Observation of Stochastic Coherence in Coupled Map Lattices

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Chaotic evolution of *structures* in Coupled map lattice driven by identical noise on each site is studied (a structure is a group of neighbouring lattice–sites for whom values of dynamical variable follow certain predefined pattern). Number of structures is seen to follow a *power–law decay* with length of the structure for a given noise–strength. An interesting phenomenon, which we call *stochastic coherence*, is reported in which a rise of *bell–shaped* type in abundance and lifetimes of these structures is observed for a range of noise–strength values.

Coupled map lattice (CML) model has been observed to exhibit diverse spatio–temporal patterns and structures [1], and serves as a simple model for experimental systems such as Rayleigh–Bénard convection, Taylor–Couette flow, B–Z reaction etc. [2]. A major interest is in understanding formation of structures, localised in both space and time, in turbulent fluid [3]. Role of fluctuations in onset, selection and evolution of such patterns and structures has been studied in some detail [4,5]. In this communication we report a novel phenomenon observed in the dynamics of structures in a chaotically evolving one dimensional CML driven by identical noise. By a *structure* we mean a group of neighbouring sites whose variable–values follow certain prespecified spatial pattern. Distribution of these structures during evolution of the lattice shows that their numbers exhibit *power–law decay* with length of the structure for a given noise–strength, with an exponent which is a function of noise–strength. It is observed that average length of these structures shows a bell–shaped curve with a characteristic peak, as a function of noise–strength. Similar behaviour is observed for average lifetime of these structures during their evolution, within same range of noise values. We call this phenomenon *stochastic coherence*.

We consider a one dimensional CML of the form

$$x_{t+1}(i) = (1-\varepsilon)F(x_t(i))$$
$$+ \frac{\varepsilon}{2}\Big[F(x_t(i-1)) + F(x_t(i+1))\Big] + \eta_t , \quad (1)$$

where $x_t(i)$, $i = 1, 2, \cdots, L$, is value of the variable located at site i at time t, η_t is the additive noise, ε is the (nearest neighbour) coupling strength, and L is the size of the lattice. Logistic function $F(x) = \mu x(1-x)$ is used as local dynamics governing nonlinear site–evolution with nonlinearity parameter μ. We have used both open–boundary conditions $x_t(0) = x_t(1)$, $x_t(L+1) = x_t(L)$, and periodic–boundary conditions $x_t(L+i) = x_t(i)$, for our system. For noise η_t we have used a uniformly distributed random number bounded between $-W$ and $+W$, where W is defined as *noise–strength* parameter.

We define a *structure* as a region of space in which the dynamical variables at sites within this region follow a predefined spatial pattern [6]. To study coherence in the system we choose a spatial pattern where the difference in the values of the variables of neighbouring sites within the structure is less than a predefined small positive number say δ, i.e., $|x_t(i) - x_t(i \pm 1)| \leq \delta$. We call δ the *structure parameter*. We look for such patterns, or coherent structures, to appear in course of evolution of the model given by Eq. 1.

We now present our main observations. Values of μ, ε and L are chosen so that the resultant dynamics of the system is chaotic. Coherent structures with length < 3 (sites) and lifetime < 2 (timesteps) are disregarded in our observation. Values for W are chosen within the range $[0, 1]$. Figure 1 shows plot (on log–log scale) of distribution of number $n(l)$ vs. length l of structures for different values of W, with $\mu = 4$, $\varepsilon = 0.6$, $\delta = 0.0001$ and $L = 10000$, and open–boundary conditions are used. Power–law nature of decay of $n(l)$ is clearly evident, which has a form

$$n(l) \propto l^{-\alpha} , \quad (2)$$

where α is power–law exponent. This indicates that the system does not have any intrinsic length scale. It may be noted that in absence of noise ($W = 0$) the decay is manifestly exponential [6]. Exponent α (i.e., slope of the log–log plot in Fig. 1) is seen to depend on noise–strength W. This fact is corroborated in Fig. 2 which shows plot of α vs. W for the same parameter values as in Fig. 1. The exponent is exhibiting a clear minimum for values of W around 0.6. We define average length $\bar{l}$ of a structure as $\bar{l} = \sum l\, n(l)/\sum n(l)$. In Fig. 3 we plot variation of $\bar{l}$ with W for values of parameters as in Fig. 1. The plot exhibits a *bell–shaped* nature within a fairly narrow range of W around value 0.6. It may be noted that the minimum of α in Fig. 2 also occurs for W quite close to this value.

We call the phenomenon observed above *stochastic coherence*. This is similar to stochastic resonance phenomenon which shows a bell–shaped behaviour of temporal response as function of noise–strength [7]. However one may note that our system does not have any intrinsic length–scale, whereas in stochastic resonance noise resonates with a given time–scale; hence our use of word coherence rather than resonance [8]. In stochastic reso-

nance noise transfers energy to the system at a characteristic frequency, whereas in stochastic coherence noise *induces coherence* to the system. At this stage it is not clear to us as to how noise is inducing coherence to the system. However, absence of any length–scale over entire range of noise seems to indicate existence of weak self–organisation in the system induced by noise.

In this context it should be noted that if a uniform–deviate random number is taken as site–variable, then the probability of a site to belong to a structure of length l is $p_\delta(l) \approx l(2\delta)^{l-1}(1-2\delta)^2$, where δ is the structure parameter introduced earlier. Hence number of such structures in a lattice of size L is given by $n_\delta(l) \approx L(2\delta)^{l-1}(1-2\delta)^2$. This shows an exponential decay of number of such structures with length, which is to be contrasted with the power–law form (2) obtained for our lattice.

It is possible to show how noise helps reducing instability of the structures, thereby increasing their abundance. Let us consider stability matrix M of a homogeneous state $\{\cdots, x_t, x_t, x_t, \cdots\}$ (this may be thought of as a large structure with $\delta = 0$ for simplicity). At time $t+1$ the matrix takes the simple form $M_{t+1} = JF_t'$, where J stands for the familiar tridiagonal matrix (with diagonal elements $1-\varepsilon$ and offdiagonal elements $\varepsilon/2$ on either sides) and $F_t' \equiv F'(x_t)(\equiv dF/dx) = \mu(1-2x_t)$. After two timesteps, we get the stability matrix for three time steps as $S_3 = M_{t+1}M_{t+2}M_{t+3} = J^3 F_t' F_{t+1}' F_{t+2}' = J^3 F_t' \mu^2 [(1-2F_t)\{1-2\mu F_t(1-F_t)\} + 6\mu\eta_t^2(1-2F_t) - 2\eta_t\{1 + \mu(1-6F_t + 6F_t^2)\} - 4\mu\eta_t^3 - 2\eta_{t+1}(1-2F_t) + 4\eta_t\eta_{t+1}]$. For delta–correlated noise with uniform distribution and zero mean (which is the case here), after averaging the above expression over noise–distribution one gets nonzero contribution due to noise only from the term quadratic in η :

$$< S_3 >= J^3 (F_t')^2 \mu \Big[1 - 2\mu F_t(1-F_t) + 6\mu <\eta_t^2> \Big], \quad (3)$$

where $<>$ denotes averaging over noise. For our lattice with localised structures in backdrop of spatiotemporal chaos we found the invariant density to be no longer symmetric, with larger weightage for values of variable above 0.5. Averaging expression (3) over this density makes the term $1-2\mu F_t(1-F_t)$ negative. Adding positive contribution of noise to it results in reduction of eigenvalue of the matrix S_3 and hence consequent reduction in instability of the state.

To study evolutionary aspects of these coherent structures we obtained distribution of number $n(\tau)$ of structures vs. their lifetimes τ for different W. This is shown in Fig. 4 (on log–linear scale). It exhibits decrease of $n(\tau)$ with τ with a *stretched exponential* type of decay having a form

$$n(\tau) \propto \exp\big(-(\text{const.})\tau^\beta\big), \quad (4)$$

where β depends on W. We define average lifetime $\bar\tau$ of a structure as $\bar\tau = \sum \tau n(\tau)/\sum n(\tau)$. In Fig. 5 we plot $\bar\tau$ vs. W for parameter values as in Fig. 4. The graph

also shows a bell–shaped feature with maximum for W around 0.6. This W value is close to that corresponding to the extrema in figures 2 and 3.

In order to ascertain chaotic nature of system evolution we have calculated lyapunov exponent spectra for our system. We find a number of lyapunov exponents to be positive, implying that the underlying evolution is chaotic. Maximum lyapunov exponent $\lambda_{\max}$ shows a minimum around noise–strength 0.6 [9]. We have studied variation of λ–spectrum with coupling strength ε. $\lambda_{\max}$ remains fairly constant for $0.2 \le \varepsilon \le 0.8$ for entire range of W. Behaviour of $\bar l$ with ε is also investigated. It is found that $\bar l$ increases monotonically with ε for all W. This fact is quite contradictory to what is expected from $\lambda_{\max}$. This indicates that lyapunov exponent alone cannot be used for proper characterisation of spatio–temporal features of the system. We have also obtained power spectrum of the time–series of the dynamical variable for a given site. The plot does not show any characteristic frequency and supports chaotic behaviour. In addition, we have calculated power spectrum of values $x_t(i)$, $i = 1, \cdots, L$, at a given time. Nature of the spectrum confirms our earlier observation that the system does not have any intrinsic length–scale. It may appear that the behaviour of our system is similar to spatio–temporal intermittency phenomenon [10]. However, our system does not show any regular burst–type feature (indicative of intermittency) in time; as already noted, power spectrum of time–series does not have any distinctive peak for entire range of noise–strength. Thus what we observe is development of appreciable coherence induced by noise in the system undergoing *spatially intermittent and temporally chaotic* evolution.

On quite a few occasions the entire lattice is seen to evolve as a single coherent structure to within the structure parameter δ. We have studied lifetime of these coherent structures. Full lattice coherence appears for W above a value around 0.4. Interesting thing is that occasionally this state is seen to persist for fairly long durations (at times as long as 200 timesteps or more), but eventually the coherence is seen to break up. This demonstrates that for our system synchronised state is not a stable attractor.

However, in rare instances our system has been found to get into an apparently synchronised state after a very large time (larger than 10^7 steps). This is obtained because of finite accuracy of computation which cannot distinguish unstable synchronised state from stable one [11]. As demonstrated in the following, for our system this type of behaviour is an artifact resulting due to combination of finite size of the lattice and finite accuracy. Let us denote by $\bar T$ the average of timesteps required for first occurrence of full lattice coherence, the average taken over different initial conditions. We have studied variation of $\bar T$ with structure parameter δ for a fixed lattice–size L, which shows a definitive power–law of the form $\bar T_L \propto \delta^{-\gamma}$, where γ is power–law exponent. This indicates that the synchronised state, i.e., full lattice co-

herent structure with $\delta = 0$, cannot be reached. We have also investigated $\bar{T}$ vs. L variation for a fixed δ. We see a power–law behaviour of form

$$\bar{T}_\delta \propto L^\nu \ , \tag{5}$$

where ν is power–law exponent. Thus even full lattice coherence with finite δ cannot be achieved as $L \to \infty$.

We now show that existence of full–lattice coherence with nonzero δ (distinct from synchronised state) in a finite lattice is essentially a consequence of power–law variation (2) of $n(l)$ with l. From relation (2) probability of a site to belong to a structure of length l is $p_\delta(l) \propto l.l^{-\alpha} = l^{1-\alpha}$. Therefore, the probability of a site to belong to a structure of length $\geq L$ is $P_0 \equiv P_\delta(\geq L) \propto \sum_L^\infty l^{1-\alpha} \approx \int_L^\infty dl\, l^{1-\alpha} \propto L^{2-\alpha}$ (for $\alpha > 2$, which is the case in the entire range of our observation as can be seen from Fig. 2). The probability that for the first time a site belongs to a structure of length $\geq L$ at timestep T is then $P_\delta(T) \propto (1 - P_0)^{T-1} P_0$. Therefore average of T is given as $\bar{T}_\delta(\geq L) = \sum_{T=1}^\infty T\, P_\delta(T) \propto (P_0)^{-1}$, i.e.,

$$\bar{T}_\delta(\geq L) \propto L^{\alpha-2} \ . \tag{6}$$

This demonstrates that full–lattice coherence (with finite δ) will be observed in a finite lattice. Fig. 2 tells us that $\alpha - 2 \approx 0.22$ for $W = 0.6$, whereas we obtain ν(relation(5)) ≈ 0.3. We believe the discrepancy to be due to boundary corrections, since (6) is obtained for an infinite lattice whereas (5) holds for finite lattices.

Above observations were also carried out for several values of ε ranging from 0.1 to 0.9, as well as for nonlinearity parameter μ between 3.6 and 4. All the features remain essentially the same. We have also observed similar behaviour with periodic–boundary conditions for the lattice.

In conclusion, we have reported a new phenomenon observed in a chaotically evolving one–dimensional CML driven by identical noise, which we termed *stochastic coherence*. It is observed that there is a phenomenal increase in abundance of coherent structures of all scales due to noise. By considering stability matrix during three time steps, we have been able to show that noise can reduce instability of these structures. Distribution of these structures shows a power–law decay with length of the structure, with an exponent which shows a minimum at some intermediate noise–strength. Average length as well as average lifetime of these structures exhibit characteristic maxima at a noise–strength quite close to previous value. This bell–shaped feature is similar to that of stochastic resonance which is a temporal phenomenon. However, we emphasise that our system does not have any intrinsic length–scale, whereas stochastic resonance is associated with a particular time–scale. These observations demonstrate that noise can play a major role in formation as well as in evolutionary dynamics of structures in spatially extended systems.

[1] Y. Oono and S. Puri, Phys. Rev. Lett. **58**, 836 (1986); G. L. Oppo and R. Kapral, Phys. Rev. A **36**, 5820 (1987); P. M. Gade and R. E. Amritkar, Phys. Rev. E **47**, 143 (1993); R. E. Amritkar and P. M. Gade, Phys. Rev. Lett. **70**, 3408 (1993); *Theory and applications of coupled map lattices* (edited by K. Kaneko), John Wiley and Sons, Chichester, England (1993); Qu Zhilin and Hu Gang, Phys. Rev. E **49**, 1099 (1994); S. Raghavachari and James A. Glazier, Phys. Rev. Lett. **74**, 3297 (1995).

[2] R. Kapral, in *Theory and applications of coupled map lattices* (edited by K. Kaneko), p. 135; M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[3] A. K. M. F. Hussain, Phys. Fluids **26**, 2816 (1983). *Topological Fluid Mechanics* (edited by H. K. Moffatt et. al.), proceedings of the IUTAM symposium, 13–18 August 1989, Cambridge University Press (1990); *Turbulence And Stochastic Processes: Kolmogorov's Ideas 50 Years On* (edited by J. C. R. Hunt et. al.), The Royal Society, London (1991).

[4] *Noise in Dynamical Systems* (edited by F. Moss and P. V. E. McClintock), Cambridge University Press, Cambridge, U.K. (1989); P. C. Hohenberg and J. B. Swift, Phys. Rev. A **46**, 4773 (1992); X–G Wu and R. Kapral, J. Chem. Phys. **100**, 1 (1994); S. Fahy and D. R. Hamann, Phys. Rev. Lett. **69**, 761 (1992); D. A. Kurtze, Phys. Rev. Lett. **77**, 63 (1996); A. Maritan and J. R. Banavar, Phys. Rev. Lett. **72**, 1451 (1994). Effect of noise on evolution of patterns is also investigated by Kapral et al. in [1].

[5] P. Jung and G. Mayer–Kress, Phys. Rev. Lett. **74**, 2130 (1995).

[6] J. K. John and R. E. Amritkar, Phys. Rev. E **51**, 5103 (1995).

[7] R. Benzi, G. Parisi, A. Sutera and A. Vulpiani, Tellus 34, 10 (1982); B. McNamara, K. Wiesenfeld and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988); R. F. Fox and Yan–nan Lu, Phys. Rev. E **48**, 3390 (1993).

[8] Existence of spatio–temporal stochastic resonance in extended system is observed leading to enhancement of *temporal characteristics* of the system due to a Gaussian noise, as reported in [5].

[9] K. Matsumoto and I. Tsuda, J. Stat. Phys. **31**, 87 (1983); **34**, 111 (1984).

[10] H. Chaté and P. Manneville, Phys. Rev. Lett. **58**, 112 (1987); Physica D **32**, 409 (1988).

[11] A. S. Pikovsky, Phys. Rev. Lett. **73**, 2931 (1994).

FIGURE CAPTIONS

Fig. 1. Plot of variation of number $n(l)$ of structures with length l for a lattice with size $L = 10000$, for different values of noise–strength W as indicated. Parameters chosen are coupling strength parameter $\varepsilon = 0.6$, structure parameter $\delta = 0.0001$, and non-linearity parameter $\mu = 4$. Open–boundary conditions are used. Data are obtained for 100000 iterates per initial condition and 4 initial conditions.

Fig. 2. Variation of exponent α (relation (2)) with noise–strength W plotted for parameter values as in Fig. 1.

Fig. 3. Plot of variation of average length $\bar{l}$ of structure with noise–strength W, with parameters as stated in Fig. 1.

Fig. 4. Plot of variation of number $n(\tau)$ of structures with lifetime τ, with conditions as in Fig. 1.

Fig. 5. Variation of average lifetime $\bar{\tau}$ of structures with noise–strength W shown for parameters as stated earlier.

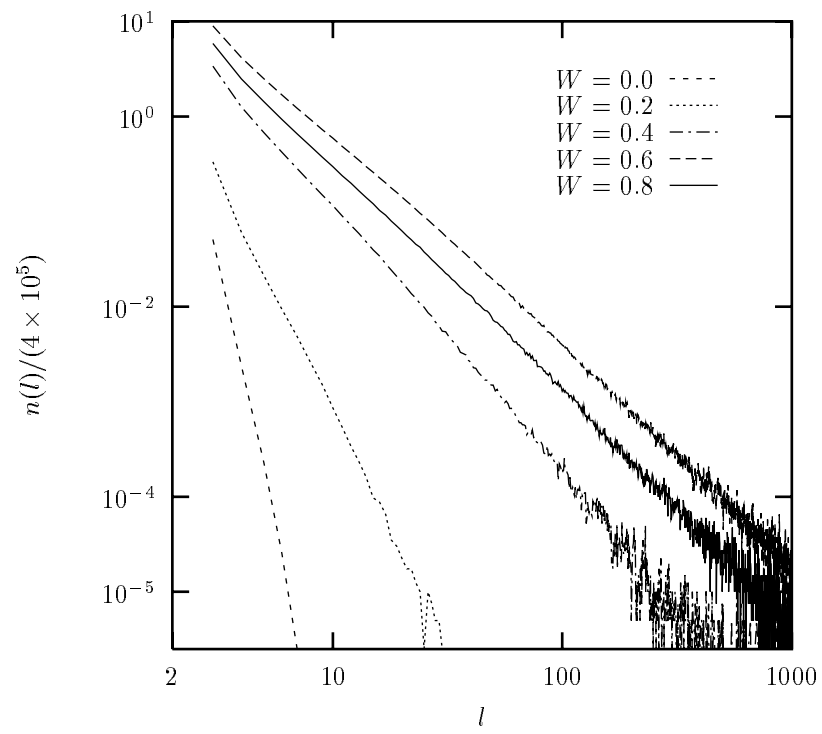Fig. 1   MR & REA

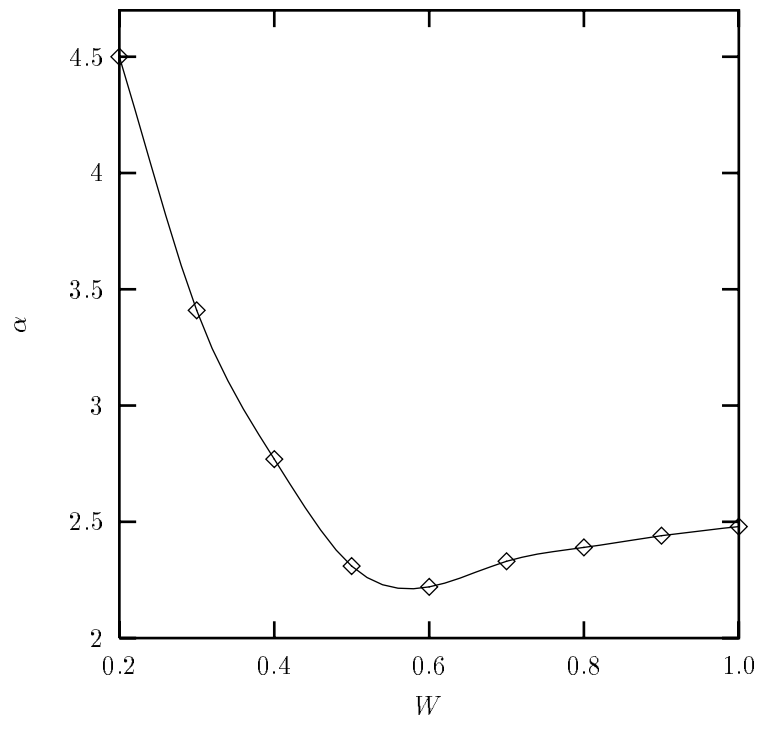Fig. 2   MR & REA

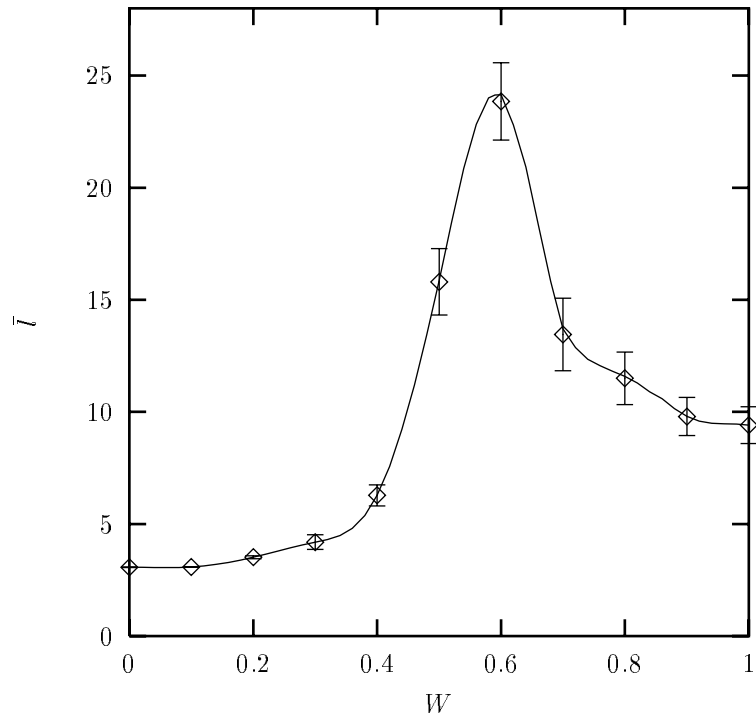Fig. 3   MR & REA
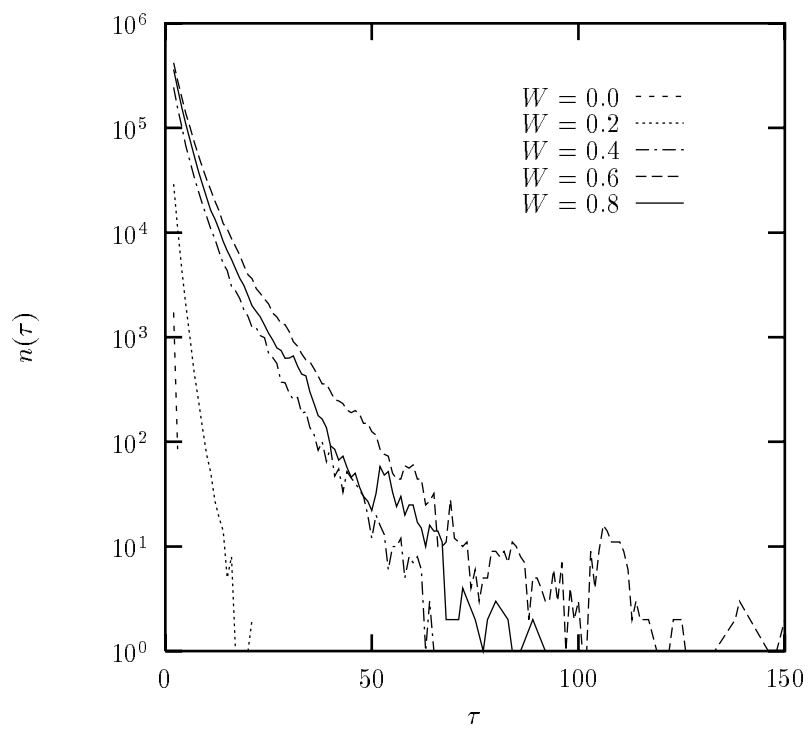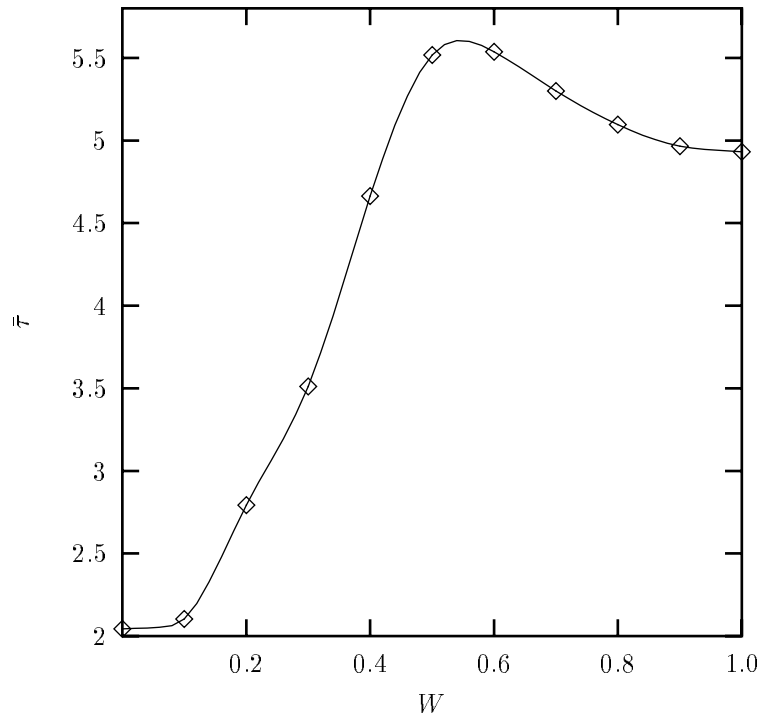
Fig. 4   MR & REA

Fig. 5   MR & REA