Wavelength-doubling bifurcations in one-dimensional coupled logistic maps

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We discuss in detail the interesting phenomenon of wavelength-doubling bifurcations in the model of coupled-map lattices reported earlier [Phys. Rev. Lett. **70**, 3408 (1993)]. We take nearest-neighbor coupling of logistic maps on a one-dimensional lattice. With the value of the parameter of the logistic map, μ, corresponding to the period-doubling attractor, we see that the wavelength and the temporal period of the observed pattern undergo successive wavelength- and period-doubling bifurcations with decreasing coupling strength ϵ. The universality constants α and δ appear to be the same as in the case of the period-doubling route to chaos in the uncoupled logistic map. The phase diagram in the ϵ-μ plane is investigated. For large values of μ and large periods, regions of instability are observed near the bifurcation lines. We also investigate the mechanism for the wavelength-doubling bifurcations to occur. We find that such bifurcations occur when the eigenvalue of the stability matrix corresponding to the eigenvector with periodicity of twice the wavelength exceeds unity in magnitude.

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I. INTRODUCTION

Modeling of spatially extended dynamical systems by various means has attracted much attention in the recent past [1–5]. The extensive studies that have been carried out in nonlinear dynamical systems have led to a reasonable understanding of the low-dimensional chaotic systems. However, our understanding of the spatially extended dynamical systems with large dimensions is still not satisfactory. The modeling and characterization of spatiotemporal systems and spatiotemporal chaos are very important for the study of many systems, such as turbulence in fluids [6], reaction-diffusion systems (e.g., [7]), etc. One of the heuristic ways in which the understanding of low-dimensional nonlinear dynamical systems can be utilized in understanding spatially extended systems is to couple such systems on a lattice and study the coupled system, e.g., the oscillator chains [3], coupled-map lattices [4], and cellular automata [2]. Several applications of such coupled systems have been studied. These systems are simpler to study and easily tractable. In particular, coupled-map lattices after their introduction by Kaneko [8] have been studied widely [9–24]. The coupled-map lattice models show many interesting phenomena, such as kink dynamics, solitons, frozen random patterns, periodic patterns, traveling-wave solutions, intermittency, chaos, etc. [5]. The phenomenon of spatiotemporal intermittency in Rayleigh-Bénard convection has been modeled by coupled-map lattices [25]. Also, they have been used in modeling spiral waves in the *B-Z* reaction [7] and the phenomenon of crystal growth [24]. Another interesting application is the simulation of the kinetics of important equations in phase-ordering processes such as the Cahn-Hilliard-Cook (CHC) and time-dependent Ginzburg-Landau (TDGL) equations [20–23]. Several studies in the model of coupled-map lattices with different kinds of local dynamics, couplings, lattice dimensionality, etc., have been carried out in the recent past [8–13].

There have been many studies of temporal period doublings in coupled-map lattices [8]. A renormalization-group approach for these period doublings has been developed by Kuznetsov [15] and recently by Kook, Ling, and Schmidt [16]. In these papers they have shown that any small coupling is essentially characterized by two parameters; one is the inertial coupling associated with the scaling factor $\alpha=-2.50\ldots$ and the other is the one corresponding to the dissipative coupling with the scaling factor 2.0.

In this paper we discuss in detail an interesting phenomenon of spatiotemporal period doubling reported recently [19] in the model of coupled-map lattices. The phenomenon is a spatial analog of the well-known route to chaos via temporal period doubling. In this phenomenon we observe successive bifurcations in which the wavelength (spatial period) and the temporal period keep doubling as the coupling parameter is changed. Our model system is a one-dimensional coupled-map lattice with logistic maps coupled symmetrically. This system has several spatially and temporally periodic stable solutions [17,18]. Starting with a stable solution with a spatial period of 2 we find that the temporal and spatial periods of the observed patterns undergo successive period-doubling bifurcations as the coupling strength is decreased. The patterns that are observed are of the traveling-wave type. Using the standard procedure, the universality constants α and δ are obtained and they appear to be the same as in the case of the period-doubling route to chaos in an uncoupled logistic map [26]. We also analyze the stability matrix and determine the condition for spatial period-doubling bifurcations to occur.

The phase diagram in the ϵ-μ plane is interesting. Here ϵ is the coupling parameter and μ is the parameter of the logistic map. For $\mu\le\mu_\infty$, where μ_∞ is the parameter value for the period-doubling attractor, we get the wavelength-doubling bifurcations as described above.

For $\mu > \mu_\infty$ the situation is more complicated. For large periods and large μ there are regions of instability near the bifurcation lines. We study the conditions under which such regions of instability occur.

II. WAVELENGTH-DOUBLING BIFURCATIONS

Let us consider the following model of a one-dimensional coupled-map lattice with symmetric nearest-neighbor couplings,

$$x_{t+1}(i)=(1-\epsilon)f(x_t(i))+\frac{\epsilon}{2}f(x_t(i+1))+\frac{\epsilon}{2}f(x_t(i-1)) , \quad (1)$$

where $x_t(i)$ is the variable associated with the ith lattice point at time t taking values in a suitably bounded phase space, and $i=1,\ldots,m$. The map f is the logistic map,

$$f(x)=\mu x(1-x); \quad x\in[0,1] \text{ and } \mu\in[0,4] .$$

The parameter ϵ represents the coupling strength and $0\leq\epsilon\leq 1$.

One can see that for $\epsilon=0$ the dynamics of the lattice is one of the uncoupled logistic maps. The logistic map undergoes a period-doubling sequence in time leading to the period-doubling attractor at $\mu=\mu_\infty=3.569\ldots$.

Let $S_\tau(N)$ denote a solution of Eq. (1) with time period τ and space period N. Consider the solution $S_2(2)=\{x_1(1),x_1(2)\}$ with $x_1(1)\neq x_1(2)$. It is possible to show that there is a range of parameter values where $S_2(2)$ is a stable solution and is given by

$$x_1(1)=\frac{(\mu+1-2\mu\epsilon)+\sqrt{(\mu+1-2\mu\epsilon)(\mu-3-2\mu\epsilon+4\epsilon)}}{2\mu(1-2\epsilon)} ,$$
$$x_1(2)=\frac{(\mu+1-2\mu\epsilon)}{\mu(1-2\epsilon)}-x_1(1) . \quad (2)$$

This solution has time period 2 and $x_1(1)=x_1(2)$, $x_2(2)=x_1(1)$. It may be also treated as a traveling-wave solution with velocity 1. The stability of this solution can be determined by the eigenvalues of the stability or the Jacobian matrix. The stability criterion is discussed in Sec. III.

Let us first consider the case when $\mu=\mu_\infty$, which is the accumulation point of the period-doubling cascade in an uncoupled logistic map and the coupling parameter ϵ is allowed to vary. The phase diagram in the ϵ-μ plane is discussed in Sec. IV. The period-2 solution $S_2(2)$ is stable in the range $(\epsilon_0=0.134\,18\ldots)$ to $(\epsilon_1=0.038\,890\ldots)$ [see Eqs. (9) and (10)]. These values are listed in Table I. For $\epsilon<\epsilon_1$ the solution $S_2(2)$ becomes unstable and undergoes a period-doubling bifurcation. A new solution $S_4(4)$ with space period 4 and time period 4 becomes stable. The solution $S_4(4)$ is stable in the range ϵ_1 to $\epsilon_2=0.009\,764\,9\ldots$. At ϵ_2 we have one more period-doubling bifurcation leading to the period-8 solution $S_8(8)$ for $\epsilon<\epsilon_2$. Further numerical investigations show that the period-doubling cascade continues and probably leads to the accumulation point at $\epsilon_\infty=0.0$.

TABLE I. The values of ϵ_n at successive bifurcation points at $\mu=\mu_\infty$ are given for different n. The table also lists values of d_n, α_n, and δ_n.

n	ϵ_n	d_n	α_n	δ_n
1	0.038 8908	0.349 32	−3.0396	3.855
2	0.009 765	−0.114 92	−2.5682	4.434
3	0.002 182	0.044 75	−2.4355	4.621
4	0.000 472	−0.018 37	−2.5216	4.642
5	0.000 102	0.007 28	−2.4884	
6	0.000 0223	−0.002 92		

At each bifurcation point both the space and the time periods double. Since we have a spatial period-doubling cascade starting with space period 2, it was necessary to choose the lattice size in powers of 2 in numerical simulations. The maximum lattice size used was 2048 with cyclic boundary condition. The stability of the solutions was checked by giving small perturbations and also by checking the eigenvalues of the matrices $M(\theta)$ [Eq. (5)] discussed in the next section.

In Table I the ϵ_n values at the successive bifurcation points are listed. Let δ_n be given by

$$\delta_n=\frac{\epsilon_n-\epsilon_{n+1}}{\epsilon_{n+1}-\epsilon_{n+2}} . \quad (3)$$

The values of δ_n are listed in Table I. Though these values are still inadequate to conclude about the asymptote $\delta=\delta_\infty$, they are clearly consistent with the value $\delta=4.6692\ldots$ obtained from the period-doubling sequence of an uncoupled logistic map as a function of μ [26].

In Fig. 1 we plot the values of $x_t(i)$ for different values of i, as a function of ϵ. The bifurcation diagram has a striking similarity to the one in the case of an uncoupled logistic map as a function of μ. To determine the scaling parameter α_n, we determine the value of ϵ for each

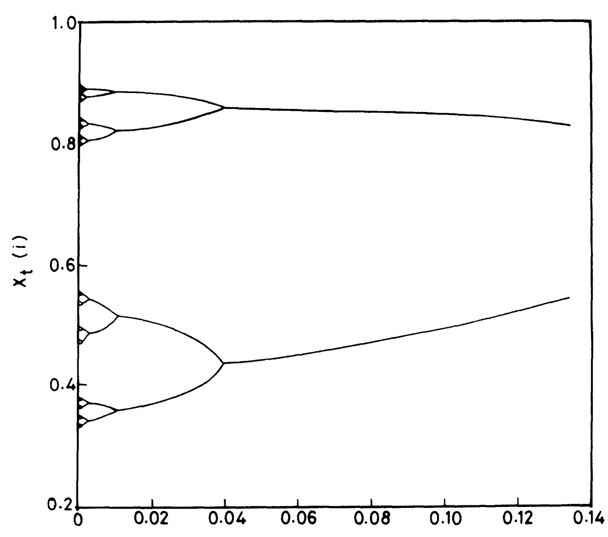

FIG. 1. The variables $x_t(i)$ at a given time at different sites are plotted as a function of ϵ at $\mu=\mu_\infty$.

period for which one value of $x_t(i)$ is 0.5. This defines the analog of the superstable orbit for an uncoupled logistic map. Let d_n be the separation of the point $x_t(i)=0.5$ from the nearest x value (see Fig. 1) for the period 2^n. Define the scaling parameter α_n by

$$\alpha_n = \frac{d_n}{d_{n+1}} . \quad (4)$$

The values of d_n and α_n are listed in Table I. We again note that the values of α_n are consistent with the asymptotic value $\alpha=\alpha_\infty=2.5029\ldots$ for the uncoupled logistic map as a function of μ [26].

The period-doubling solutions that we observe can also be treated as traveling-wave solutions. The velocities of the solutions with periods 2, 4, and 8 that we have obtained are 1, 3, and 5, respectively. For higher-order periods the observed velocity is 11. We note that the sequence of traveling-wave speeds (1,3,5,11) corresponds to the succession $v_{n+1}=v_n+2v_{n-1}$. This is characteristic of the main frequency in the frequency-doubling cascade in nonorientable manifolds [27].

III. LINEAR STABILITY ANALYSIS

We now consider the stability of the periodic solution $S_\tau(N)$. This problem can be simplified by using the results of Refs. [17,18]. We first consider a one-dimensional lattice chain $\mathcal{C}_M$ of length M with cyclic boundary conditions, i.e., the first and the Mth lattice points are neighbors of each other.

Let $R_t=(x_t(1),\ldots,x_t(N))$ denote the state of the system of the chain $\mathcal{C}_N$ at time t. Let $S_\tau(N,1)$ denote a solution of Eq. (1) with temporal periodicity τ for the chain $\mathcal{C}_N$, i.e.,

$$S_\tau(N,1)=\{R_1,R_2,\ldots,R_\tau,R_1,R_2,\ldots\} .$$

Now consider a closed chain $\mathcal{C}_{kN}$ of length kN, $k=1,2,\ldots$. Obviously the spatially periodic sequence

$$S_\tau(N,k)=\{\langle R_1,\ldots,R_1\rangle_k,\ldots,\langle R_\tau,\ldots,R_\tau\rangle_k,\langle R_1,\ldots,R_1\rangle_k,\ldots\}$$

of wavelength N built from the states $\{R_t\}$ as the building blocks is a solution of Eq. (1) for the closed chain $\mathcal{C}_{kN}$ with temporal periodicity τ. Here the ordered pair $\langle R_t,\ldots,R_t\rangle_k$ represents a state made up of k replicas of the state R_t. We call $S_\tau(N,k)$ the k-replica solution of $S_\tau(N,1)$. The stability criterion for the k-replica solution was discussed in Ref. [17]. It was shown that the problem of eigenvalues of a $kN\times kN$ stability matrix of the k replica solution can be simplified to the analysis of k matrices of size $N\times N$ which are constructed using the stability matrix for the solution $S_\tau(N,1)$, the building block of spatial periodicity. The problem can be further simplified for a traveling-wave solution [18]. If v is the velocity of the traveling wave then the problem of stability analysis of the k-replica solution reduces to the analysis of the eigenvalues of the $N\times N$ matrices $M(\theta)$ given by [18]

$$M(\theta)=(\Pi_\theta)^v J_\theta , \quad (5)$$

where $\theta=0,2\pi/k,\ldots,(k-1)2\pi/k$. Here Π_θ and J_θ are $N\times N$ matrices given by

$$\Pi_\theta=\begin{bmatrix} 0 & 0 & \cdots & 0 & e^{i\theta} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (6)$$

and

$$J_\theta=\begin{bmatrix} (1-\epsilon)f'(x_1(1)) & \frac{\epsilon}{2}f'(x_1(2)) & \ldots & \frac{\epsilon}{2}f'(x_1(N))e^{i\theta} \\ \frac{\epsilon}{2}f'(x_1(1)) & (1-\epsilon)f'(x_1(2)) & \ldots & 0 \\ \vdots & & & \vdots \\ \frac{\epsilon}{2}f'(x_1(2))e^{-i\theta} & 0 & \ldots & (1-\epsilon)f'(x_1(N)) \end{bmatrix} . \quad (7)$$

Now the k-replica solution is stable if all the eigenvalues of the matrices $M(\theta)$ have magnitude less than 1. As $k\to\infty$ or as the size of the lattice increases, θ takes continuous values between 0 and 2π. It is easy to show that it is sufficient to check the eigenvalues of $M(\theta)$ in the range $0\le\theta\le\pi$ to determine the stability of the solution as $k\to\infty$, i.e., an infinite lattice. Let

$$S_\tau(N)=\lim_{k\to\infty} S_\tau(N,k) .$$

Let us apply the above stability analysis to the period-2 solution $S_2(2,k)$ [Eq. (2)], which has velocity 1. For $N=2$ and $v=1$, matrices $M(\theta)$ [Eq. (5)] are given by

$$M(\theta)=\begin{pmatrix}0 & e^{i\theta}\\ 1 & 0\end{pmatrix}\begin{pmatrix}(1-\epsilon)f'(x_1(1)) & \frac{\epsilon}{2}(1+e^{i\theta})f'(x_1(2))\\ \frac{\epsilon}{2}(1+e^{-i\theta})f'(x_1(1)) & (1-\epsilon)f'(x_1(2))\end{pmatrix} . \quad (8)$$

We first consider the stability of the solution for $k=1$, i.e., the solution $S_2(2,1)$. The stability criterion is that the eigenvalues of the matrix $M(0)$ have magnitude less than 1. From the eigenvalues we find that the solution $S_2(2,1)$ is stable in the range $\epsilon_0<\epsilon<\epsilon'$, where

$$\epsilon_0=\frac{1}{2}\left[1-\left[\frac{3}{\mu(\mu-2)}\right]^{1/2}\right] ,$$
$$\epsilon'=\frac{2\mu^2-4\mu-3-\sqrt{8\mu^2-16\mu+9}}{4\mu(\mu-2)} . \quad (9)$$

At the lower limit ϵ', one of the eigenvalues of $M(0)$ becomes -1, while at the upper limit ϵ_0, both the eigenvalues are complex and have unit magnitude. For $k=2$ we must consider both $\theta=0$ and π, i.e., the matrices $M(0)$ and $M(\pi)$. The analysis of the eigenvalues of $M(\pi)$ shows that the stability range of ϵ values shrinks with the upper limit ϵ_0 remaining unchanged and the lower limit shifting to ϵ_1, which is one of the solutions of the equation

$$4(1-\epsilon)^3-(1-2\epsilon)^2-\mu(\mu-2)(1-\epsilon)^2(1-2\epsilon)^2=0 . \quad (10)$$

At ϵ_1 the eigenvalues of $M(\pi)$ are ± 1. For $k>2$ the eigenvalues of $M(\theta)$ with θ in the range $0-\pi$ must be considered. By obtaining eigenvalues for θ values for $k=3,4,\ldots$, we find that there is no further reduction in the stability range (ϵ_1,ϵ_0) of ϵ values for the solution $S_2(2,k)$ as $k\to\infty$ [17]. We have also numerically checked the stability of the solution in this range by random perturbations of the solution.

Let us consider the point ϵ_1, where we have a period-doubling bifurcation and for $\epsilon<\epsilon_1$ we have a stable solution $S_4(4)$ of period 4. At $\epsilon=\epsilon_1$ the eigenvalues of the matrix $M(\pi)$ are ± 1. We will now show that this indicates a tendency towards wavelength doubling. In fact, in general a solution of spatial periodicity N develops a tendency to wavelength doubling when the eigenvalues corresponding to matrix $M(\pi)$ become greater than 1 in magnitude.

To understand how an eigenvalue of $M(\pi)$ being greater than 1 in magnitude shows a tendency towards spatial period doubling, consider the full $Nk\times Nk$ matrix W whose eigenvalues determine the stability of the solution $S_N(Nk,Nk)$ [18],

$$W=\Pi_0^v\begin{pmatrix}(1-\epsilon)f'(x_1(1)) & \frac{\epsilon}{2}f'(x_1(2)) & 0 & \cdots & \frac{\epsilon}{2}f'(x_1(N))\\ \frac{\epsilon}{2}f'(x_1(1)) & (1-\epsilon)f'(x_1(2)) & \frac{\epsilon}{2}f'(x_1(1)) & \cdots & 0\\ 0 & \frac{\epsilon}{2}f'(x_1(2)) & (1-\epsilon)f'(x_1(1)) & \cdots & 0\\ \vdots & & & & \vdots\\ \frac{\epsilon}{2}f'(x_1(1)) & 0 & 0 & \cdots & (1-\epsilon)f'(x_1(N))\end{pmatrix} . \quad (11)$$

Let us consider the eigenvalue equation for the $N\times N$ matrix $M(\theta)$ [Eq. (8)],

$$M(\theta)v_i(\theta)=\Lambda_i(\theta)v_i(\theta) , \quad (12)$$

where $v_i(\theta)$, $i=1,2,\ldots,N$, are the eigenvectors with eigenvalues $\Lambda_i(\theta)$. It is easy to verify that the full matrix W [Eq. (11)] has the same eigenvalues $\Lambda_i(\theta)$ with the eigenvectors $V_i(\theta)$ given by

$$V_i(\theta)=(v_i(\theta),v_i(\theta)e^{i\theta},v_i(\theta)e^{i2\theta},\ldots,v_i(\theta)e^{i(k-1)\theta})^T ,$$

where T represents the transpose. Now let us consider a small deviation $\Delta_1=\{\delta_1(1),\delta_1(2),\ldots,\delta_1(Nk)\}$ from the initial periodic solution $\langle R_1,\ldots,R_1\rangle_k$, $R_1=(x_1(1),\ldots,x_1(N))$. We expand this deviation in terms of the eigenvectors of W

$$\Delta_1=\sum_{\theta=0}^{(k-1)2\pi/k}\sum_{i=1}^{N}c_i(\theta)V_i(\theta) ,$$

where $c_i(\theta)$ are the expansion coefficients. After time t the deviation from the periodic solution is given by

$$\Delta_{t+1}=\sum_{\theta=0}^{(k-1)2\pi/k}\sum_{i=1}^{N}(\Lambda_i(\theta))^t c_i(\theta)V_i(\theta) .$$

It is clear that the contributions to the deviation by the eigenvectors with eigenvalues having magnitude greater than 1 will increase while the remaining contributions will decrease. Thus if the only eigenvalues greater than 1 are those corresponding to $\theta=\pi$, then the contribution to the deviation from the corresponding eigenvectors $V_i(\pi)$ will survive. The wavelength of the new solution will thus correspond to that of $V_i(\pi)$ which is twice the original one. We note that we have carried out the above analysis using linear stability analysis. Thus the above result only gives a tendency towards wavelength doubling. The exact final solution cannot be obtained using this analysis.

In general, if an eigenvalue of the matrix $M(\theta)$ exceeds 1 in magnitude then it shows a tendency towards a new solution of wavelength $2\pi N/\theta$, if $\theta \neq 0$, and of the same wavelength N if $\theta = 0$. All the wavelength-doubling bifurcations that we observed are consistent with the above analysis. At each bifurcation point one or more of the eigenvalues of the matrix $M(\pi)$ have magnitude 1 and the eigenvalues of the matrices $M(\theta)$ with other values of θ are still less than 1.

Let Λ denote the eigenvalue with the largest magnitude of the matrices $M(\theta)$. We define the Lyapunov exponent λ as

$$\lambda = \ln|\Lambda| \ . \tag{13}$$

In Fig. 2 we plot the value of the Lyapunov exponent as a function of ϵ. We observe a graph similar to the one in the case of the period-doubling transition to chaos in an uncoupled logistic map as a function of μ with the difference that here λ remains finite since the largest magnitude eigenvalue is never zero. Starting from zero at a bifurcation point the Lyapunov exponent decreases as ϵ decreases, reaches a minimum, and then again rises to zero at the next bifurcation point.

IV. BEHAVIOR IN THE ϵ-μ PLANE

We now consider the bifurcation diagram in the ϵ-μ plane (Fig. 3). First let us consider the case $3.0 \leq \mu < \mu_\infty$. For $\mu < 3.0$ the inhomogeneous periodic solutions $S_N(N)$ that we are discussing do not exist. [See Eq. (9) for ϵ_0, which expresses the upper limit of stability for the solution $S_2(2)$.] The behavior in this region of the ϵ-μ plane is similar to that observed at $\mu = \mu_\infty$ except that the bifurcations do not continue *ad infinitum* as ϵ is decreased. For a given value of μ, we start from the solution $S_2(2)$, which is stable in the range $\epsilon_0 > \epsilon > \epsilon_1$ [Eqs. (9) and (10)]. As ϵ is decreased, we observe successive period-doubling bifurcations until we obtain the correct periodicity of the uncoupled logistic map in the limit $\epsilon \rightarrow 0$.

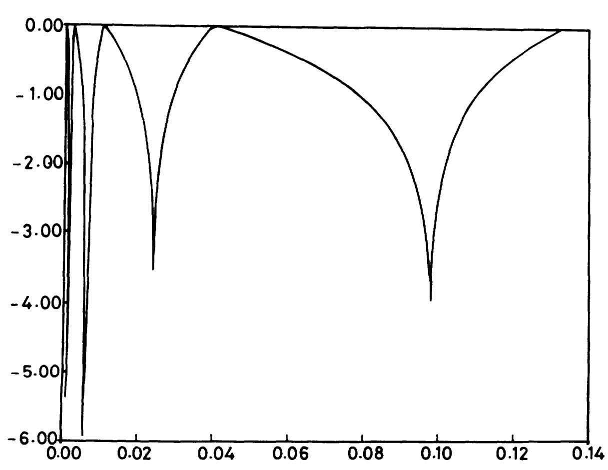

FIG. 2. The Lyapunov exponent λ is plotted as a function of ϵ at $\mu = \mu_\infty$.

We now consider the case $\mu_\infty < \mu \leq 4.0$. Here the bifurcation diagram shows interesting behavior. Bifurcation lines for the transitions from $S_2(2)$ to $S_4(4)$ and from $S_4(4)$ to $S_8(8)$ continue up to $\mu = 4.0$ (Fig. 3). However, the bifurcation line for the transition from $S_8(8)$ to $S_{16}(16)$ does not continue to $\mu = 4.0$. The line continues up to $\mu = 3.770\ldots$ and then a small region of instability is encountered for larger values of μ. For any $\mu > 3.770\ldots$ there is a small range of ϵ values for which neither of the solutions $S_8(8)$ and $S_{16}(16)$ is stable and the behavior appears to be chaotic. This region is not easy to resolve graphically and a schematic diagram of this region in the ϵ-μ plane is shown in Fig. 4. The range of stability of the solution $S_{16}(16)$ extends up to $\mu = 4.0$. The next bifurcation line for the transition $S_{16}(16)$ to $S_{32}(32)$ continues up to $\mu = 3.90\ldots$ to within our numerical accuracy. For larger values of μ the solution $S_{32}(32)$ is not stable for any ϵ. The bifurcation line for the transition $S_{32}(32)$ to $S_{64}(64)$ continues up to $\mu = 3.57\ldots$. For larger values of μ there is a region of instability similar to the one shown in Fig. 4. The upper boundary of the region of instability meets the bifurcation line for the transition $S_{16}(16)$ to $S_{32}(32)$ at $\mu = 3.90\ldots$. The lower boundary meets a similar line coming from below at $\mu = 3.61\ldots$. Thus the solution $S_{64}(64)$ is not stable for $\mu > 3.61\ldots$.

The linear stability analysis near the unstable regions described above can be done by calculating the eigenvalues of the matrices $M(\theta)$ [Eq. (5)]. As discussed in the preceding section, at the wavelength-doubling bifurcation an eigenvalue of the matrix $M(\pi)$ becomes 1 in magnitude and the eigenvalues of the matrices with other values of θ are smaller in magnitude. On the other hand, it is found that as we approach the region of instability (Fig. 4) with decreasing ϵ an eigenvalue of the matrix $M(0)$ becomes unity in magnitude while the eigenvalues

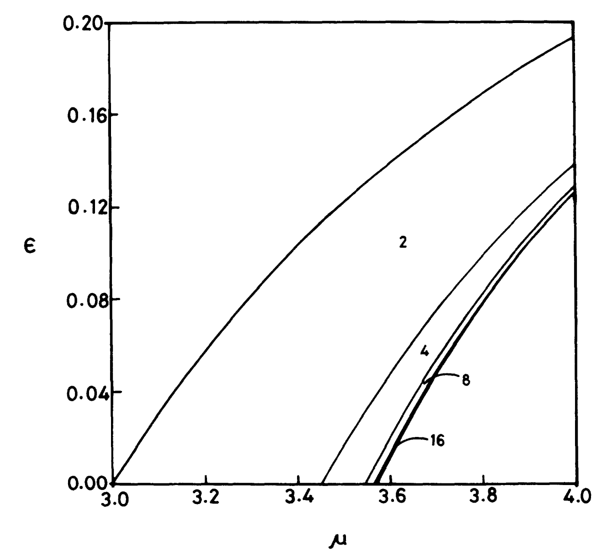

FIG. 3. The phase diagram showing the wavelength-doubling bifurcation lines in the ϵ-μ plane. The numbers represent the wavelength and the temporal periods of the stable solutions. Higher periods cannot be resolved on this scale.

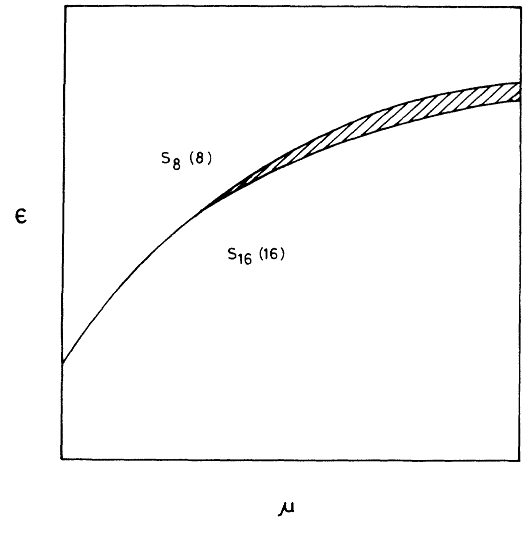

FIG. 4. A schematic diagram of the bifurcation line for the transition from $S_8(8)$ to $S_{16}(16)$. The shaded region is the one in which both the solutions are unstable and the behavior appears to be chaotic.

of all other matrices including the one for $\theta=\pi$ have magnitude less than 1.

We have also investigated the region in the ϵ-μ plane near the period-3 window in an uncoupled logistic map. Wavelength-doubling bifurcations similar to those reported above are seen in this case also.

V. DISCUSSION AND CONCLUSIONS

We have found a phenomenon of wavelength-doubling bifurcation in coupled-map lattices. This is the spatial analog of the normal temporal period-doubling route to chaos. We find that the wavelength and the temporal period keep doubling as the coupling parameter is decreased. We have also obtained the condition for the wavelength-doubling bifurcations to occur by using the linear stability analysis. The universality constants appear to be the same as the ones in the case of the well-known period-doubling route to chaos. This wavelength-doubling route to spatiotemporal chaos can be very important in our understanding of different phenomena seen in spatiotemporal systems. Experiments on large-aspect-ratio cells in Rayleigh-Bénard convection with annular geometry may be one of the systems where the phenomenon described above may be observed [28].

The behavior in the ϵ-μ plane is interesting. For $\mu<\mu_\infty$ finite wavelength doublings are observed and the solutions smoothly go over to those corresponding to uncoupled logistic maps as $\epsilon\to0$. For large periods and $\mu>\mu_\infty$ we observe regions of instabilities where our periodic solutions are not stable.

For $\mu=\mu_\infty$ there appear to be infinite wavelength-doubling bifurcations as $\epsilon\to0$. It is interesting to ask the following question. Is the point ($\epsilon=0$, $\mu=\mu_\infty$) unique in the ϵ-μ plane? In other words, we would like to know whether it is possible to have values of $\mu>\mu_\infty$ for which infinite wavelength-doubling bifurcations can be obtained? Clearly, this question is difficult to answer numerically and would require an analytic treatment.

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