L$, given by *conjugation by* g, induces a map $\chi_L(B) : H'_A(A) \longrightarrow H_B(B)$. The crucial property to note is the following one:

Given a rational point $\xi_k \in H'_k(k)$, there exists a point $\xi_A \in H'_A(A)$, and hence in $H'_B(B)$, which extends ξ_k, since H'_A is smooth over A and k is algebraically closed.

The proposition will follow by the following Lemma. Let A, B etc be as above.

Lemma 10.2. Let A be a complete discrete valuation ring with quotient field K. Let Z_A and Y_A be A-schemes with Z_A smooth. Let $\chi_L : Z_L \longrightarrow Y_L$ be a L-morphism such that $\chi_L(B) : Z_A(A) \longrightarrow Y_B(B)$. Then, the L-morphism χ_K extends to a B-morphism $\chi_B : U_B \longrightarrow Y_B$, where U_B is an open dense subscheme of Z_B which intersects all the irreducible components of the closed fibre Z_k.

In particular, if Z_A and Y_A are smooth and separated group schemes and if χ_L is a morphism of L-group schemes then there exists an extension $\chi_B : Z_B \longrightarrow Y_B$ as a morphism of B-group schemes.

Proof. Consider the graph of χ_L and denote its schematic closure in $Z_B \times_B Y_B$ by Γ_B. Let $p : \Gamma_B \longrightarrow Z_B$ be the first projection. Then p is an isomorphism on generic fibres. So, it is enough if we prove that p is invertible on an open dense B-subscheme U_B of Z_B, which intersects all the components C, of the closed fibre Z_k.

We claim that, the map $p_k : \Gamma_k \longrightarrow Z_k$ is surjective onto the subset of k-rational points of each components, and this will imply that p_k is surjective since k is algebraically closed. Note that Z_A is assumed to be smooth and so, the closed fibre is reduced and also k is algebraically closed. Thus, each $z_k \in Z_k(k)$ lifts to a point $z \in Z_A(A) \subset Z_B(B)$, A, being a complete discrete valuation ring. Since $\chi_L(B)$ maps $Z_A(A) \longrightarrow Y_B(B)$, we see that, there exists a $y \in Y_B(B)$ such that $(z, y) \in \Gamma_B(B)$. Thus, z_k lies in the image of p_k. This proves the claim.

In particular, by the well-known result of Chevalley on images of morphisms, the generic points, α's, of all the components C of Z_k , lie in the image of p_k. Let $p_k(\xi) = \alpha$. Consider the local rings $\mathcal{O}_{\Gamma_B,\xi}$ and $\mathcal{O}_{Z_B,\alpha}$. Then by the above claim, the local ring $\mathcal{O}_{\Gamma_B,\xi}$ dominates $\mathcal{O}_{Z_B,\alpha}$. Since Z_B is smooth and hence normal, for every α the local rings, $\mathcal{O}_{Z_B,\alpha}$ are all discrete valuation ring's. Further, since Γ_B is the schematic closure of Γ_L, it implies that Γ_B is B-flat and Γ_L is open dense in Γ_B. Moreover, since p is an isomorphism on generic fibres both local rings have the same quotient rings. Finally, since $\mathcal{O}_{Z_B,\alpha}$ is a discrete valuation ring, we have an isomorphism of local rings. Therefore since the schemes are of finite type over B, we have open subsets $V_{i,B}$ and $U_{i,B}$ for each component of Z_k, which we index by i, such that p induces an isomorphism between $V_{i,B}$ and $U_{i,B}$. This gives an extension of χ to open subsets $U_{i,B}$ for every i, with the property that these maps agree on the generic fibre. Since Y_B is separated these extensions glue together to give an extension χ_B on an open subset, which we denote by U_B; this open subset will of course intersect all the components of the closed fibres of Z_k.

The second part of the lemma follows immediately, if Y_A is affine (which is our case). More generally, we appeal to the general theorem of A.Weil on morphisms into group schemes, which says that if a rational map ψ_B is defined in codimension ≤ 1 and if the target space is a group scheme then it extends to a global morphism. (cf. for example [BLR] pp 109). As we have checked above this holds in our case and this implies that as a morphism of schemes, ψ_L extends to give $\psi_B : Z_B \longrightarrow Y_B$.

Further, by assumption χ_L is already a morphism of L-group schemes and hence it is easy to see that the extension χ_B is also a morphism of B-group schemes. This concludes the proof of the lemma.

Remark 10.3. Larsen in ([La], (2.7) p 619), concludes from $(*)$, in the l-adic case the statement of Proposition 10.1. However, we give a complete proof.

Remark 10.4. In this section, by the assumption on the separability index of the affine (G, H)-module we were able to conclude that the flat closure H'_A is indeed smooth. We observe that since we are over char.p, in general the limiting fibre of the flat closure H'_A need not be *reduced*. This, as one knows is true in char.0 by virtue of Cartier's theorem. Indeed more generally, given a flat group scheme H'_A with smooth generic fibre H'_K, there is a construction due to Raynaud of what he calls the Neron-smoothening of H'_A. This exists as a smooth group scheme H''_A with generic fibre $H''_K \simeq H'_K$ with the following universal property: given any smooth A-scheme D_A and an A-morphism $D_A \longrightarrow H'_A$, this map factors uniquely via an A-morphism $D_A \longrightarrow H''_A$. In particular, $H''_A(A) = H'_A(A)$. Thus more generally without any separability index assumptions, the proof of Proposition 10.1 gives a morphism $H''_B \longrightarrow H_B$. One is unable to make use of this since the principal bundle E'_A has structure group H'_A and there is no natural reason for its lifting to a principal H''_A-bundle. (see [BLR]).

11 Semistable reduction theorem

Let H be a semi-simple algebraic group over k an algebraically closed field of char. p. Let $H \subset G = SL(V)$, be the representation we have fixed in

§1. We retain all the notations of §7. The aim of this section is to prove the following theorem.

Theorem 11.1. (*Semistable reduction)* Let W be a finite dimensional *affine* (G, H)*-module* associated to H and G and let $p > \psi_G(W)$. Let H_K denote the group scheme $H \times \operatorname{Spec} K$, and P_K be a semistable H_K-bundle on X_K. Then there exists a finite extension L/K, with B as the integral closure of A in L such that the bundle P_K, after base change to $\operatorname{Spec} B$, extends to a semistable H_B-bundle P_B on X_B.

Proof. First by Proposition 9.1 we have an H'_A-bundle which extends the H_K-bundle upto isomorphism. Then, by Proposition 10.1, by going to the extension L/K we have a morphism of B-group schemes $\chi_B : H'_B \longrightarrow H_B$ which is an isomorphism over L. Therefore, one can extend the structure group of the bundle E'_B to obtain an H_B-bundle E_B which extends the H_K-bundle E_K.

We need only prove that the fibre of E_B over the closed point is indeed *semistable*. This is precisely the content of Proposition 11.3 below. Q.E.D.

Remark 11.2. We remark that this is fairly straightforward in char.0 since it comes as the extension of structure group of E'_l by the map $\chi_k : H'_k \longrightarrow H_k$. We note that in char.0, E'_l is the H'_k-bundle obtained as the reduction of structure group of the polystable vector bundle $E(V_A)_l$ and so remains semistable by any associated construction (cf. Proposition 2.6 of [BS]). In our situation this becomes much more complex and we isolate it in the following proposition.

Proposition 11.3. The limiting bundle, namely the fibre of E_B over the closed point is semistable.

Proof. Recall from Proposition 9.2, that the limiting bundle of the family E'_B namely E'_l, had the property that it had a further reduction of structure group to a reductive and saturated group R_w of H'_k and hence of $G_k = G$. Thus the representation $R_w \longrightarrow G_k$ is also *low height* by ([S2] pp 25). Further, by the low height property, the representation $R_w \longrightarrow G = SL(V)$ is completely reducible.

Observe further that since H'_k is *not reductive* (cf. Remark 6.5 above), there exists a proper parabolic subgroup $P \subset G_k$ such that $H'_k \subset P$. This follows by the theorem of Morozov-Borel-Tits (cf. [BoT]). Therefore the subgroup $R_w \subset H'_k \subset P$. Now since $R_w \longrightarrow G$ is completely reducible, $R_w \subset P$ implies that $R_w \subset L$ for a Levi subgroup $L \subset P$.

Now R_w is a saturated reductive subgroup of G. Therefore, since $p > \psi_G(W)$, by Lemma 11.7 (and Remark 11.8) below we see that the modules $LieG_k$ and $LieH'_k$ are low height modules for R_w and in particular completely reducible.

Now R_w is a saturated group and the connected component of identity, R^0_w, is reductive by Proposition 9.2. Since R_w is saturated as a subgroup of G by height considerations, the modules $LieG_k$ and $LieH'_k$ are low height modules for R^0_w as well. (cf. Remark 5.6)

Thus by Remark 3.4, we have the following:

$$H^i(R^0_w, Lie(H'_k)) = H^i(R^0_w, Lie(G_k)) = 0$$

for all $i \geq 1$.

Recall that by Remark 5.6 the *saturatedness* of R_w implies that the index $[R_w : R^0_w]$ is prime to the characteristic p.

Therefore if we denote R_w/R^0_w by I_w, we see that the order of I_w is prime to p. Hence we have the following vanishing of cohomology:

$$H^i(I_w, Lie(H'_k)) = H^i(I_w, Lie(G_k)) = 0$$

for all $i \geq 1$. (For this classical result cf. [CE] p. 237.)

Putting together the above results, we can conclude the following:

$$H^i(R_w, Lie(H'_k)) = H^i(R_w, Lie(G_k)) = 0$$

for all $i \geq 1$.

This implies, by the infinitesimal lifting property of ([SGA 3] Exp.III Cor 2.8) that if we consider the product group scheme $R_{w,B} = R_w \times Spec(B)$, then the inclusion

$$i_k : R_w \hookrightarrow H'_k \hookrightarrow G_k$$

lifts to an inclusion

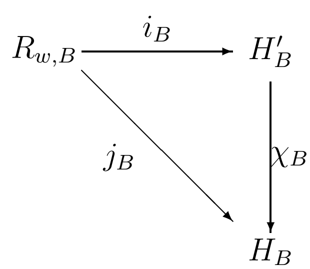
$$i_B : R_{w,B} \hookrightarrow H'_B \hookrightarrow G_B$$

where the generic inclusion is defined upto conjugation by the inclusion over the residue field.

Denote the above composite by:

$$i_{1,B} : R_{w,B} \hookrightarrow G_B$$

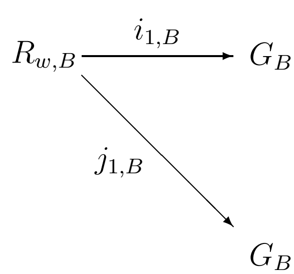
By Proposition 10.1, we also have a morphism $\chi_B : H'_B \longrightarrow H_B$, which is an isomorphism over the function field L. We have the following diagram:



We note that we also have an inclusion $H_B \hookrightarrow G_B$ coming from the original representation $H \hookrightarrow G$. In other words we have another morphism

$$j_{1,B} : R_{w,B} \longrightarrow G_B$$

Thus, we get the following diagram:



(We remark that there is no vertical arrow to complete the above diagram!)

Note that over the function field L the maps $j_{1,L}$ and $i_{1,L}$ coincide upto conjugation. Thus by the cohomology vanishing stated above and the rigidity of maps ([SGA 3] Exp III Cor 2.8), the maps over the residue fields are also conjugates.

Consider the bundle E'_l which comes equipped with a reduction to R_w and is semistable as an R_w-bundle (cf. Prop 9.2).

Since the representations $i_{1,k} : R_w \hookrightarrow G_k$ and $j_{1,k} : R_w \hookrightarrow G_k$ are conjugate it follows that the associated G_k-bundles $E'_l(j_{1,k})$ and $E'_l(i_{1,k})$ are isomorphic. Therefore since $E'_l(i_{1,k})$ is semistable so is $E'_l(j_{1,k})$. In particular, since the morphism $j_{1,k} : R_w \hookrightarrow G_k$ factors via H_k, the associated H_k-bundle $E'_l(j_k)$ is semistable. This implies that the induced bundle E_B is a family of semistable H_B-bundles. This completes the proof of the Theorem 11.1. Q.E.D.

Remark 11.4. Let $H \subset G$, where G is a linear group. In the notation of §2 let F_H and F_G stand for the functors associated to families of semistable bundles of degree zero. (cf. Proposition 3.13). The inclusion of H in G induces a morphism of functors $F_H \longrightarrow F_G$. We remark that the semistable reduction theorem for principal H-bundles **need not** imply that the induced morphism $F_H \longrightarrow F_G$ is a proper morphism of functors. Indeed, this does not seem to be the case. However, it does imply that the associated morphism at the level of moduli spaces is indeed proper (cf. Theorem 4.6).

11.1 *Some remarks on low height modules*

Lemma 11.5. Let $H \subset G = SL(V)$ be a low height representation. Let W be a low height G-module such that G/H is embedded as a closed orbit in W (cf. Def 3.12). Suppose that the subspace $V^H \subset V$ of H-fixed vector in V is the zero subspace. Then W contains direct summand different from V and V^*. (Note that by the low height assumptions all modules are completely reducible.)

Proof. For if $W = \oplus V$, then the vector $w \in W$ which has a closed G-orbit and whose isotropy is H projects onto a vector $v \in V$ fixed by H. But by assumption, the subspace $V^H = 0$. Hence W cannot be a direct sum of copies of V. We also observe that this implies $(V^*)^H = 0$ as well and therefore W is not the direct sum of V^*'s alone.

Lemma 11.6. Let $R \subset G = SL(V)$ be a reductive saturated subgroup of G that is contained in the Levi of a parabolic subgroup of G. Let W be a low height G-module that contains a component not isomorphic to V and V^*. Then $Lie(G)$ and $Lie(H)$ are low height R-modules, in particular completely reducible.

Proof. Let $n = dim(V)$. Since W contains a component other than V and V^*, $ht_G(W) \geq 2(n-2)$.

Hence W being a low height G-module we have $p > 2(n-2)$. Since R is not irreducible in that R is contained in a certain Levi subgroup $L \subset P$ of a parabolic subgroup $P \subset G$, it follows that $ht_R(V) < ht_G(V) = n-1$.

Hence $ht_R(V \otimes V^*) \leq 2(n-2) < p$. In other words, $V \otimes V^* = Lie(G)$ is a low height R-module. Note also that $ht_R(Lie(H)) \leq ht_R(Lie(G))$. Q.E.D.

Lemma 11.7. Let $H \subset G = SL(V)$ be a low height representation. Let W be a low height G-module such that G/H is embedded as a closed orbit in W (cf. Def 3.12). Let $R \subset G = SL(V)$ be a reductive saturated subgroup of G that is contained in the Levi of a parabolic subgroup of G. Assume that $V^H \subset V^R$. Then $Lie(H)$ and $Lie(G)$ are low height R-modules.

Proof. Let V' be the subspace complementary to V^H in V. Let $n = dim(V)$ and $n' = dim(V')$. Let $G' = SL(V') \subset G$. Then the representation $H \hookrightarrow SL(V) = G$ factors through $H \hookrightarrow SL(V') = G'$. Moreover G' is saturated in G (being the semisimple part of a Levi subgroup of a parabolic subgroup) and therefore V and W are low height G'-modules (by Remark 5.6).

By the choice of V', we have $(V')^H = 0$. Since $V^H \subset V^R$ we see that $R \subset G'$. Therefore the G'-orbit gives a closed embedding of G'/H in W. It follows by Lemma 11.5 that W contains summands other than V' and V'^*.

Hence by Lemma 11.6 $Lie(G')$ and $Lie(H)$ are low height R-modules. Now the result follows because $ht_H(V) = ht_H(V')$ and hence $ht_R(V) = ht_R(V')$. This works for the duals as well, i.e $ht_R(V^*) = ht_R(V'^*)$. By additivity of heights we see that

$$ht_R(V \otimes V^*) = ht_R(V' \otimes V'^*) < p$$

since $Lie(G')$ is a low height R-module. Q.E.D.

(cf. [S2] p. 27 for some of the computations made here)

Remark 11.8. We note that the subgroup R_w to which we apply Lemma 11.7 satisfies the condition of the Lemma, especially the condition that $V^H \subset V^{R_w}$. This follows since H'_A is the *flat closure* of H'_K in G_A and since $R_w \subset H'_k$. In fact, for the purposes of Prop 11.3 or the semistable reduction theorem one could have worked with $G' = SL(V')$ instead of G. In that case it is clear that the flat closure of H'_K is actually realised in G'_A itself.

11.2 *Irreducibility of the moduli space*

We first remark that the semistable reduction theorem Theorem 11.1 holds in fact in a slightly more general setting as well.

Corollary 11.9. Let $\mathcal{X} \longrightarrow \mathcal{S}$ be a smooth family of curves parametrised by $S = SpecA$ where A is a complete discrete valuation ring with char.$K = 0$ and residue characteristic p. Suppose further that $p > \psi(W)$ as in Theorem 11.1. Let H_S be a reductive group scheme obtained from a split Chevalley group scheme $H_{\mathbf{Z}}$. Suppose further that we are given a family of semistable principal H_K-bundles E_K over $\mathcal{X}_K$. Then, there exists a finite cover $S' \longrightarrow S$ such that the family after pull-back to S' extends to a semistable family $E_{S'}$.

Proof. The proof of Theorem 11.1 goes through with some minor modifications.

We then have

Corollary 11.10. Let H be simply connected. Then for $p > \psi_G(W)$ the moduli spaces $M(H)$ of principal H-bundles is irreducible.

Proof. The proof of this is now standard once Cor 11.9 is given and one knows the fact over fields of char 0. The argument very briefly runs as follows: The first point is to observe that given the prime bounds, namely $p > \psi_G(W)$, the moduli scheme can be constructed as in §4 over $S = \mathbf{Z} - \{p \leq \psi_G(W)\}$. Call this scheme $M(H)_S$. Then Cor 11.9 implies that $M(H)_S$ is projective and further, GIT (cf. [Ses1]) implies that the canonical map $M(H) \longrightarrow M(H)_S \otimes k$ is a bijection on k-valued points. Further, since $M(H)_S$ is projective and connected over the generic fibre (by char 0 theory), Zariski's connectedness theorem implies that the closed fibre $M(H)_S \otimes k$ is also connected and hence so is $M(H)$. Now observe that the quot scheme Q'' constructed in §4 is easily seen to be *smooth* by some standard deformation theory. Hence $M(H)$ is normal and connected and therefore *irreducible*.

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