STOCHASTIC BOUNDS FOR THE MEAN CHARACTERISTICS OF AN M/G/1 QUEUE WITH GENERAL RETRIAL TIMES

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The main goal of this paper is to investigate various monotonicity properties of a single server retrial queue with a first-come-first-served (FCFS) orbit and general retrial times using the stochastic order relations (strong stochastic ($\leq_{st}$), increasing convex ($\leq_{icx}$), and Laplace ordering ($\leq_L$)) in order to derive performance indices bounds.

Keywords: Retrial queues, Stochastic ordering, Monotonicity, Ageing distributions.

1. INTRODUCTION

The retrial queueing system has been studied extensively due to its wide applicability. Apart from theoretical interest, it has been successfully applied in telephone switching systems, telecommunication networks, and computer networks [1].

In almost all models of retrial queues, the time between retrials for any customer is assumed to be exponentially distributed. In recent years, retrial queueing systems with general service times and nonexponential retrial time distribution have received little attention [3, 6]. An important characteristic of the general retrial times policy is that we always obtain analytical solutions in terms of closed-form expressions. The general retrial times policy arises naturally in problems where the server is required to search for customers, that is, this policy is related to many service systems where, after each service completion, the processor will spend a random amount of time in order to find the next item to be processed.

Many efforts have been devoted to derive performance measures such as queue length distribution, waiting times distribution, busy period distribution etc. in retrial queues. In many cases the behavior of the retrial queue is described by the Markov chain with spatially inhomogeneous infinitesimal generator (or transition probability matrix) caused by transitions due to repeated attempts. This spatial inhomogeneity often leads the analytical complexity and bounds and/or approximations are used instead.

Stochastic comparison methods have been used to produce bounds and approximations for queue length processes and waiting times in many queueing systems. For the detailed results about the comparison methods and their applications, e.g. see [4, 5]

In this paper we study some monotonicity properties similar to Boualem et al. [2], for an $M/G/1$ queue with general service times and nonexponential retrial time distribution under FCFS orbit discipline. The performance characteristics of such a system are available in Gómez-Corral [3]. We prove the monotonicity of the transition operator of the embedded Markov chain relative to strong stochastic ordering and convex ordering. We obtain comparability conditions for the distribution of the number of customers in the system. Inequalities are derived for the mean characteristics of the busy period, number of customers served during a busy period, number of orbit busy periods and waiting times.

2. DESCRIPTION OF THE QUEUEING SYSTEM

We consider a single server retrial queue with general service times and nonexponential retrial time distribution under FCFS orbit discipline. Primary customers arrive in a Poisson process with rate λ. If the server is free, the primary customer is served immediately and leaves the system after service completion. Otherwise, the customer leaves the service area and enters the retrial group in accordance with an FCFS discipline. We will assume that only the customer at the head of the orbit is allowed for access to the server. If the server is busy upon retrial, the customer joins the orbit again. Such a process is repeated until the customer finds the server idle and gets the requested service at the time of a retrial. Successive inter-retrial times of any customer follow an arbitrary law with common probability distribution function $A(x)$, Laplace-Stieltjes transform $L_A(s)$ and first moment α_1. The service times are independently and identically distributed with probability distribution function $B(x)$, Laplace-Stieltjes transform $L_B(s)$ and first two moments β_1, β_2. We suppose that inter-arrival times, retrial times and service times are mutually independent. The performance characteristics of such a system are available in Gómez-Corral [3].

Let τ_n be the time of the nth departure and Q_n the number of customers in the orbit just after the time τ_n. We have the following fundamental recursive equation:

$$Q_{n+1} = Q_n + v^{n+1} - \delta_{Q_{n+1}},$$

where v^{n+1} is the number of primary customers arriving at the system during the service time which ends at τ_{n+1}. Its distribution is given by $k_j = \int_0^\infty (\lambda x)^j (j!)^{-1} e^{-\lambda x} dB(x),\ j \geq 0$, with generating function $k(z) = \sum_{j\geq 0} k_j z^j = L_B(\lambda(1-z))$.

The Bernoulli random variable $\delta_{Q_{n+1}}$ is equal to 1 or 0 depending on whether the customer who leaves the system at time τ_{n+1} proceeds from the orbit or otherwise.

The sequence of random variables $\{Q_n\}$ forms an embedded Markov chain for our queueing system which is irreductible and aperiodic on the state-space $\mathbb{N}$. The inequality $\lambda\beta_1 < L_A(\lambda)$ is a necessary and suffcient condition for the system to be stable [3].

2.1. Some useful lemmas. This subsection presents several useful lemmas which will be used later in establishing the main results. Consider two $M/G/1$ retrial queues with classical retrial policy and feedback with parameters $\lambda^{(i)}$, $\theta^{(i)}$ and $B^{(i)}$, $i = 1, 2$. Let $k_j^{(i)}$ be the distribution of the number of primary calls which arrive during the service time of a call in the ith system.

The following two lemmas turns out to be a useful tool for showing the monotonicity properties of the embedded Markov chain.

Lemma 1. *If $\lambda^{(1)} \leq \lambda^{(2)}$ and $B^{(1)} \leq_s B^{(2)}$, then $\{k_n^{(1)}\} \leq_s \{k_n^{(2)}\}$, where $\leq_s$ is one of the symbols $\leq_{st}$ or $\leq_{icx}$.*

Proof. the proof is known in the more general setting of a random summation. □

Lemma 2. *If $\lambda^{(1)} \leq \lambda^{(2)}$ and $B^{(1)} \leq_L B^{(2)}$, then $\{k_n^{(1)}\} \leq_L \{k_n^{(2)}\}$.*

Proof. We have

$$k^{(i)}(z) = \sum_{n \geq 0} k_n^{(i)} z^n = L_{B^{(i)}}(\lambda^{(i)}(1-z)), \ i = 1, 2$$

where $k^{(1)}(z)$, $k^{(2)}(z)$ are the corresponding distributions of the number of new arrivals during a service time.
Let $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_L B^{(2)}$. To prove that $\{k_n^{(1)}\} \leq_L \{k_n^{(2)}\}$, we have to establish that

$$L_{B^{(1)}}(\lambda^{(1)}(1-z)) \geq L_{B^{(2)}}(\lambda^{(2)}(1-z)).$$

□

3. MONOTONICITY PROPERTIES OF THE EMBEDDED MARKOV CHAIN

The one-step transition probabilities of the embedded Markov chain $\{Q_n, n \geq 1\}$ is defined in the following formulae
$p_{nm} = (1 - L_A(\lambda))k_{m-n} + L_A(\lambda)k_{m-n+1}$, for $n \neq 0$ and $m \geq 0$,
$p_{0m} = k_m$, for $m \geq 0$.

Let $\mathbf{T}$ be the transition operator of an embedded Markov chain which associates to every distribution $\omega = \{\omega_m\}_{m \geq 0}$ a distribution $\mathbf{T}\omega = \{\nu_m\}_{m \geq 0}$ such that $\nu_m = \sum_{n \geq 0} \omega_n p_{nm}$. From Stoyan [5], T is monotone with respect to $\leq_{st}$ if and only if

$$\overline{p}_{nm} - \overline{p}_{n-1m} \geq 0 \ \text{ for all } \ n \ \text{ and } \ m,$$

and is monotone with respect to $\leq_v$ if and only if

$$\overline{\overline{p}}_{n-1m} + \overline{\overline{p}}_{n+1m} - 2\overline{\overline{p}}_{nm} \geq 0 \ \text{ for all } \ n \ \text{ and } \ m.$$

Here, $\bar{p}_{n,m} = \sum_{l=m}^{\infty} p_{n,l}$ and $\bar{\bar{p}}_{n,m} = \sum_{l=m}^{\infty} \bar{p}_{n,l}$.

Theorem 1. $\mathbf{T}$ *is monotone with respect to the orders* $\leq_{st}$ *and* $\leq_{icx}$.

Proof. In our case:

$$\begin{aligned}\overline{p}_{nm} &= (1-L_A(\lambda))\overline{k}_{m-n} + L_A(\lambda)\overline{k}_{m-n+1} = (1-L_A(\lambda))k_{m-n} + \overline{k}_{m-n+1},\\ \overline{\overline{p}}_{nm} &= (1-L_A(\lambda))\overline{k}_{m-n} + \overline{\overline{k}}_{m-n+1}.\end{aligned}$$

Thus

$$\begin{aligned}\overline{p}_{nm} - \overline{p}_{n-1m} &= (1-L_A(\lambda))k_{m-n} + L_A(\lambda)k_{m-n+1} \geq 0,\\ \overline{\overline{p}}_{n-1m} + \overline{\overline{p}}_{n+1m} - 2\overline{\overline{p}}_{nm} &= (1-L_A(\lambda))k_{m-n-1} + L_A(\lambda)k_{m-n} \geq 0.\end{aligned}$$

□

In the following two theorems, we give comparability conditions of two transition operators. Consider two $M/G/1$ retrial queues with nonexponential retrial times with parameters $\lambda^{(1)}$, $A^{(1)}$, $B^{(1)}$ and $\lambda^{(2)}$, $A^{(2)}$, $B^{(2)}$ respectively. Let $\mathbf{T}^1$ and $\mathbf{T}^2$ be the transition operators of the corresponding embedded Markov chains.

Theorem 2. *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ *and* $A^{(1)} \leq_L A^{(2)}$, *then* $\mathbf{T}^1 \leq_s \mathbf{T}^2$, *i.e. for any distribution* ω, *we have* $\mathbf{T}^1\omega \leq_s \mathbf{T}^2\omega$, *where* $\leq_s$ *is one of the symbols* $\leq_{st}$ *or* $\leq_{icx}$.

Proof. From Stoyan [5], it is well known that to prove $\mathbf{T}^1 \leq_s \mathbf{T}^2$, we have to show the following numerical inequalities for the one-step transition probabilities $p^{(1)}_{nm}, p^{(2)}_{nm}$:

$$\overline{p}^{(1)}_{nm} \leq \overline{p}^{(2)}_{nm}, \quad \forall n,\ m, \quad (\text{for } \leq_s = \leq_{st}), \tag{1}$$

$$\overline{\overline{p}}^{(1)}_{nm} \leq \overline{\overline{p}}^{(2)}_{nm}, \quad \forall n,\ m, \quad (\text{ for } \leq_s = \leq_{icx}), \tag{2}$$

To prove inequality (1), we have

$$\overline{p}^{(1)}_{nm} = (1 - L_{A^{(1)}}(\lambda^{(1)}))k^{(1)}_{m-n} + \overline{k}^{(1)}_{m-n+1}.$$

Since $\lambda^{(1)} \leq \lambda^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $L_{A^{(1)}}(\lambda^{(1)}) \geq L_{A^{(2)}}(\lambda^{(2)})$ and

$$\overline{p}^{(1)}_{nm} \leq (1 - L_{A^{(2)}}(\lambda^{(2)}))k^{(1)}_{m-n} + \overline{k}^{(1)}_{m-n+1}.$$

But

$$(1 - L_{A^{(2)}}(\lambda^{(2)}))k^{(1)}_{m-n} + \overline{k}^{(1)}_{m-n+1} = (1 - L_{A^{(2)}}(\lambda^{(2)}))\overline{k}^{(1)}_{m-n} + L_{A^{(2)}}(\lambda^{(2)})\overline{k}^{(1)}_{m-n+1}.$$

By Lemma 1, we have $\overline{k}^{(1)}_n \leq \overline{k}^{(2)}_n$, $\forall n \geq 0$.
Using these inequalities we get:

$$\overline{p}^{(1)}_{nm} \leq (1 - L_{A^{(2)}}(\lambda^{(2)}))\overline{k}^{(2)}_{m-n} + L_{A^{(2)}}(\lambda^{(2)})\overline{k}^{(2)}_{m-n+1} = \overline{p}^{(2)}_{nm}.$$

Following the technique above it is possible to establish inequality (2). □

Theorem 3. *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_L B^{(2)}$ *and* $A^{(1)} \leq_L A^{(2)}$, *then* $\mathbf{T}^1 \leq_L \mathbf{T}^2$.

Proof. Let $\omega = (\omega_m)$ be a distribution and $\mathbf{T}_\omega = \nu = (\nu_m)$, where
$\nu_m = \sum\limits_{n\geq 0} \omega_n p_{nm} = \omega_0 k_m + \sum\limits_{n\geq 1} \omega_n p_{nm}$, for all $m \geq 0$.
Let $k(z) = \sum\limits_{n\geq 0} k_n z^n$ and $\omega(z) = \sum\limits_{n\geq 0} \omega_n z^n$ be the generating functions of (k_n) and (ω_n) respectively. The generating function of ν is given by

$$\begin{aligned} G(z) &= \sum_{m\geq 0} \nu_m z^m = \sum_{m\geq 0}\sum_{n\geq 0} \omega_n p_{nm} z^m = \sum_{m\geq 0} [\omega_0 k_m + \sum_{n\geq 1} \omega_n p_{nm}] z^m \\ &= \omega_0 k(z) + \frac{1}{z} k(z)(\omega(z) - \omega_0)(z + (1-z)L_A(\lambda)). \end{aligned}$$

If the conditions of Theorem 3 are fulfilled, then $k^{(1)}(z) \geq k^{(2)}(z)$ by Lemma 2 and $(1-z)L_{A^{(1)}}(\lambda^{(1)}) \geq (1-z)L_{A^{(2)}}(\lambda^{(2)})$, $\forall\ z \in [0,1]$. Hence $G^{(1)}(z) \geq G^{(2)}(z)$. □

4. BOUNDS FOR THE MEAN CHARACTERISTICS OF THE SYSTEM

The main characteristics of a system busy period, the orbit busy period and waiting time are:
L: the length of a system busy period,
I: the number of service completions occurring during $(0, L]$,
N_b: the number of orbit busy periods which take place in $(0, L]$,
W: the waiting time.
Gómez-Corral [3] shows that, if $\lambda\beta_1 < L_A(\lambda)$, then
$E(L) = \frac{\beta_1}{L_A(\lambda)-\lambda\beta_1}$, $E(I) = \frac{L_A(\lambda)}{L_A(\lambda)-\lambda\beta_1}$, $E(N_b) = \frac{1-L_B(\lambda)}{L_B(\lambda)}$, and $E(W) = \frac{\lambda\beta_2+2\beta_1(1-L_A(\lambda))}{2(L_A(\lambda)-\lambda\beta_1)}$.

Suppose once more that we have two $M/G/1$ retrial queues with nonexponential retrial times with parameters $\lambda^{(1)}$, $A^{(1)}$, $B^{(1)}$ and $\lambda^{(2)}$, $A^{(2)}$, $B^{(2)}$, respectively. Let $L^{(i)}$, $I^{(i)}$, $N_b^{(i)}$ and $W^{(i)}$ be the length busy period, the number of customers served during a busy period, the number of orbit busy periods which take place in $(0, L^{(i)}]$ and the waiting time respectively, in the i-th system, $i = 1, 2$.

Theorem 4. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $E(L^{(1)}) \leq E(L^{(2)})$, and $E(I^{(1)}) \leq E(I^{(2)})$, where $\leq_s$ is one of the symbols $\leq_{st}$, $\leq_{icx}$, $\leq_L$.*

Proof. The quantities $E(L)$ and $E(I)$ which are increasing with respect to λ and β_1, decreasing with respect to $L_A(.)$. Under conditions of Theorem 4, we obtain the desired inequalities. Recall that $X \leq_s Y$ implies $E(X^n) \leq E(Y^n)$ for all n. □

Theorem 5. *For any $M/G/1$ retrial queue,*
$E(L) \leq \frac{\beta_1}{e^{-\lambda\alpha_1}-\lambda\beta_1}$, *and* $E(I) \leq \frac{e^{-\lambda\alpha_1}}{e^{-\lambda\alpha_1}-\lambda\beta_1}$.
If A and B are $\mathcal{L}$, then $E(L) \geq \frac{\beta_1(1+\lambda\alpha_1)}{1-\lambda\beta_1(1+\lambda\alpha_1)}$, *and* $E(I) \geq \frac{1}{1-\lambda\beta_1(1+\lambda\alpha_1)}$.

Proof. We consider auxiliary $M/D/1$ and $M/M/1$ retrial queues with the same arrival rates λ, mean service times β_1 and mean retrial times α_1. A is Dirac distribution at α_1 for the $M/D/1$ system, and is exponential distribution for the $M/M/1$ system. Using the theorem above we obtain the stated results. □

Theorem 6. *If* $\lambda^{(1)} \leq \lambda^{(2)}$*,* $B^{(1)} \leq_{st} B^{(2)}$ *and* $A^{(1)} \leq_L A^{(2)}$*, then*

$$E(N_b^{(1)}) \leq E(N_b^{(2)}), \ \ and \ \ E(W^{(1)}) \leq E(W^{(2)}).$$

Proof. The quantities $E(N_b)$ and $E(W)$ are increasing with respect to λ, β_1 and β_2, decreasing with respect to $L_B(.)$ and $L_A(.)$. Under the conditions of Theorem 6 we obtain the desired inequalities. □

Theorem 7. *For any* $M/G/1$ *retrial queue,*
$E(N_b) \leq e^{\lambda\beta_1} - 1$*, and* $E(W) \leq \frac{\lambda\beta_2 + 2\beta_1(1-e^{-\lambda\alpha_1})}{2(e^{-\lambda\alpha_1} - \lambda\beta_1)}$.
If B *and* A *are* $\mathcal{L}$*, then*
$E(N_b) \geq \lambda\beta_1$*, and* $\frac{\lambda\beta_2(1+\lambda\alpha_1)+2\lambda\beta_1\alpha_1}{2(1-\lambda\beta_1(1+\lambda\alpha_1))} \leq E(W) \leq \frac{2\lambda\beta_1^2 + 2\beta_1(1-e^{-\lambda\alpha_1})}{2(e^{-\lambda\alpha_1} - \lambda\beta_1)}$.

Proof. The proof is similar to that of Theorem 5. In addition, if a given distribution F is $\mathcal{L}$ then $\beta_2 \leq 2\beta_1^2$. □

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