ON THE $M^X/G/1$ RETRIAL QUEUE WITH IMPATIENT CUSTOMERS

N.V. Djellab[1], K.N. Arrar[1], J.B. Baillon[2]

[1] *University of Annaba*

[2] *University Paris 1 Pantheon Sorbone*

[1] *Annaba, Algeria*

[2] *UFR 27, CERMSEM CNRS-UMR 8095, France*

djellab@yahoo.fr

In this work, we carry out a stochastic analysis of the $M/G/1$ retrial queue with batch arrivals and impatient customers. Our study includes: steady state joint distribution of the server state and the number of customers in retrial group, embedded Markov chain and stochastic decomposition for the number of customers in the system. We investigate also the asymptotic behaviour of the system under high and low retrial intensities.

Keywords: retrial queue, batch arrival, impatient customer, decomposition property, low retrial rate, convergence.

1. INTRODUCTION

In telephone networks, we can observe that a calling subscriber after some unsuccessful retrials gives up further repetitions and abandons the system. In queueing systems with repeated attempts (retrial queues), this phenomenon is represented by the set of probabilities $\{H_k,\ k \geq 1\}$, called the persistence function, where H_k is the probability that after the kth attempt fails, a customer will make the $(k+1)$th one. In general, it is assumed that the probability of a customer reinitiating after failure of a repeated attempt does not depend on the number of previous attempts (i.e. $H_2 = H_3 = ...$) [2]. In this paper, we consider a single server queueing system at which primary customers arrive in batches of size k (with probability $c_k(t),\ \ k \geq 1$) according to a Poisson stream with rate $\lambda > 0$. If the server is busy at the arrival epoch, then with probability $1 - H_1$ all these customers leave the system without service and with probability $0 < H_1 < 1$ join the retrial group (orbit); whereas if the server is idle, then one of the arriving customers begins his service and leaves the system after the service and the others go to the orbit. Let $c_k = \lim\limits_{k \to \infty} c_k(t),\ k \geq 1$. Thus, $C(z) = \sum\limits_{k=1}^{\infty} c_k z^k$ is the generating function of the steady state distribution of the batch size and $\overline{c} = C^{'}(1)$ is the mean batch size. The impatience phenomenon is represented by the probability $1 - H_1$ (the probability $H_2 = 1$). Any orbiting customer will repeatedly retry until the time at which he finds the server idle and starts his service. The retrial times are exponentially distributed with distribution function $T(x) = 1 - e^{-\theta x}, x \geq 0$, having finite mean $\frac{1}{\theta}$. The service times follow a general distribution with distribution function $B(x)$ and Laplace-Stieltjes transform $\widetilde{B}(s), Re(s) > 0$. Let $\beta_k = (-1)^k \widetilde{B}^{(k)}(0)$ be the kth moment of the service time about the origin and $\rho = \lambda H_1 \beta_1 \overline{c}$ be the traffic intensity. Finally, we admit the hypothesis of mutual independence between all random variables defined above.

2. STEADY STATE DISTRIBUTION OF THE SYSTEM STATE AND DECOMPOSITION PROPERTY

The state of the considered system at time t can be described by means of the process $\{C(t), N_o(t), \zeta(t), t \geq 0\}$, where $N_o(t)$ is the number of customers in the orbit, $C(t)$ is the state of the server at time t. We have that $C(t)$ is 0 or 1 depending on whether the server is idle or busy. If $C(t) = 1$, $\zeta(t)$ represents the elapsed service time of the customer in service at time t.
Let $\rho < 1$. Define

$$p_{0n} = \lim_{t\to\infty} P(C(t) = 0, N_o(t) = n),$$
$$p_{1n} = \int_0^{\infty} \left[\lim_{t\to\infty} \frac{d}{dx} P(C(t) = 1, \zeta(t) \leq x, N_o(t) = n)\right] dx.$$

In the previous work [1], we found the partial generating functions

$$P_0(z) = \sum_{n=0}^{\infty} z^n p_{0n} = \frac{H_1(1-\rho)}{\rho + H_1(1-\rho)} \mathbf{e}^{\left[\frac{\lambda}{\theta}\int_1^z \frac{1-\widetilde{B}(\lambda H_1(1-C(u)))\frac{C(u)}{u}}{\widetilde{B}(\lambda H_1(1-C(u)))-u} du\right]},$$
$$P_1(z) = \sum_{n=0}^{\infty} z^n p_{0n} = \frac{1-\widetilde{B}(\lambda H_1(1-C(z)))}{(\widetilde{B}(\lambda H_1(1-C(z)))-z)H_1} P_0(z).$$

We also investigated the steady state queue size distribution at departure epochs ξ_k (the time when the server enters the idle state for the k-th time). The obtained generating function of the steady state distribution of the embedded Markov chain has the following form:

$$\varphi(z) = \frac{1-\rho}{\overline{c}\rho} \frac{\widetilde{B}(\lambda H_1(1-C(z)))(1-C(z))}{\widetilde{B}(\lambda H_1(1-C(z)))-z} \mathbf{e}^{\left[\frac{\lambda}{\theta}\int_1^z \frac{1-\widetilde{B}(\lambda H_1(1-C(u)))\frac{C(u)}{u}}{\widetilde{B}(\lambda H_1(1-C(u)))-u} du\right]}. \quad (1)$$

At present, we establish that the right hand part of (1) can be decomposed into two factors: the first factor $\frac{1-\rho}{\overline{c}\rho}\frac{\widetilde{B}(\lambda H_1(1-C(z)))(1-C(z))}{\widetilde{B}(\lambda H_1(1-C(z)))-z}$ is the generating function for the number of customers at departure epochs in the ordinary $M^X/G/1$ queue with impatient customers (model M_∞), and the second one $\mathbf{e}^{\left[\frac{\lambda}{\theta}\int_1^z \frac{1-\widetilde{B}(\lambda H_1(1-C(u)))\frac{C(u)}{u}}{\widetilde{B}(\lambda H_1(1-C(u)))-u} du\right]} = \frac{P_0(z)}{P_0(1)}$ represents the generating function for the number of customers at the departure epochs in the corresponding retrial queue (model M_θ) given that the server is idle. Thus, stochastic decomposition property of the considered system can be expressed in the following manner:

$$\{0, N_{o\theta}(t), t \geq 0\} = \{0, N_{q\infty}(t), t \geq 0\} + \{0, R_\theta(t), t \geq 0\}.$$

The processes $\{0, N_{o\theta}(t), t \geq 0\}$ and $\{0, R_\theta(t), t \geq 0\}$ are related to the model M_θ, where $R_\theta(t)$ represents the number of customers in the orbit at time t given that the server is idle. The process $\{0, N_{q\infty}(t), t \geq 0\}$ is associated with the model M_∞, where $N_{q\infty}(t)$ is the number of customers in the waiting line at time t.

3. ASYMPTOTIC BEHAVIOUR OF THE SYSTEM

Now, we investigate the asymptotic behaviour of the system under low retrial rate $(\theta \to 0)$. To this end, the following result is established:

Theorem 1. *If $\beta_2 < \infty$, then as $\theta \to 0$, the number of customers in the orbit is asymptotically Gaussian with mean* $\frac{\lambda}{\theta}\frac{\overline{c}+\rho-1}{1-\rho}$ *and variance* $\frac{\lambda}{\theta}\frac{\beta_1+\rho\overline{c}\beta_1^2+C''(1)\beta_1^2-\beta_1\rho^3-\rho^2\overline{c}\beta_1^2+\overline{c}\rho^2\beta_2}{2(1-\rho)^2\beta_1^2}$.

In general, under high retrial intensity $(\theta \to \infty)$ the steady state distribution of a retrial queue converges to a limit, which is usually the steady state distribution of a certain limit system. In our case, it is intuitive that this is the model M_∞. We prove this heuristic argument with the help of the stochastic decomposition property.
Let

$$\begin{aligned} \pi_n(\theta) &= \lim_{k\to\infty} P(N_{o\theta}(\xi_k) = n), \\ \pi_n(\infty) &= \lim_{k\to\infty} P(N_{q\infty}(\xi_k) = n), \\ q_n(\theta) &= \lim_{k\to\infty} P(R_\theta(\xi_k) = n). \end{aligned}$$

Theorem 2. *The following inequalities take place*

$$2\frac{1-\rho}{\overline{c}\rho\widetilde{B}(\lambda H_1)}(1-q_0(\theta)) \leq \sum_{n=0}^{\infty}|\pi_n(\theta)-\pi_n(\infty)| \leq 2(1-q_0(\theta)), \quad \textit{where}$$

$$q_0(\theta) = \mathbf{e}^{\left[\frac{\lambda}{\theta}\int_1^z \frac{1-\widetilde{B}(\lambda H_1(1-C(u)))\frac{C(u)}{u}}{\widetilde{B}(\lambda H_1(1-C(u)))-u}du\right]} \quad \textit{and} \quad \rho=\lambda H_1\beta_1\overline{c}.$$

$$As \quad \theta \to \infty, \quad \sum_{n=0}^{\infty}|\pi_n(\theta)-\pi_n(\infty)| = 0\left(\frac{1}{\theta}\right).$$

REFERENCES

1. *Djellab N., Arrar K. N., Baillon J. B.* Asymptotic behaviour of the number of customers in retrial group of M/G/1 retrial queue with batch arrivals // International Conference Modern Stochastics: Theory and Applications II, Kyev, Ukraine. 2010. P. 7–11.
2. *Falin G. I., Templeton J. G. C.* Retrial Queues // Chapman and Hall 1997.