ASYMPTOTIC ANALYSIS OF MARKOV QUEUEING NETWORK WITH UNRELIABLE SYSTEMS

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The closed exponentional queueing network with unreliable systems with the large number of messages is investigated. We have received the systems of differential equations for average number of messages and serviceable channels of network systems.

Keywords: unreliable queueing systems, approximation.

1. INTRODUCTION

Let us examine the closed exponential queueing network with the K messages of the same type which consist of $n+1$ queueing systems (QS) $S_0, S_1, \dots, S_n$. The system S_i includes m_i identical service channels, $i = \overline{1,n}$, and $m_0 = K$.

Considering that service channels of the system S_0 are absolutely reliable and in the other systems systems $S_1, S_2, \dots, S_n$ the service channels are exposed to random failure; besides the time of the proper functionality of each S_i system's channel has the exponential distribution with the parameter β_i, $i = \overline{1,n}$. After the breakage the channel starts to reconstruct immediately. The time of reconstruction also has the exponential distribution with the parameter γ_i, $i = \overline{1,n}$. After servicing in system S_i the message immediately transfer into the system S_j with probability p_{ij}, $i,j = \overline{0,n}$, $p_{00} = 0$, $\sum_{i=0}^{n} p_{ij} = 1$. The matrix $P = \|p_{ij}\|_{(n+1)\times(n+1)}$ is transition probability matrix of irreducible Markov chains. If the arrived in the system S_j message finds at least one service channel operable and free from the other messages it is immediately serviced and the time of service is a random variable with the parameter μ_i, $i = \overline{1,n}$. Otherwise the message expects the beginning of service without restriction on duration of waiting. Let's assume that if the service channel would fail while completing some message, then after the restoration the interrupted message will be completed. Disciplines of the message processing in the network systems are FIFO.

Our aim is to receive the system of the differential equations for the average number of messages and serviceable channels in the network QS at the large values of K. It should be noted that the presented technics of the results reception has been offered for the first time in the works [1, 2] for the exponentional networks without the specified features (with reliable QS).

2. THE SYSTEM OF EQUATIONS FOR THE STATES PROBABILITIES

Assuming that the service time of messages, durations of serviceable work of channels and restoration time of service channels are independent random variables. The state of such network at the moment t could be described through vector

$$z(t) = (d(t), k(t)) = (d_1(t), d_2(t), \ldots, d_n(t), k_1(t), k_2(t), \ldots, k_n(t)), \tag{1}$$

where $d_i(t)$ and $k_i(t)$ are the numbers of serviceable channels and the messages numbers in the system S_i at the moment t accordingly, $0 \le d_i(t) \le m_i$, $0 \le k_i(t) \le K$, $t \in [0, +\infty)$. It is obvious that $k_0(t) = K - \sum_{i=1}^{n} k_i(t)$ is the number of messages in the system S_0 at the moment t.

Vector $z(t)$ describes $2n-$dimensional Markov process with the continuous time and the definite number of states. Let's consider, that

$$P(d, k, t) = P(d(t) = d, k(t) = k),$$

where $d = (d_1, d_2, \ldots, d_n)$, $0 \le d_i \le m_i$ and $k = (k_1, k_2, \ldots, k_n)$, $0 \le k_i \le K$, $i = \overline{1, n}$. Let's denote I_i as $n-$vector with zero components excluding i, that is equals to 1. Let's describe the possible passages of Markov process $z(t)$ in the state $z(t+\Delta t) = (d, k, t+\Delta t)$ at the time Δt:

- from the state $(d, k + I_i - I_j, t)$ the passage is possible with the probability
$$\mu_i p_{ij} \min\left(d_i(t), k_i(t) + 1\right)\Delta t + o(t), \quad i, j = \overline{1, n};$$
- from the state $(d, k - I_i, t)$ with the probability
$$\mu_i p_{ij} \left(K - \sum_{i=1}^{n} k_i(t) + 1\right) \Delta t + o(t), \quad i = \overline{1, n};$$
- from the state $(d, k + I_i, t)$ with the probability
$$\mu_i p_{i0} \min\left(d_i(t), k_i(t) + 1\right)\Delta t + o(t), \quad i = \overline{1, n};$$
- from the state $(d - I_i, k, t)$ with the probability
$$\gamma_i(m_i - d_i(t) + 1)\Delta t + o(t), \quad i = \overline{1, n};$$
- from the state $(d + I_i, k, t)$ with the probability
$$\beta_i(d_i(t) + 1)\Delta t + o(t), \quad i = \overline{1, n};$$
- from the state (d, k, t) with the probability
$$1 - \left[\mu_0 \left(K - \sum_{i=1}^{n} k_i(t)\right) + \sum_{i=1}^{n} \mu_i \min(d_i(t), k_i(t)) + \right.$$
$$\left. + \sum_{i=1}^{n} \gamma_i(m_i - d_i(t)) + \sum_{i=1}^{n} \beta_i d_i(t)\right] \Delta t + 0(\Delta t);$$

• from all other states with the probability $o(\Delta t)$.

Then, the usage of the formula of total probability makes it possible to write the system of difference equations for the probabilities of states from which at $\Delta t \to 0$ we receive the system of difference-differential equations of Kolmogorov for the states probabilities

$$\frac{dP(d,k,t)}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i p_{ij} \min(d_i(t), k_i(t)) \left[P(d, k - I_i + I_j, t) - P(d,k,t)\right] +$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i p_{ij} \left[\min(d_i(t), k_i(t)+1) - \min(d_i(t), k_i(t))\right] P(d, k - I_i + I_j, t) +$$

$$+\mu_0 \left(K - \sum_{i=1}^{n} k_i(t)\right) \left[P(d, k - I_j, t) - P(d,k,t)\right] + \mu_0 P(d, k - I_j, t) +$$

$$+\sum_{i=1}^{n} \mu_i p_{i0} \min(d_i(t), k_i(t)) \left[P(d, k + I_i, t) - P(d,k,t)\right] +$$

$$+\sum_{i=1}^{n} \mu_i p_{i0} \left[\min(d_i(t), k_i(t)+1) - \min(d_i(t), k_i(t))\right] P(d, k + I_i, t) +$$

$$+\sum_{j=1}^{n} \gamma_i (m_i - d_i(t)) \left[P(d - I_j, k, t) - P(d,k,t)\right] + \sum_{i=1}^{n} \gamma_i P(d - I_i, k, t) +$$

$$+\sum_{i=1}^{n} \beta_i d_i(t) \left[P(d + I_i, k, t) - P(d,k,t)\right] + \sum_{i=1}^{n} P(d + I_i, k, t). \qquad (2)$$

The solution of this system in the analytical form is generally inconvenient. Therefore we will consider the important case of the large number of messages in the network, $K >> 1$. In order to determine probability distribution of the random vector $z(t)$, it is convenient to switch to the relative variables, considering vector

$$\xi(t) = \left(\frac{d_1(t)}{K}, \frac{d_2(t)}{K}, \ldots, \frac{d_n(t)}{K}, \frac{k_1(t)}{K}, \frac{k_2(t)}{K}, \ldots, \frac{k_n(t)}{K}\right),$$

In this case possible values of this vector at the fixed t will belong to the bounded closed set

$$G = \left\{ (y,k) = (y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i \leq 1, 0 \leq y_i \leq \frac{m_i}{K} \right\} \qquad (3)$$

in which they place in the nodes of the $2n-$dimensional grid at the distance $\varepsilon = \frac{1}{K}$ from each other. While magnifying K "the charging density" of the multiple G with the possible components of vector $\xi(t)$ will increase, and it is possible to consider, that it has a continuous distribution with the probabilities density $p(y,x,t)$, and $K^{2n}P(d,k,t) \to p(y,x,t)$ if $K \to \infty$. Therefore it is possible to use the approximation of the function $P(d,k,t)$, using the relation $K^{2n}P(d,k,t) = K^{2n}P(yK,xK,t) = p(y,x,t)$, $(y,x) \in G$.

Let denote that $e_i = \varepsilon I_i$, $i = \overline{1,n}$, $c(b) = \begin{cases} 1, & b > 0 \\ 0, & b \le 0 \end{cases}$, and

$$\min(b, a+1) = \min(b,a) + c(b-a), \quad c(b-a) = \frac{\partial \min(b,a)}{\partial a}, \tag{4}$$

thus $\min(b,a) = \begin{cases} a, & b \ge a \\ b, & b < a \end{cases}$. Using the relative variables $y_i = \dfrac{d_i}{K}$, $x_i = \dfrac{k_i}{K}$, $l_i = \dfrac{m_i}{K}$ for $i = \overline{1,n}$, expression (4) and that at $K \to +\infty$, $\varepsilon \to 0$, system (2) can be written as follows:

$$\frac{\partial p(y,x,t)}{\partial t} = \sum_{i=1}^{n}\sum_{j=1}^{n} K\mu_i p_{ij} \min(y_i,x_i)\,[p(y,x+e_i-e_j,t) - p(y,x,t)] +$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i p_{ij} \frac{\partial \min(y_i,x_i)}{\partial x_i} p(y,x+e_i-e_j,t) +$$

$$+K\mu_0\left(1 - \sum_{i=1}^{n} x_i\right)[p(y,x-e_j,t) - p(y,x,t)] + \mu_0 p(y,x-e_j,t)$$

$$+\sum_{i=1}^{n} K\mu_i p_{i0} \min(y_i,x_i)\,[p(y,x+e_i,t) - p(y,x,t)] +$$

$$+\sum_{i=1}^{n} \mu_i p_{i0} \frac{\partial \min(y_i,x_i)}{\partial x_i} p(y,x+e_i,t) +$$

$$+\sum_{i=1}^{n} K\gamma_i(l_i - y_i)\,[p(y-e_i,x,t) - p(y,x,t)] + \sum_{j=1}^{n} \gamma_i p(y-e_i,x,t) +$$

$$+\sum_{i=1}^{n} K\beta_i y_i\,[p(y+e_i,x,t) - p(y,x,t)] + \sum_{j=1}^{n} \beta_i p(y+e_i,x,t). \tag{5}$$

3. THE SYSTEM OF DE FOR EXPECTED CHARACTERISTICS

Let's present the right part (5) with the accuracy of term ε^2. If $p(y,x,t)$ is twice continuously differentiated at y and x, than

$$p(y,x \pm e_i,t) = p(y,x,t) \pm \varepsilon \frac{\partial p(y,x,t)}{\partial x_i} + \frac{\varepsilon^2}{2}\frac{\partial^2 p(y,x,t)}{\partial x_i^2} + o(\varepsilon^2),$$

$$p(y,x+e_i-e_j,t) = p(y,x,t) + \varepsilon\left(\frac{\partial p(y,x,t)}{\partial x_i} - \frac{\partial p(y,x,t)}{\partial x_j}\right) +$$

$$+\frac{\varepsilon^2}{2}\left(\frac{\partial^2 p(y,x,t)}{\partial x_i^2}-2\frac{\partial^2 p(y,x,t)}{\partial x_i \partial x_j}+\frac{\partial^2 p(y,x,t)}{\partial x_j^2}\right)+o(\varepsilon^2),$$

$$p(y\pm e_i,x,t)=p(y,x,t)\pm\varepsilon\frac{\partial p(y,x,t)}{\partial y_i}+\frac{\varepsilon^2}{2}\frac{\partial^2 p(y,x,t)}{\partial y_i^2}+o(\varepsilon^2),\quad i=\overline{1,n}. \qquad (6)$$

Using them and that $\varepsilon K=1$, it is possible to receive that the density $p(y,x,t)$ satisfies with the accuracy within the term ε^2 to the Kolmogorov-Fokker-Planc equation:

$$\frac{\partial p(y,x,t)}{\partial t}=-\sum_{i=1}^{n}\frac{\partial}{\partial y_i}\left(A_i^{(y)}(y)p(y,x,t)\right)-\sum_{i=1}^{n}\frac{\partial}{\partial_i}\left(A_i^{(x)}(y,x)p(y,x,t)\right)+$$

$$+\frac{\varepsilon}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial y_i y_j}\left(B_{ij}^{(y)}(y)p(y,x,t)\right)+\frac{\varepsilon}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i x_j}\left(B_{ij}^{(x)}(y,x)p(y,x,t)\right), \qquad (7)$$

where

$$A_i^{(y)}(y)=\gamma_i\left(l_i-y_i\right),i=\overline{1,n}, \qquad (8)$$

$$A_i^{(x)}(y,x)=\sum_{j=1}^{n}\mu_j p_{ji}^{*}\min(y_j,x_j)+\mu_0 p_{0i}\left(1-\sum_{i=1}^{n}x_i\right); \qquad (9)$$

$$p_{ji}^{*}=\begin{cases}p_{ji}, & j\neq i,\\ p_{ii}-1, & j=i;\end{cases}\quad B_{ii}^{(y)}(y)=\gamma_i(l_i-y_i)+\beta_i y_i;\quad B_{ij}^{(y)}(y)=0,\quad i\neq j;$$

$$B_{ii}^{(x)}(y,x)=\sum_{j=1}^{n}\mu_j p_{ji}^{**}\min(y_j,x_j)+\mu_0 p_{0i}\left(1-\sum_{i=1}^{n}x_i\right),$$

$$p_{ji}^{**}=\begin{cases}p_{ji}, & j\neq i,\\ 1-p_{ii}, & j=i;\end{cases}\quad B_{ij}^{(x)}(x)=-\mu_i p_{ij}\min(y_i,x_i),\quad i\neq j,\quad i,j=\overline{1,n}.$$

As the density $p(y,x,y)$ satisfies the Kolmogorov-Fokker-Planc equation and $A_i^{(y)}(y)$, $A_i^{(x)}(x)$ piecewise linear functions on y, x, according to [3], the mathematical expectations $w_i(t)=M\left\{\frac{d_i(t)}{K}\right\}$, $n_i(t)=M\left\{\frac{k_i(t)}{K}\right\}$, $i=\overline{1,n}$, with the accuracy within the terms of infinitesimal order $O(\varepsilon^2)$ are defined from the systems of the equations

$$\frac{dw_i(t)}{dt}=A_i^{(y)}(w_i(t))=\gamma_i(l_i-w_i(t))-\beta_i w_i(t),\quad i=\overline{1,n}, \qquad (10)$$

$$\frac{dn_i(t)}{dt}=A_i^{(x)}(w_i(t),n_i(t))=\sum_{j=1}^{n}\mu_j p_{ji}^{*}\min(w_j(t),n_j(t))-$$

$$+\mu_0 p_{0i}\left(1-\sum_{i=1}^{n}n_i(t)\right),\quad i=\overline{1,n}. \qquad (11)$$

The right hand sides of system (11) are continuous piecewise linear functions. By segmentation of phase space and obtaining solutions of system (11) in ranges of right hand sides linearity it is possible to solve whole system.

Let $\Omega(t) = \{1, 2, \ldots, n\}$ be set of vector $n(t)$ component indices. Let's divide $\Omega(t)$ into two disjoint sets $\Omega_0(t)$ and $\Omega_1(t)$:

$$\Omega_0(t) = \{i : w_i(t) < n_i(t) \leq 1\}, \quad \Omega_1(t) = \{j : 0 \leq n_j(t) \leq w_j(t)\}.$$

Each partitioning specifies disjoint regions $G_\tau(t)$ in set

$$G(t) = \left\{ n(t) : n_i(t) \geq 0, \sum_{i=1}^{n} n_i(t) \leq 1 \right\},$$

such that:

$$G_\tau(t) = \left\{ n(t) : w_i(t) < n_i(t) < 1, i \in \Omega_0(t); \right.$$

$$\left. 0 \leq n_j(t) \leq w_j(t), j \in \Omega_1(t); \sum_{c=1}^{n} n_c(t) \leq 1 \right\}, \tau = 1, 2, \ldots, 2^n, \quad \bigcup_{\tau=1}^{2^n} G_\tau(t) = G(t).$$

Then system of equations (11) of explicit form is:

$$\frac{dn_i(t)}{dt} = \sum_0 \mu_j p_{ji}^* w_j(t) + \sum_1 \mu_j p_{ji}^* n_j(t) +$$

$$+ \mu_0 p_{0i} \left(1 - \sum_{i=1}^{n} n_i(t) \right), i = \overline{1, n}, \qquad (12)$$

for each region $G_\tau(t)$.

The solution of uniform system of equations (10), (12) allows obtaining average relative number of messages and serviceable channels at any queueing system of queueing network.

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