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LINEAR-FRACTIONAL PROGRAMMING: PROBLEMS OF OPTIMIZATION OF INHOMOGENEOUS FLOWS IN THE GENERALIZED NETWORKS

L.A. Pilipchuk

Belarussian State University
4, Nezalezhnosti Ave., 220050, Minsk, BELARUS

Abstract: Here we consider the linear-fractional non-homogeneous flow programming optimization problem with additional constraints of general kind. We obtain the increment of the objective function using network properties of the problem and principles of decomposition of a support. In the received formulas for calculation of reduced costs only the part of system of potentials is used.

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1. Mathematical Model

Let $G = (I, U)$ be a finite oriented connected multigraph (multinetwork) without multiple arcs and loops, where I is a set of nodes and $U \subset I \times I$ is a set of multiarcs. Assume that a finite non-empty set $K = \{1, \ldots, |K|\}$ of different products (commodities) is transported through the multinetwork G. Let us denote a connected network corresponding to a certain type of flow $k \in K$: $G^k = (I^k, U^k)$, $I^k \subseteq I$, $U^k = \{(i,j)^k : (i,j) \in \widehat{U}^k\}$, $\widehat{U}^k \subseteq U$ – a set of arcs of the network G carrying the flow of type $k \in K$, $I^k = I(U^k), I(U^k) = \{i \in I : i \in I^k\}$ is the set of nodes used for transporting (producing/consuming/transiting) the

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k^{th} product. In order to distinguish the products, which can simultaneously pass through an multiarc $(i,j) \in U$, we introduce the set $K(i,j) = \{k \in K : (i,j) \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$. Now let us define a set U_0 as an arbitrary subset of multiarcs of the multinetwork G, $U_0 \subseteq U$. Each multiarc $(i,j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i,j) \subseteq K(i,j), |K_0(i,j)| > 1$. For all multiarcs $(i,j) \in U$ we assume the amount of each product $k \in K(i,j)$ to be nonnegative. For products from a set $K_1(i,j)$, such that $K_1(i,j) = K(i,j) \backslash K_0(i,j)$, if $(i,j) \in U_0$ and $K_1(i,j) \subseteq K(i,j)$, if $(i,j) \in U \setminus U_0$. Moreover, each multiarc $(i,j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i,j)$, where $K_1(i,j) \subseteq K(i,j)$ is an arbitrary subset of products transported through the multiarc (i,j).

On the described multinetwork G we consider the linear-fractional nonhomogeneous flow programming optimization problem with additional constraints of general kind:

$$f(x) = \frac{p(x)}{q(x)} = \frac{\sum\limits_{(i,j)\in U} \sum\limits_{k\in K(i,j)} p_{ij}^k x_{ij}^k + \beta}{\sum\limits_{(i,j)\in U} \sum\limits_{k\in K(i,j)} q_{ij}^k x_{ij}^k + \gamma} \longrightarrow \max, \tag{1}$$

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = a_i^k, i \in I^k, k \in K; \tag{2}$$

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, p = \overline{1,l}; \tag{3}$$

$$\sum_{k\in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, x_{ij}^k \geq 0, k \in K_0(i,j), (i,j) \in U_0; \tag{4}$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, k \in K_1(i,j), (i,j) \in U; \tag{5}$$

$$x_{ij}^k \geq 0, k \in K(i,j) \setminus K_1(i,j), (i,j) \in U \setminus U_0, \tag{6}$$

where $I_i^+(U^k) = \{j \in I^k : (i,j) \in U^k\}, I_i^-(U^k) = \{j \in I^k : (j,i) \in U^k\}$; x_{ij}^k – amount of the k^{th} product transported through an multiarc (i,j); d_{ij}^k – carrying capacity of an multiarc (i,j) for the k^{th} product; d^0_{ij} – aggregate capacity of an multiarc $(i,j) \in U_0$ for a total amount of products $K_0(i,j)$; λ^{kp}_{ij} – weight of a unit of the k^{th} product transported through an multiarc (i,j) in the p^{th} additional constraint; μ^k_{ij} – a flow transformation coefficient for arc $(i,j)^k$, $\mu^k_{ij} \in]0,1]$; α_p – total weighted amount of products imposed by the p^{th} additional constraint; a^k_i – intensity of a node i for the k^{th} product, $p^k_{ij}, q^k_{ij}, \beta, \gamma, a^k_i \in \mathbf{R}$.Let's assume, that the denominator

$$q(x) = \sum_{(i,j)\in U} \sum_{k\in K(i,j)} q^k_{ij} x^k_{ij} + \gamma$$

of objective function $f(x)$ does not change a sign for the set of multiflows $x \in X$. Without restriction of a generality we shall assume, that $q(x) > 0, \forall x \in X$.

2. The General Solution of the Homogeneous System

Let x be a multiflow of the problem $(1)-(6)$ i. e. components of the vector x meet the conditions $(2)-(6)$. Along with the multiflow x let us define support multiflow $\{x, U_S\}$ as a pair [1], containing of an arbitrary multiflow x and a support [5] U_S of multigraph $G = \{I, U\}$ of the problem (1) – (6), $U_S = \{U^k_S, k \in K, U^*\}$, $U^k_S \subset U^k, k \in K$; $U^* \subseteq \overline{U}_0, \overline{U}_0 = \left\{(i,j) \in U_0 : |K^0_S(i,j)| > 1\right\}$, $K_S(i,j) = \{k \in K(i,j) : (i,j)^k \in U^k_S\}$, $(i,j) \in U$, $K^0_S(i,j) = K_S(i,j) \bigcap K_0(i,j), (i,j) \in U_0$ of the problem $(1)-(6)$. Let's consider some other multiflow

$$\overline{x} = \left(\overline{x}^k_{ij} = x^k_{ij} + \Delta x^k_{ij} : (i,j) \in U, k \in K(i,j)\right)$$

Then $\Delta x = \left(\Delta x^k_{ij}, (i,j) \in U, k \in K(i,j)\right)$ is the vector of flow increments along the multiarc $(i,j) \in U$. Let us denote

$$\begin{aligned} z_{ij} &= \sum_{k\in K_0(i,j)} x^k_{ij}, \\ \overline{z}_{ij} &= \sum_{k\in K_0(i,j)} \overline{x}^k_{ij}, \\ \Delta z_{ij} = \overline{z}_{ij} - z_{ij} &= \sum_{k\in K_0(i,j)} \Delta x^k_{ij}, \ (i,j) \in U_0. \end{aligned} \tag{7}$$

Since the multiflow $\overline{x}$ meets the conditions $(2)-(6)$ then the following relations are true:

$$\sum_{j\in I_i^+(U^k)} \overline{x}_{ij}^k - \sum_{j\in I_i^-(U^k)} \mu_{ji}^k \overline{x}_{ji}^k = a_i^k,\ i\in I^k,\ k\in K, \tag{8}$$

$$\sum_{(i,j)\in U}\sum_{k\in K(i,j)} \lambda_{ij}^{kp}\overline{x}_{ij}^k = \alpha_p,\ p=\overline{1,l}, \tag{9}$$

$$\sum_{k\in K_0(i,j)} \overline{x}_{ij}^k \le d_{ij}^0,\ (i,j)\in U^*. \tag{10}$$

where the part of the constraints (4) are written down only for the support multiarcs U^* [2, 5]. Subtracting from (8)–(10) the corresponding constraints (2)–(4), we obtain:

$$\sum_{j\in I_i^+(U^k)} \Delta x_{ij}^k - \sum_{j\in I_i^-(U^k)} \mu_{ji}^k \Delta x_{ji}^k = 0,\ i\in I^k,\ k\in K, \tag{11}$$

$$\sum_{(i,j)\in U}\sum_{k\in K(i,j)} \lambda_{ij}^{k}\Delta x_{ij}^k = 0,\ p=\overline{1,l}, \tag{12}$$

$$\sum_{k\in K_0(i,j)} \Delta x_{ij}^k = \Delta z_{ij},\ (i,j)\in U^*, \tag{13}$$

where Δz_{ij} is defined by formula (7).

Let us order the components Δx of solution of system (11)–(13) according [5] of the following way:

$$\Delta x = (\Delta x_L, \Delta x_B, \Delta x_N),$$

where

$$\Delta x_L = (\Delta x_{ij}^k, (i,j)^k\in U_L^k, k\in K), \Delta x_B = (\Delta x_{ij}^k, (i,j)^k\in U_B^k, k\in K),$$

$$\Delta x_N = (\Delta x_{ij}^k, (i,j)^k\in U_N^k, k\in K), U_N^k = U^k\backslash \left(U_L^k \bigcup U_B^k\right).$$

In [5] we investigated theoretical-graphical properties of the structure of the support $U_S = \{U_S^k, k\in K\}$ of the multigraph $G=\{I,U\}$ for the problem (1) – (6). The aggregation of sets $U_S = \{U_S^k, k\in K\}$ includes the support $U_L = \{U_L^k, k\in K\}$ of multigraph G for system (2) and the set $U_B = \{U_B^k, k\in K\}$ of bicycling arcs [2, 5].

The general solution of the homogeneous system (11) is the following [2, 5]:

$$\Delta x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} \Delta x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho), (i,j)^k \in U_L^k, k \in K, \tag{14}$$

where $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho),\ (i,j)^k \in U^k)$ – characteristic vector, entailed by arc $(\tau,\rho)^k \in U^k \setminus U_L^k$ concerning a support U_L^k for system (2), $k \in K$ [1, 2, 5].

3. Formulae for Increment of Linear-Fractional Objective Function

We obtain the increment of the objective function (1) the extreme linear-fractional non-homogeneous problem (1)–(6) of flow programming with additional constraints.

$$\Delta f = f(\overline{x}) - f(x) =$$

$$= f(x + \Delta x) - f(x) = \frac{p(x+\Delta x)}{q(x+\Delta x)} - f(x) =$$

$$= \frac{p(x+\Delta x) - f(x)q(x+\Delta x)}{q(x+\Delta x)} =$$

$$= \left[\left(\sum_{(i,j)\in U}\sum_{k\in K(i,j)} p_{ij}^k(x_{ij}^k + \Delta x_{ij}^k) + \beta\right) -\right.$$

$$\left. -f(x)\left(\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k(x_{ij}^k + \Delta x_{ij}^k) + \gamma\right)\right] \Big/$$

$$\Big/ \left(\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k(x_{ij}^k + \Delta x_{ij}^k) + \gamma\right) = \Big(p(x) +$$

$$+ \sum_{(i,j)\in U}\sum_{k\in K(i,j)} p_{ij}^k \Delta x_{ij}^k - f(x)\Big(q(x) + \sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k \Delta x_{ij}^k\Big)\Big) \Big/$$

$$\Big/ \left(\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k(x_{ij}^k + \Delta x_{ij}^k) + \gamma\right)$$

or

$$\Delta f = \frac{\sum\limits_{(i,j)\in U}\sum\limits_{k\in K(i,j)} p_{ij}^k \Delta x_{ij}^k - f(x)\sum\limits_{(i,j)\in U}\sum\limits_{k\in K(i,j)} q_{ij}^k \Delta x_{ij}^k}{\sum\limits_{(i,j)\in U}\sum\limits_{k\in K(i,j)} q_{ij}^k (x_{ij}^k + \Delta x_{ij}^k) + \gamma} \tag{15}$$

Let's transform expressions in a numerator and a denominator for (15).

$$\sum_{(i,j)\in U}\sum_{k\in K(i,j)} p_{ij}^k \Delta x_{ij}^k = \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \Delta x_{\tau\rho}^k \Delta_P^k(\tau,\rho), \tag{16}$$

where

$$\Delta_P^k(\tau,\rho) = p_{\tau\rho}^k + \sum_{(i,j)^k\in U_L^k} p_{ij}^k \delta_{ij}^k(\tau,\rho), \tag{17}$$

and

$$\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k \Delta x_{ij}^k = \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \Delta x_{\tau\rho}^k \Delta_Q^k(\tau,\rho), \tag{18}$$

where

$$\Delta_Q^k(\tau,\rho) = q_{\tau\rho}^k + \sum_{(i,j)^k\in U_L^k} q_{ij}^k \delta_{ij}^k(\tau,\rho). \tag{19}$$

As $\overline{x} = \Big(\overline{x}_{ij}^k : (i,j)\in U,\ k\in K(i,j))\Big)$ is multiflow, $\overline{x}_{ij}^k = x_{ij}^k + \Delta x_{ij}^k$, that for the denominator $g(\overline{x})$ of the objective function (1) the inequality is true:

$$g(\overline{x}) = \sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k (x_{ij}^k + \Delta x_{ij}^k) + \gamma > 0$$

Hence the sign on an increment Δf of the objective function depends only on numerator.

Let's substitute (16) and (18) into numerator of the increment of the objective function (15). We have

$$N_{\Delta f} = \sum_{(i,j)\in U}\sum_{k\in K(i,j)} p_{ij}^k \Delta x_{ij}^k - f(x)\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k \Delta x_{ij}^k =$$

$$= \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \Delta^k(\tau,\rho)\Delta x_{\tau\rho}^k, \tag{20}$$

where

$$\Delta^k(\tau,\rho) = \Delta_P^k(\tau,\rho) - f(x)\Delta_Q^k(\tau,\rho),\ (\tau,\rho)^k \in U^k\setminus U_L^k,\ k\in K. \tag{21}$$

For performance of decomposition of an increment of the objective function we shall substitute (14) into (12).

$$\sum_{k\in K}\sum_{(i,j)^k\in U^k}\lambda_{ij}^{kp}\Delta x_{ij}^k=\sum_{k\in K}\sum_{(i,j)^k\in U_L^k}\lambda_{ij}^{kp}\Delta x_{ij}^k+\sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k}\lambda_{\tau\rho}^{kp}\Delta x_{\tau\rho}^k=$$

$$=\sum_{k\in K}\sum_{(i,j)^k\in U_L^k}\lambda_{ij}^{kp}\left[\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k}\Delta x_{\tau\rho}^k\delta_{ij}^k(\tau,\rho)\right]+\sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k}\lambda_{\tau\rho}^{kp}\Delta x_{\tau\rho}^k=$$

$$=\sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k}\left(\lambda_{\tau\rho}^{kp}+\sum_{(i,j)^k\in U_L^k}\lambda_{ij}^{kp}\delta_{ij}^k(\tau,\rho)\right)\Delta x_{\tau\rho}^k=0, \tag{22}$$

$$\Lambda_{\tau\rho}^{kp}=\lambda_{\tau\rho}^{kp}+\sum_{(i,j)^k\in U_L^k}\lambda_{ij}^{kp}\delta_{ij}^k(\tau,\rho),(\tau,\rho)^k\in U^k\backslash U_L^k,\ p=\overline{1,l}.$$

The equations (22) are transformed to the following kind:

$$\sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k}\Lambda_{\tau\rho}^{kp}\Delta x_{\tau\rho}^k=0,\ p=\overline{1,l}. \tag{23}$$

Similarly we shall transform the equations of system (13).

$$\sum_{k\in K_0(i,j)}\sum_{(\tau,\rho)^k\in U^k\backslash U_L^k}\delta_{ij}(B_{\tau\rho}^k)\Delta x_{\tau\rho}^k=\Delta z_{ij},(i,j)\in U^*, \tag{24}$$

where

$$\delta_{ij}(B_{\tau\rho}^k)=\begin{cases}\delta_{ij}^k(\tau,\rho),k\in K_0(i,j),\\ 0,k\notin K_0(i,j),\\ (i,j)\in U_0,(\tau,\rho)^k\in U^k\backslash U_L^k,k\in K.\end{cases}$$

or

$$\begin{gathered}\sum_{k\in K_0(i,j)}\sum_{(\tau,\rho)^k\in U_B^k}\delta_{ij}(B_{\tau\rho}^k)\Delta x_{\tau\rho}^k=\\ =\Delta z_{ij}-\sum_{k\in K_0(i,j)}\sum_{(\tau,\rho)^k\in U_N^k}\delta_{ij}(B_{\tau\rho}^k)\Delta x_{\tau\rho}^k,\ (i,j)\in U^*.\end{gathered} \tag{25}$$

Thus, we obtain the system of linear algebraic equations (23), (25). Let's present system (23), (25) in the matrix form:

$$D\Delta x_B=\tilde\beta,\ \tilde\beta=\begin{pmatrix}\tilde\beta_p, & p=\overline{1,l},\\ \tilde\beta_{l+\xi(i,j)}, & (i,j)\in U^*\end{pmatrix}, \tag{26}$$

$$\Delta x_B = \left(\Delta x_{ij}^k,\ (i,j)^k \in U_B^k,\ k \in K\right),$$

$$\tilde{\beta}_p = -\sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U_N^k} \Lambda_{\tau\rho}^{kp}\Delta x_{\tau\rho}^k,\ p=\overline{1,l},$$

$$\tilde{\beta}_{l+\xi(i,j)} = \Delta z_{ij} - \sum_{k\in K_0(i,j)}\ \sum_{(\tau,\rho)^k\in U_N^k} \delta_{ij}(B_{\tau\rho}^k)\Delta x_{\tau\rho}^k,\ (i,j)\in U^*.$$

where

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, D_1 = (\Lambda_{\tau\rho}^{kp}, p=\overline{1,l},\ t(\tau,\rho)^k = \overline{1,|U_B|}),$$

$$D_2 = (\delta_{ij}(B_{\tau\rho}^k), \xi(i,j) = \overline{1,|U^*|}, t(\tau,\rho)^k = \overline{1,|U_B|}),$$

$\xi = \xi(i,j)$ – number of the arc $(i,j)\in U^*, \xi\in\{1,2,\ldots,|U^*|\}$. The numbers $\delta_{ij}(B_{\tau\rho}^k)$ are calculated as follows:

$$\delta_{ij}(B_{\tau\rho}^k) = \begin{cases} \delta_{ij}^k(\tau,\rho), k\in K_0(i,j), \\ 0, k\notin K_0(i,j), \\ (i,j)\in U_0, (\tau,\rho)^k \in U^k\backslash U_L^k, k\in K. \end{cases}$$

where $B_{\tau\rho}^k = U_L^k \bigcup (\tau,\rho)^k$ – a bicycle, entailed by the arc $(\tau,\rho)^k \in U^k\setminus U_L^k, k\in K$. Also, for each bicycle $B_{\tau\rho}^k$, entailed by an arc $(\tau,\rho)^k\in U_B^k$ we consider the determinants $\Lambda_{\tau\rho}^{kp}$ [5, 6] of the bicycle $B_{\tau\rho}^k$ for the equations (3) with numbers $p=\overline{1,l}$:

$$\Lambda_{\tau\rho}^{kp} = \sum_{(i,j)^k\in B_{\tau\rho}^k} \lambda_{ij}^{kp}\delta_{ij}^k(\tau,\rho), (\tau,\rho)^k\in U^k\backslash U_L^k,$$

or

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k\in U_L^k} \lambda_{ij}^{kp}\delta_{ij}^k(\tau,\rho), (\tau,\rho)^k\in U^k\backslash U_L^k.$$

Since $U_S = \{U_S^k, k\in K, U^*\}$ is a support of the multigraph $G=\{I,U\}$ for the problem (1) – (6) [4, 5], then $\det D \neq 0$. We have

$$\Delta x_B = D^{-1}\tilde{\beta}. \tag{27}$$

Let's denote through $D^{-1} = = (\nu_{zq} : z = \overline{1,\widetilde{t}},\ q=\overline{1,\widetilde{t}})$ – elements of a inverse matrix, $\widetilde{t} = \sum\limits_{k\in K} |U_B^k|$.

Let's present (27) as (28).

$$\Delta x^k_{\tau\rho} = \sum_{p=1}^{l} \nu_{t(\tau,\rho)^k,p}\widetilde{\beta}_p + \sum_{(i,j)\in U^*} \nu_{t(\tau,\rho)^k,l+\xi(i,j)}\widetilde{\beta}_{l+\xi(i,j)},$$
$$(\tau,\rho)^k \in U^k_B,\ k\in K,\ \widetilde{\beta} = \begin{pmatrix} \widetilde{\beta}_p, & p=\overline{1,l}, \\ \widetilde{\beta}_{l+\xi(i,j)}, & (i,j)\in U^*, \end{pmatrix}, \tag{28}$$

$$\widetilde{\beta}_p = -\sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_N} \Lambda^{kp}_{\tau\rho}\Delta x^k_{\tau\rho},\ p=\overline{1,l}, \tag{29}$$

$$\widetilde{\beta}_{l+\xi(i,j)} = \Delta z_{ij} - \sum_{k\in K_0(i,j)}\ \sum_{(\tau,\rho)^k\in U^k_N} \delta_{ij}(B^k_{\tau\rho})\Delta x^k_{\tau\rho}, (i,j)\in U^*. \tag{30}$$

Let's substitute (28) into numerator of the increment of the objective function (20):

$$\sum_{(i,j)\in U}\ \sum_{k\in K(i,j)} p^k_{ij}\Delta x^k_{ij} - f(x)\sum_{(i,j)\in U}\ \sum_{k\in K(i,j)} q^k_{ij}\Delta x^k_{ij} =$$
$$= \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k\setminus U^k_L} \Delta^k(\tau,\rho)\Delta x^k_{\tau\rho} =$$
$$= \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_B} \Delta^k(\tau,\rho)\Delta x^k_{\tau\rho} + \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_N} \Delta^k(\tau,\rho)\Delta x^k_{\tau\rho} =$$
$$= \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_B} \Delta^k(\tau,\rho)\times$$
$$\times\left(\sum_{p=1}^{l} \nu_{t(\tau,\rho)^k,p}\widetilde{\beta}_p + \sum_{(i,j)\in U^*} \nu_{t(\tau,\rho)^k,l+\xi(i,j)}\widetilde{\beta}_{l+\xi(i,j)} \right) +$$
$$+ \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_N} \Delta^k(\tau,\rho)\Delta x^k_{\tau\rho}. \tag{31}$$

Let us introduce the following denotation:

$$r_p = \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_B} \Delta^k(\tau,\rho)\nu_{t(\tau,\rho)^k,p},\ \ p=\overline{1,l};$$
$$r_{ij} = \sum_{k\in K}\ \sum_{(\tau,\rho)^k\in U^k_B} \Delta^k(\tau,\rho)\nu_{t(\tau,\rho)^k,l+\xi(i,j)},\ \ (i,j)\in U^*.$$

Let's substitute (29) and (30) into (31). In result the numerator will be transformed to a kind (32):

$$N_{\Delta f} = \sum_{k\in K} \sum_{(\tau,\rho)^k\in U_N^k} \left(\Delta^k(\tau,\rho) - \sum_{p=1}^{l} r_p \Lambda_{\tau\rho}^{kp} - \sum_{(i,j)\in U^*} r_{ij}\delta_{ij}(B_{\tau\rho}^k) \right) \times$$

$$\times \Delta x_{\tau\rho}^k + \sum_{(i,j)\in U^*} r_{ij}\Delta z_{ij} \qquad (32)$$

or

$$N_{\Delta f} = \sum_{k\in K} \sum_{(\tau,\rho)^k\in U_N^k} \widetilde{\Delta}^k(\tau,\rho)\Delta x_{\tau\rho}^k + \sum_{(i,j)\in U^*} r_{ij}\Delta z_{ij}, \qquad (33)$$

$$\widetilde{\Delta}^k(\tau,\rho) = \Delta^k(\tau,\rho) - \sum_{p=1}^{l} r_p \Lambda_{\tau\rho}^{kp} - \sum_{(i,j)\in U^*} r_{ij}\delta_{ij}(B_{\tau\rho}^k). \qquad (34)$$

Theorem. *The general solution of nonhomogeneous system (2) is calculated from the formulae* (35) *for the fixed* $k \in K$

$$x_{ij}^k = \sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \left(\tilde{x}_{ij}^k - \sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \tilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) \right),$$

$$(i,j)^k \in U_L^k; \quad x_{\tau\rho}^k \in \mathbf{R}, \qquad (35)$$

where $\tilde{x}^k = (\tilde{x}_{ij}^k, (i,j)^k \in U^k)$ *– is any partial solution of the nonhomogeneous system* (2) *and* $\delta^k(\tau,\rho) = (\delta_{ij}^k(\tau,\rho), (i,j)^k \in U^k)$, $(\tau,\rho)^k \in U^k \setminus U_L^k, k \in K$ *is the system of characteristic vectors, entailed by an arc* $(\tau,\rho)^k \in U^k \setminus U_L^k, k \in K$ *for the fixed* $k \in K$*, see [5].*

Remark. Further, we shall use the partial solution

$$\tilde{x}^k = (\tilde{x}_{ij}^k, (i,j)^k \in U^k), k \in K$$

which is constructed to the following rules: non-supporting elements $(\tau,\rho)^k \in U^k \setminus U_L^k, k \in K$ are equal to zeros and supporting elements $(i,j)^k \in U_L^k, k \in K$ satisfy system (2).

Let's transform a denominator of the increment of the objective function (15). For it we transform the sum $\sum_{(i,j)\in U} \sum_{k\in K(i,j)} q_{ij}^k x_{ij}^k$, using (35) for arcs $(i,j)^k \in U_L^k, k \in K :$

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} q_{ij}^k x_{ij}^k = \sum_{k\in K} \sum_{(i,j)^k\in U^k} q_{ij}^k x_{ij}^k = \sum_{k\in K} \sum_{(i,j)^k\in U_L^k} q_{ij}^k x_{ij}^k +$$

$$+\sum_{k\in K}\sum_{(i,j)^k\in U^k\setminus U_L^k} q_{ij}^k x_{ij}^k = \sum_{k\in K}\sum_{(i,j)^k\in U_L^k} q_{ij}^k \left[\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} x_{\tau\rho}^k \delta_{ij}^k(\tau,\rho) + \right.$$

$$\left. + \left(\tilde{x}_{ij}^k - \sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \tilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho)\right)\right] + \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} q_{\tau\rho}^k x_{\tau\rho}^k =$$

[change the order of summation:]

$$= \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} x_{\tau\rho}^k \left(q_{\tau\rho}^k + \sum_{(i,j)^k\in U_L^k} q_{ij}^k \delta_{ij}^k(\tau,\rho)\right) +$$

$$+ \sum_{k\in K}\sum_{(i,j)^k\in U_L^k} q_{ij}^k \left[\widetilde{x}_{ij}^k - \sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \widetilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho)\right] =$$

Let's denote

$$Q = \sum_{k\in K}\sum_{(i,j)^k\in U_L^k} q_{ij}^k \left[\widetilde{x}_{ij}^k - \sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \widetilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho)\right].$$

We have:

$$\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k x_{ij}^k = \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \Delta_Q^k(\tau,\rho) x_{\tau\rho}^k + Q, \tag{36}$$

where

$$\Delta_Q^k(\tau,\rho) = q_{\tau\rho}^k + \sum_{(i,j)^k\in U_L^k} q_{ij}^k \delta_{ij}^k(\tau,\rho),$$

$$Q = \sum_{k\in K}\sum_{(i,j)^k\in U_L^k} q_{ij}^k \left(\widetilde{x}_{ij}^k - \sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \widetilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau,\rho)\right) \tag{37}$$

Let's substitute (37) into a denominator of the increment of the objective function (15).

$$\sum_{(i,j)\in U}\sum_{k\in K(i,j)} q_{ij}^k (x_{ij}^k + \Delta x_{ij}^k) + \gamma =$$

$$= \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \Delta_Q^k(\tau,\rho) x_{\tau\rho}^k + \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\setminus U_L^k} \Delta_Q^k(\tau,\rho) \Delta x_{\tau\rho}^k + Q + \gamma =$$

$$= \sum_{k\in K} \sum_{(\tau,\rho)^k \in U^k \setminus U_L^k} \Delta_Q^k(\tau,\rho) \left(x_{\tau\rho}^k + \Delta x_{\tau\rho}^k \right) + Q + \gamma \qquad (38)$$

Using (33) and (38) we have transformed the formulas of the increment of the objective function (15):

$$\Delta f = \frac{\sum\limits_{k\in K} \sum\limits_{(\tau,\rho)^k \in U_N^k} \widetilde{\Delta}^k(\tau,\rho)\Delta x_{\tau\rho}^k + \sum\limits_{(i,j)\in U^*} r_{ij}\Delta z_{ij}}{\sum\limits_{k\in K} \sum\limits_{(\tau,\rho)^k \in U^k \setminus U_L^k} \Delta_Q^k(\tau,\rho) \left(x_{\tau\rho}^k + \Delta x_{\tau\rho}^k \right) + Q + \gamma}, \qquad (39)$$

$$\widetilde{\Delta}^k(\tau,\rho) = \Delta^k(\tau,\rho) - \sum_{p=1}^{l} r_p \Lambda_{\tau\rho}^{kp} - \sum_{(i,j)\in U^*} r_{ij}\delta_{ij}(B_{\tau\rho}^k), \qquad (40)$$

where

$$\Delta^k(\tau,\rho) = \Delta_P^k(\tau,\rho) - f(x)\Delta_Q^k(\tau,\rho),$$

$$\Delta_P^k(\tau,\rho) = p_{\tau\rho}^k + \sum_{(i,j)^k \in U_L^k} p_{ij}^k \delta_{ij}^k(\tau,\rho),$$

$$\Delta_Q^k(\tau,\rho) = q_{\tau\rho}^k + \sum_{(i,j)^k \in U_L^k} q_{ij}^k \delta_{ij}^k(\tau,\rho),$$

$$r_p = \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_B^k} \Delta^k(\tau,\rho)\nu_{t(\tau,\rho)^k,p}, \ \ p = \overline{1,l},$$

$$r_{ij} = \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_B^k} \Delta^k(\tau,\rho)\nu_{t(\tau,\rho)^k,l+\xi(i,j)}, \ \ (i,j) \in U^*.$$

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