# SOME MARTINGALE RELATIONS FOR M/G/1 RETRIAL QUEUE

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**Abstract**

In this paper we apply the martingale method due to Baccelli and Makowski for analysing the $M/G/1$ retrial queue. Using the recursive equation of the process embedded at depart epochs, we construct a discrete-time martingale stopped at the first passage time where the system becomes empty. We derive the stability condition and study the busy period of this system.

*Keywords:Retrial queues, Embedded Markov Chain, Martingale, Busy Period.*

## 1. INTRODUCTION

During the past years, an important research effort has been devoted to retrial queues due to their specificity and their ability in modeling several systems. Our queuing system is characterized by the phenomenon that an arriving customer who finds the server busy upon arrival is obliged to leave the service area and repeat his demand after some time called retrial time. Between trials, a blocked customer that remains in a retrial group is said to be orbit. The major analytic results and techniques used in retrial systems area are summarized in Yang and Templeton [2] and Falin [5] survey papers. We also refer to the synthesis presented by A.Aissani [4]. Regarding analysis or optimization for which stability problems are studied, often under specific assumptions, the martingale method represents an alternative approach; although, this approach was not very often used Queuing Theory. The purpose of this work is to enlarge the scope of applicability of martingale method to M/G/1 retrial systems.

## 2. THE MATHEMATICAL MODEL

We consider an $M/G/1$ retrial queuing system where the primary customers arrive according to a Poisson process with arrival rate $\lambda$ and the service times are independent and identically distributed with arbitrary probability distribution $B(.)$ and Laplace - Stieljes transforms $B^*(.), Re(s) \geq 0$. The time between successive repeated attempts are exponentially distributed with rate $\nu$. If the server is free at the instant of a primary call, the arriving customer begins service immediately and leaves the system after service completion. Otherwise, if the customer finds the server busy, then it enters orbit and becomes a source of repeated calls (secondary call source). We define the number of customers in the system at time $t$ to be $X(t)$. Note that this process is not a Markov chain since it depends on the history of the process and not just upon the current state. So, we consider the discrete time process $\{X_n, n \geq 1\}$, where $X_n$ is the number of customers in the system as seen by the $nth$ departing customer. More precisely,

$$\{X_n = X(t_n); n = 1, 2, ...\}$$

where $t_n$ is the departure time of the $nth$ customer. According to the results established in [6],

$$X_{n+1} = X_n + A_{n+1} - \delta_{X_n}, n \geq 1. \tag{1}$$

where $A_n$ is the number of primary calls during the $nth$ customer service time; $\delta_{X_n} = 1$ If the $(n+1)th$ customer comes from the orbit and $\delta_{X_n} = 0$ otherwise. The random variables $A_n$ are mutually independent and their joint distribution is given by

$$P(A_n = k) = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dB(t), k = 0, 1, 2, ...; n \geq 1. \tag{2}$$

The generating function of this distribution is

$$a(z) = \sum_{k=0}^{\infty} z^k P(A_n = k) = B^*(\lambda - \lambda z)), 0 \leq z \leq 1. \tag{3}$$

where $B*(s)$ is the Laplace-Stieltjes transform of the service time probability distribution. Let $(\Omega, \Im, P)$ be the basic probability space, where $\Im$ is the $\sigma$-algebra generated by the input parametric sequences. We also define the increasing sequence of sigma-algebras $\Im_n$ by $\Im_n = \sigma\{A_m : 0 < m < n\}$ generated by the sequence of events $\{A_m\}$. Consequently the random variables $X_n$ are $\Im_n$-measurable and the random variable $A_{n+1}$is independent from the $F_n$ $\sigma$-algebras. With the above notations and using the properties of the conditional expectations

$$E(z^{X_{n+1}}/\Im_n) = z^{X_n - \delta_{X_n}} a(z) a.s. \tag{4}$$

## 3. THE MARTINGALE

We can define a martingale $M_n(z)$ with filtration $(\Im_n)$ by

$$M_0(z) = z^{X_0}$$

$$M_n(z) = z^{X_n} \frac{z^{\sum_{k=0}^{n-1} \delta_{X_k}}}{a(z)^n}, 0 \leq z \leq 1 \tag{5}$$

is an integrally positive martingale.

*Proof.* It is not difficult to see that the sequence $\{M_n(z), n \in N\}$ is a positive martingale since

$$E(M_{n+1}(z)/\Im_n) = M_n(z) \tag{6}$$

Moreover, from the Martingale theorem [7]it is integrable.

The quantity $\rho = \lambda\alpha$ is called the traffic intensity i.e. the mean number of arrivals per mean service time. □

## 4. STABILITY OF THE M/G/1 RETRIAL SYSTEM

We first study the instability of the M/G/1 retrial system.

**Theorem 1.** *Under the assumption $\rho > 1$, the M/G/1 retrial system is unstable, and additionally, we have a.s.*

$$lim_{n\to\infty} X_n = \infty$$

*Proof.* For every $0 < z \leq 1$ and every $n \in N$, the relation (5) implies that

$$E(Z^{X_{n+1}}/\Im_n) \leq \frac{a(z)}{z} z^{X_n} a.s. \tag{7}$$

Under the assumption $\rho > 1$ and since $a(.)$ is convex, then according to Takacs lemma [6] ( p .47), we can find $z_0$ such $c_0 = \frac{a(z_0}{z_0} < 1$. Consequently $E(z_0^{X_n}/\Im_n) \leq C_0 z_0^{X_n} \leq z_0^{X_n}$ a.s., which proves that the sequence $\{z_0^{X_n}, n \in N\}$ is an a.s. majorized by a constant $a = 1$ positive super martingale. According to [6] ( th II-2-9, p26), this sequence converges then a.s. On the other hand, by using the Dominated Convergence Theorem, we obtain

$$lim_n E(z_0^{X_n}) = E(lim_n z_0^{X_n} \tag{8}$$

By recurrence on $n$, we deduce that

$$E(z_0^{X_n}) \leq c_0^n E(z_0^{X_0} \leq c_0^n \tag{9}$$

Passing to the limit when $n$ tends to the infinity, we get

$$lim_n E(z_0^{X_n}) = E(lim_n z_0^{X_n}) = 0 \tag{10}$$

which implies that $lim_n E(z_0^{X_n}) = 0$ a.s. for $0 < z_0 < 1$. This leads to the result. □

The case where $\rho \leq 1$ remains for study.

We consider $\sigma$ as an arbitrary stopping time for $\Im_n$ and we define the random variable $\nu(\sigma)$ as the first instant after the time $\sigma$where the system comes back to its empty state. That is

$$\nu(\sigma) = inf\{n \geq 1 : X_{\sigma+n} = 0\}, if \sigma < \infty, \tag{11}$$

$$\nu(\sigma) = 0, otherwise,$$

with the convention that $inf\{\emptyset\} = +\infty$. The following theorem formulates a conservation law for the recall $M/G/1$ system and that is essential to prove the stability result.

**Theorem 2.** *For $0 < z \leq 1$ and under the assumption $\rho \leq 1$,*

$$E(1_{[\sigma<\infty,\nu(\sigma)<\infty]}\frac{z\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}{a(z)\nu(\sigma)}/\Im_\sigma) = 1_{[\sigma<\infty]}z^{X_\sigma}a.s. \tag{12}$$

*Proof.* We consider $\tau(\sigma) = \sigma + \nu(\sigma)$ a stopping time for $\{\Im_t, t \in N\}$. For every $t \geq 0$ , $\tau(\sigma)\wedge t$ and $\sigma\wedge t$ will still be the stopping times. It is clear that $\forall \sigma\wedge t \leq \tau(\sigma)\wedge t$. Since $\{M_n(z)\}_0^\infty$ is an integrable positive martingale, then according to [7] ( cor.IV-2-6,p.67), we have for every $0 < z \leq 1$ and $t \in N$

$$E(M_{\tau(\sigma)\wedge t}(z)/\Im_{\sigma\wedge t}) = M_{\sigma\wedge t}(z)a.s. \tag{13}$$

By using proposition II-1-3 of [7] ( p21) and the $\Im_{\sigma\wedge t}$ measurability of the event $[\sigma < t]$, the equality (13) writes up as follow

$$E(1_{[\sigma<t]}M_{\tau(\sigma)\wedge t}(z)/\Im_{\sigma\wedge t}M_{\sigma\wedge t}(z)a.s. \tag{14}$$

On the other hand, since $a(.)$ is convex and $\rho \leq 1$ then according to Takacs lemma [6]( p47], for $0 < z \leq 1$ we have $z < a(z)$. Consequently, for every $0 < z \leq 1$ and $t \in N$,

$$0 \leq 1_{[\sigma<t]}z_{X_{\tau(\sigma)\wedge t}}\frac{z^{\sum_{k=0}^{\tau(\sigma)\wedge t}}}{a(z)^{\tau(\sigma)\wedge t}} \leq 1_{[\sigma<t]}z^{-\sum_{k=0}^{\tau(\sigma)\wedge t}-1}\delta_{X_k} \tag{15}$$

□

## 5. MAIN RESULT

We formulate, now the main result,

**Theorem 3.** *Assume that $\rho = -\lambda B^{*'}(0) \leq 1$ , and the sequence of service times forms a renewal sequence, then there exists a sequence $\{\tau_n(n)\}_1^\infty$ of a.s. finite $\Im_n$-stopping times such that $X_{\tau_n} = 0$ on $\{\tau_n < \infty\}$ and $\tau_n + 1 \leq \tau_{n+1}, \forall n \in N*$. Additionally, if $\nu_{n+1} := \tau_{n+1} - \tau_n foralln > 0$, then the random variables $\{\tau_n(n)\}_2^\infty$ form an i.i.d. sequence independent of $\tau_1$ and*

$$E(\nu_{n+2}) = \frac{\Psi(1)}{1-\rho}, if\rho < 1, \tag{16}$$

$$E(\nu_{n+2}) = \infty, if\rho = 1, \tag{17}$$

*where* $\Psi(1) = \mathbf{e}$
$(\lambda\frac{}{\nu\int_0^1\frac{1-a(u)}{a(u)-u}du})$.

The proof of the theorem 3 follows the methodology of Baccelli and Makowski [1] using the above Martingale relations and the fact that for the M/G/1 Retrial Queue, the busy period satisfies the equation

$$E(\nu(\sigma)) = I_\infty exp(\frac{\lambda}{\nu}\int_0^1\frac{1-a(y)}{a(y)-y}dy$$

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