INVESTIGATION OF QUEUING SYSTEM $GI^{(2)}|M_2|\infty$

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We offer the model of applications parallel service in queuing system (QS) which consists of two units of service with an unlimited number of servers. A recurrent stream of binary applications enters to an input to the system. We recorded the approximate (asymptotic) equality of the first and second orders for the respective blocks of the system.

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1. INTRODUCTION

Currently, attention to the queuing theory is largely stimulated by the need to apply the results of this theory to important practical problems arising in connection with the rapid development of communication systems, the emergence of information and computing systems, appearance and complexity of various technological systems, the creation of automated control systems.

At the present stage of development of queuing theory, one of the urgent tendencies is the queuing systems investigation with batch arrival of applications and parallel service [1, 2]. Range of application of the this QS is quite extensive, for example, modeling of modern information and computer systems requires take into account the packet traffic, as well as one of the basic principles for the design of modern computer networks - parallel information processing [3, 4]. Therefore there is a need to develop new mathematical models of queuing systems, such as systems with extraordinary incoming flow and the various service options, including two or more units of service.

2. PROBLEM STATEMENT

We consider a QS $GI^{(2)}|M_2|\infty$ with two service blocks, each of which contains an unlimited number of servers.

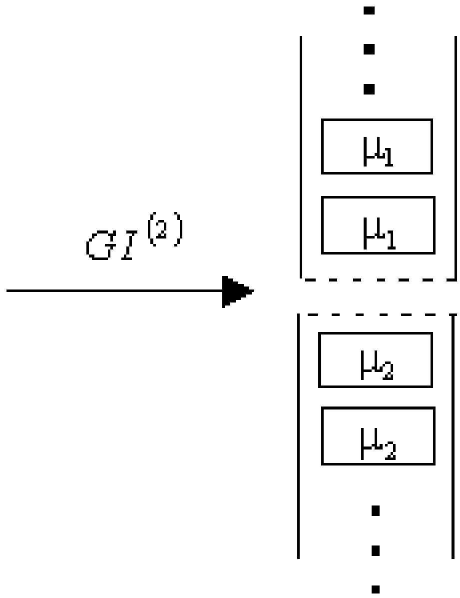

Fig. 1. QS with parallel services of multiple applications $GI^{(2)}|M_2|\infty$

A recurrent flow [5] of dual applications specified distribution function A(x) - length of intervals between the moments of events in the concerned stream enters to the input of the system. Service intervals of various applications are stochastically independent and identically distributed in each block and have an exponential distribution with μ_1 and $\mu_2$parameters, respectively. Received applications takes any of the free servers and after completion of the service the application leaves the system.

3. MATHEMATICAL MODEL

We define $i_k(t)$ - number of requests in the k-th block of service. Because the incoming flow is non-Poisson, the two-dimensional process $\{i_1(t),i_2(t)\}$ is non-Markov, and therefore we consider an additional component of z(t), equal to the length of the interval from t to themoment of the next event in the recurrent input stream.

For the concerned system a three-dimensional random process $\{z(t),i_1(t),i_2(t)\}$ is Markov, so its probability distribution

$$P\left(z,i_1,i_2,t\right)=P\{z\left(t\right)<z,i_1\left(t\right)=i_1,i_2\left(t\right)=i_2\}$$

We can write equality

$$P(z-\Delta t,i_1,i_2,t+\Delta t)=\left\{P(z,i_1,i_2,t)-P(\Delta t,i_1,i_2,t)\right\}\left(1-i_1\mu_1\Delta t\right)\left(1-i_2\mu_2\Delta t\right)+$$

$$+P(\Delta t,i_1-1,i_2-1,t)A(z)+P(z,i_1,i_2+1,t)\left(i_2+1\right)\mu_2\Delta t+$$

$$+P(z,i_1+1,i_2,t)\,(i_1+1)\,\mu_1\Delta t+o(\Delta t),$$

from which we can readily obtain a system of differential equations of Kolmogorov [6]

$$\frac{\partial P(z,i_1,i_2,t)}{\partial t}=\frac{\partial P(z,i_1,i_2,t)}{\partial z}-\frac{\partial P(0,i_1,i_2,t)}{\partial z}-P(z,i_1,i_2,t)i_1\mu_1-P(z,i_1,i_2,t)i_2\mu_2+$$

$$+P(z,i_1+1,i_2,t)\,(i_1+1)\,\mu_1+P(z,i_1,i_2+1,t)\,(i_2+1)\,\mu_2+\frac{\partial P(0,i_1-1,i_2-1,t)}{\partial z}A(z).$$

For the stationary probability distribution this system can be rewritten as

$$\frac{\partial P(z,i_1,i_2)}{\partial z}-\frac{\partial P(0,i_1,i_2)}{\partial z}-P(z,i_1,i_2)i_1\mu_1-P(z,i_1,i_2)i_2\mu_2+$$

$$+P(z,i_1+1,i_2)\,(i_1+1)\,\mu_1+P(z,i_1,i_2+1)\,(i_2+1)\,\mu_2+\frac{\partial P(0,i_1-1,i_2-1)}{\partial z}A(z)=0.$$

Defining $H(z,u_1,u_2)=\sum\limits_{i_1=0}^{\infty}\sum\limits_{i_2=0}^{\infty}e^{ju_1i_1}e^{ju_2i_2}P(z,i_1,i_2)$, where $j=\sqrt{-1}$ imaginary unit and taking into account that

$$\frac{\partial H(z,u_1,u_2)}{\partial u_1}=j\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}i_1e^{ju_1i_1}e^{ju_2i_2}P(z,i_1,i_2),$$

$$\frac{\partial H(z,u_1,u_2)}{\partial u_2}=j\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}i_2e^{ju_1i_1}e^{ju_2i_2}P(z,i_1,i_2),$$

for the functions $H(z,u_1,u_2)$ we obtain the basic equation for investigating the system $GI^{(2)}|M_2|\infty$

$$\frac{\partial H\left(z,u_1,u_2\right)}{\partial z}+\frac{\partial H\left(0,u_1,u_2\right)}{\partial z}\left\{e^{j(u_1+u_2)}A\left(z\right)-1\right\}-j\mu_1\left(e^{-ju_1}-1\right)\frac{\partial H\left(z,u_1,u_2\right)}{\partial u_1}-$$

$$-j\mu_2\left(e^{-ju_2}-1\right)\frac{\partial H\left(z,u_1,u_2\right)}{\partial u_2}=0. \tag{1}$$

We define additional conditions as

$$H(z,0,0)=R(z),$$

where $R(z)$ – stationary distribution of the $z(t)$. Solution $H(z,u_1,u_2)$ defines the characteristic function of the number of servers employed in the stationary state in each block of the system $GI^{(2)}|M_2|\infty$ as equalities

$$Me^{ju_1i_1(t)}=H(\infty,u_1,0),$$

$$Me^{ju_2i_2(t)}=H(\infty,0,u_2).$$

4. THE METHOD OF ASYMPTOTIC ANALYSIS

4.1. Asymptotic form of the first order. For a complete analysis of the studied queuing systems apply the method of asymptotic analysis [5].

We consider the basic equation for the characteristic function (1):

$$\frac{\partial H\left(z,u_1,u_2\right)}{\partial z}+\frac{\partial H\left(0,u_1,u_2\right)}{\partial z}\left(e^{j(u_1+u_2)}A\left(z\right)-1\right)-j\mu_1\left(e^{-ju_1}-1\right)\frac{\partial H\left(z,u_1,u_2\right)}{\partial u_1}-$$

$$-j\mu_2\left(e^{-ju_2}-1\right)\frac{\partial H\left(z,u_1,u_2\right)}{\partial u_2}=0,$$

which will be solved in the asymptotic conditions of the growing service-time, believing that μ_1,μ_2 $\to$0. We define $\mu_1=\varepsilon$, $\mu_2=\varepsilon$q and do changes in equation (1)

$$u_1=\varepsilon w_1,\quad u_2=\varepsilon w_2,\quad H(z,u_1,u_2)=F_1(z,w_1,w_2,\varepsilon),\tag{2}$$

In the result we get the equation for $F_1(z,w_1,w_2,\varepsilon)$ in the following form:

$$\frac{\partial F_1(z,w_1,w_2,\varepsilon)}{\partial z}+\frac{\partial F_1(0,w_1,w_2,\varepsilon)}{\partial z}\left(e^{j(w_1+w_2)}A\left(z\right)-1\right)-$$

$$-j\left(e^{-j\varepsilon w_1}-1\right)\frac{\partial F_1(z,w_1,w_2,\varepsilon)}{\partial w_1}-j\left(e^{-j\varepsilon w_2}-1\right)\frac{\partial F_1(z,w_1,w_2,\varepsilon)}{\partial w_2}=0.\tag{3}$$

We prove the following statement:

Theorem 1. *The limit (when $\varepsilon\to$0) solution of equation* (2) *has the form:*

$$F_1(z,w_1,w_2)=R(z)\mathbf{e}^{\{j(w_1+w_2)\lambda\}},$$

where R(z) - stationary probability distribution of random process values{z(t)}, and the parameter λ is given by: $\frac{\partial R(0)}{\partial z}=\lambda$

Proof. The proof 1. In the equation (3) limiting transition is executed at $\varepsilon\to 0$, we will receive that $F_1(z,w_1,w_2)$ the equation decision is

$$\frac{\partial F_1(z,w_1,w_2)}{\partial z}+\frac{\partial F_1(0,w_1,w_2)}{\partial z}\{A(z)-1\}=0,$$

which defines a vector function R(z), therefore

$$F_1(z,w_1,w_2)=R(z)\Phi_1\left(w_1,w_2\right)\tag{4}$$

We find the scalar function $\Phi_1(w_1,w_2)$ as follows. In the equation (3) limiting transition is performed at z $\to\infty$, we will receive an equality

$$\frac{\partial F_1(0,w_1,w_2,\varepsilon)}{\partial z}\left(e^{j(w_1+w_2)\varepsilon}-1\right)-j\left(e^{-j\varepsilon w_1}-1\right)\frac{\partial F_1(\infty,w_1,w_2,\varepsilon)}{\partial w_1}-$$

$$-j\left(e^{-j\varepsilon w_2}-1\right)\frac{\partial F_1(\infty,w_1,w_2,\varepsilon)}{\partial w_2}=0,$$

dividing the left and right parts of the equation by ε and executing limiting transition at $\varepsilon \to 0$, we will receive the equality

$$\frac{\partial F_1(0,w_1,w_2)}{\partial z} j\left(w_1+w_2\right)+j^2 w_1 \frac{\partial F_1(\infty,w_1,w_2)}{\partial w_1}+j^2 w_2 \frac{\partial F_1(\infty,w_1,w_2)}{\partial w_2}=0,$$

in which we substitute the expression (4) and write down the differential equation in partial derivatives

$$\frac{\partial \Phi_1\left(w_1,w_2\right)}{\partial w_1}+j^2 w_2 \frac{\partial \Phi_1\left(w_1,w_2\right)}{\partial w_2}=j\left(w_1+w_2\right)\frac{\partial R(0)}{\partial z}\Phi_1\left(w_1,w_2\right)=$$

$$=j\left(w_1+w_2\right)\lambda \Phi_1\left(w_1,w_2\right),$$

whose solution is satisfying the initial condition $\Phi_1(0,0)=1$ and it has the form

$$\Phi_1\left(w_1,w_2\right)=\mathbf{e}^{\{j\lambda(w_1+w_2)\}}.$$

We get $F_1(z,w_1,w_2)=R(z)\mathbf{e}^{\{j\lambda(w_1+w_2)\}}$.

The theorem is proved. □

Using (2) and (4) we can write the approximate (asymptotic) equality

$$H(z,u_1,u_2)=R(z)\mathbf{e}^{\left\{j\lambda\left(\frac{u_1}{\varepsilon}+\frac{u_2}{\varepsilon}\right)\right\}}=R(z)\mathbf{e}^{\left\{j\lambda\left(\frac{u_1}{\mu_1}+\frac{u_2}{\mu_2 q}\right)\right\}}$$

from which we can obtain following expression for the characteristic function of process $\{i_1(t),i_2(t)\}$ in the stationary state,

$$Me^{ju_1 i_1(t)}=H(\infty,u_1,0)=\mathbf{e}^{\left\{ju_1\frac{\lambda}{\mu_1}\right\}},\quad Me^{ju_2 i_2(t)}=H(\infty,0,u_2)=\mathbf{e}^{\left\{ju_2\frac{\lambda}{\mu_2 q}\right\}}.$$

The received equations is called the first order asymptotic form for the service blocks of systems service $GI^{(2)}|M_2|\infty$.

For a more detailed investigation, we consider the second order asymptotic form.

4.2. Asymptotic form of the second order. In equation (1) we make the substitution

$$H(z,u_1,u_2)=H_2(z,u_1,u_2)\mathbf{e}^{\left\{j\lambda\left(\frac{u_1}{\mu_1}+\frac{u_2}{\mu_2 q}\right)\right\}},$$

which later becomes

$$\frac{\partial H_2(z,u_1,u_2)}{\partial z}+\frac{\partial H_2(0,u_1,u_2)}{\partial z}\left(\mathbf{e}^{\{j(u_1+u_2)\}}A\left(z\right)-1\right)+$$

$$+j\mu_1\left(1-e^{-ju_1}\right)\frac{\partial H_2(z,u_1,u_2)}{\partial u_1}+j\mu_2 q\left(1-e^{-ju_2}\right)\frac{\partial H_2(z,u_1,u_2)}{\partial u_2}-$$

$$-\lambda\left(1-e^{-ju_1}\right)H_2(z,u_1,u_2)-\lambda\left(1-e^{-ju_2}\right)H_2(z,u_1,u_2)=0.$$

We define $\mu_1=\varepsilon^2$, $\mu_2=\varepsilon^2 q$ in the last equation and make substitutions

$$u_1=\varepsilon w_1,\quad u_2=\varepsilon w_2,\quad H_2(z,u_1,u_2)=F_2(z,w_1,w_2,\varepsilon),$$

we get the result

$$\frac{\partial F_2(z,w_1,w_2,\varepsilon)}{\partial z}+\frac{\partial F_2(0,w_1,w_2,\varepsilon)}{\partial z}\left(e^{j\varepsilon(w_1+w_2)}A\left(z\right)-1\right)+$$
$$+j\varepsilon\left(1-e^{-j\varepsilon w_1}\right)\frac{\partial F_2(z,w_1,w_2,\varepsilon)}{\partial w_1}+j\varepsilon\left(1-e^{-j\varepsilon w_2}\right)\frac{\partial F_2(z,w_1,w_2,\varepsilon)}{\partial w_2}-$$
$$-\lambda\left(1-e^{-j\varepsilon w_1}\right)\partial F_2(z,w_1,w_2,\varepsilon)-\lambda\left(1-e^{-j\varepsilon w_2}\right)\partial F_2(z,w_1,w_2,\varepsilon)=0. \quad (5)$$

The following theorem is proved similarly theorem for first order asymptotic form.

Theorem 2. *The limit (when $\varepsilon \to 0$) solution of equation* (5) *has the form*

$$F_2(z,w_1,w_2)=R(z)\mathbf{e}^{\left\{\frac{(j(w_1+w_2))^2}{2}\kappa_2\right\}},$$

where R(z) - stationary probability distribution of the random process values {z(t)}, the parameter λ is given by: $\frac{\partial R(0)}{\partial z}=\lambda$, the value $\kappa_2$is determined as $\kappa_2=\lambda+\frac{\partial f_2(0)}{\partial z}$.

The vector function $f_2(z)$ satisfies the condition $f_2(\infty)=0$ and it is a solution of the equation

$$\frac{\partial f_2(z)}{\partial z}+\frac{\partial f_2(0)}{\partial z}\left(A(z)-1\right)+\frac{\partial R(0)}{\partial z}A(z)-\lambda R(z)=0.$$

Also the asymptotic (approximate) equality can be written

$$H_2(z,u_1,u_2)=F_2(z,w_1,w_2\varepsilon)\approx F_2(z,w_1,w_2)=$$
$$=R(z)\mathbf{e}^{\left\{j^2\left[\kappa_2\frac{(w_1+w_2)^2}{2}-\frac{\lambda w_1 w_2}{2}\right]\right\}}=$$
$$=R(z)\mathbf{e}^{\left\{\frac{j^2\kappa_2}{2}\left(\frac{u_1}{\sqrt{\mu_1}}+\frac{u_2}{\sqrt{\mu_2 q}}\right)^2-j^2\frac{\lambda u_1 u_2}{2\sqrt{\mu_1\mu_2 q}}\right\}},$$

from which we can obtain following expression for the characteristic function $H(z,u_1,u_2)$

$$H(z,u_1,u_2)=R(z)\mathbf{e}^{\left\{j\lambda\left(\frac{u_1}{\mu_1}+\frac{u_2}{\mu_2 q}\right)+\frac{j^2\kappa_2}{2}\left(\frac{u_1}{\sqrt{\mu_1}}+\frac{u_2}{\sqrt{\mu_2 q}}\right)^2-j^2\frac{\lambda u_1 u_2}{2\sqrt{\mu_1\mu_2 q}}\right\}},$$

Then the marginal characteristic functions have the form

$$Me^{ju_1 i_1(t)}=H(\infty,u_1,0)=\mathbf{e}^{\left\{j\lambda\frac{u_1}{\mu_1}+\frac{(ju_1)^2}{2\mu_1}\kappa_2\right\}},$$
$$Me^{ju_2 i_2(t)}=H(\infty,0,u_2)E=\mathbf{e}^{\left\{j\lambda\frac{u_2}{\mu_2 q}+\frac{(ju_2)^2}{2\mu_2 q}\kappa_2\right\}}.$$

The received equations is called second order asymptotic form for service blocks of the system $GI^{(2)}|M_2|\infty$.

5. CONCLUSION

So we have constructed a model of multiple parallel service requests of recurrent flow. We investigate two-dimensional stochastic process characterizing the number of engaged servers in the first and the second service block. We have found expressions for the characteristic functions at the asymptotic conditions of the growing claims service-time.

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