On algebraic points in the plane near smooth curves

Vasili Bernik, Friedrich Götze, Olga Kukso

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1 Introduction

Despite major results on the distribution of rational numbers on the real line there remain a number of deep problems. Some of them can be found in the monographs of Cassels and Schmidt [1, 2]. The problem of counting integer points is a classical topic in number theory and there are various related problems like the Gauss circle problem or the problem number of divisors of natural numbers bounded by some big number [3, 4]. Some facts on counting integer points in multidimensional domains can be found in [5]. During the last 20 years considerable progress has been made concerning the number of points with rational coordinates near smooth curves by Beresnevich and Vilani [6, 7] insofar as the lower and upper bounds that have been obtained are of the same order.

In the present paper we introduce a method, which allows us to obtain bounds for the number of points with algebraic coordinates lying in a given domains of a Euclidean space. We consider algebraic points in the plane, but part of our results can be generalized to higher dimensional spaces.

Let $P \in \mathbb{Z}[x]$ be of the form

$$P(x) = P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{1}$$

$$H = H(P) = \max_{1 \leqslant j \leqslant n} |a_j|, \quad \deg P = n.$$

Let μA be the Lebesgue measure of a measurable set $A \subset \mathbb{R}^2$, and $|I|$ the length of an interval $I \subset \mathbb{R}$. In what follows $c, c(n), c_1, c_2, \ldots$ stand for some positive constants depending on n only. Let $Q > Q_0(n)$, where Q_0 is a sufficiently large number. We will use the Vinogradov symbols $f \ll g$ which means that $f \leqslant cg$. The notation $B \asymp D$ means $D \ll B \ll D$.

For some arbitrary positive constants μ_1, μ_2 consider a rectangle

$$\Pi_1 = I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2] \subset [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$$

such that

$$\Pi_1 \cap \{|x - y| \leqslant 0.1\} = \emptyset \tag{2}$$

and

$$|I_1| = b_1 - a_1 = Q^{-\mu_1}, \ |I_2| = b_2 - a_2 = Q^{-\mu_2}.$$

Note that the lengths of I_1 and I_2 are small provided that $\mu_1 > 0$, $\mu_2 > 0$ and Q is sufficiently large.

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k$ denote k real roots of P, $1 \leqslant k \leqslant n$.

We introduce the class of polynomials

$$\mathbf{P}_n(Q) = \{P_n \in \mathbb{Z}[x] : \deg P = n, n \geqslant 3, a_n \gg H(P), H(P) \leqslant Q\}. \tag{3}$$

The condition $|a_n| \gg H$ implies that the roots of $P(x)$ are bounded, see Sprindzuk [8].

Let $K_n(\Pi_1, Q)$ be the set of points (α_i, α_j), $1 \leqslant i < j \leqslant k$, such that

(i) (α_i, α_j) are real roots of $P \in \mathbf{P}_n(Q)$,

(ii) $(\alpha_i, \alpha_j) \in \Pi_1$.

Remark. Condition (ii) excludes the coincidence of the roots α_1 and α_2.

The aim of this paper is to estimate the cardinality of $K_n(\Pi_1, Q)$.

Theorem 1 *Let* $0 < \mu_i < \frac{1}{2}$, $i = 1, 2$. *Then*

$$\#K_n(\Pi_1, Q) \gg Q^{n+1-\mu_1-\mu_2}. \tag{4}$$

Remark. Consider $J_1 \times J_2 = [\frac{1}{3} - Q^{-1-\varepsilon}, \frac{1}{3} + Q^{-1-\varepsilon}] \times [\frac{1}{4} - Q^{-1-\varepsilon}, \frac{1}{4} + Q^{-1-\varepsilon}]$, where $\varepsilon > 0$. Suppose that, on the contrary, that there is a polynomial $T \in \mathbf{P}_n(Q)$ such that a pair of its roots (α_1, α_2) belongs to $J_1 \times J_2$ and T is coprime to $P(x) = (3x - 1)(4x - 1) = 12x^2 - 7x + 1$. The last assumption implies that $|R(T, P)| \geq 1$, where $R(T, P)$ is the resultant of $T(x)$ and $P(x)$. Since the roots of $T(x)$ are bounded, we have

$$\begin{aligned} 1 \leqslant |R(T, P)| &= 12^n a_n^2 \prod_{i=1}^{n} |\frac{1}{3} - \alpha_i| \prod_{j=1}^{n} |\frac{1}{4} - \alpha_j| = \\ &= 12^n a_n^2 |\alpha_1 - \frac{1}{3}||\alpha_2 - \frac{1}{4}| \prod_{i \neq 1} |\frac{1}{3} - \alpha_i| \prod_{j \neq 2} |\frac{1}{4} - \alpha_j| \ll \\ &\ll Q^2 Q^{-1-\varepsilon} Q^{-1-\varepsilon} = Q^{-2\varepsilon}. \end{aligned} \tag{5}$$

The inequality (5) yields a contradiction if Q is sufficiently large.

This remark shows that Theorem 1 cannot be considerably improved. It won't hold for $\max_j \mu_j > 1$. Improvements are possible for intervals I_1, I_2 only that don't contain algebraic numbers of small degree and height.

Corollary. Let $f(x)$ be a continuous function on the interval $I = [a, b]$ and let

$$\mathcal{L}(Q, \lambda) = \{(x, y) \ : \ x \in I, |y - f(x)| < Q^{-\lambda}\}, \quad 0 < \lambda < \frac{1}{2}. \qquad (6)$$

Then there are at least $c(n)Q^{n+1-\lambda}$ algebraic points such that $(\alpha_1, \alpha_2) \in \mathcal{L}(Q, \lambda)$.

Proof of the corollary. The set $\mathcal{L}(Q, \lambda)$ represents a strip containing the curve $y = f(x)$. Its width equals $2Q^{-\lambda}$, $0 < \lambda < \frac{1}{2}$. Let us split an interval $[a, b]$ into equal parts of length at most $Q^{-\lambda}$ choosing points

$$x_0 = a, \ x_1 = x_0 + Q^{-\lambda}, \ \ldots, \ x_j = x_{j-1} + Q^{\lambda}, \ \ldots, \ x_s = x_0 + sQ^{\lambda},$$

where $\lambda \leqslant 1$. Furthermore, inscribe rectangles of size $Q^{-\lambda} \times c(n)Q^{-\lambda}$ into every rectangle

$$\{(x, y) \ : \ |x - \frac{x_i + x_{i+1}}{2}| \leqslant \frac{1}{2}Q^{-\lambda}, \ |y - f(x)| < \frac{1}{2}Q^{-\lambda}\}.$$

By Theorem 1, every such rectangle contains at least $c(n)Q^{n+1-2\lambda}$ algebraic points (α_1, α_2). Collecting the algebraic in all rectangles we obtain

$$\#\mathcal{L}(Q, \lambda) \cap \mathbb{A}_n \gg c(n)Q^{n+1-\lambda}.$$

■

The proof of Theorem 1 is based on the construction of special polynomials $P(t) \in \mathbf{P}_n(Q)$ such that

1. $|P(x)|$ and $|P(y)|$ are small,
2. $|P'(x)|$ and $|P'(y)|$ are comparable with $H(P)$,

where $(x, y) \in B_1 \subset \Pi_1$ and $\mu B_1 > \frac{1}{2}\mu\Pi_1$.

Let $\overline{c} = (c_1, c_2, c_3, c_4)$ and $\overline{v} = (v_1, v_2)$ denote positive vectors. Let $M_n(\overline{c}, Q)$ denote the set of points $\overline{x} \in \Pi_1$ such that the following system

$$\begin{cases} |P(x)| < c_1 Q^{-v_1}, \\ |P(y)| < c_2 Q^{-v_2}, \\ |P'(x)| < c_3 Q, \\ |P'(y)| < c_4 Q, \\ v_1 + v_2 = n - 1 \end{cases} \qquad (7)$$

has a solution $P(t) \in \mathbb{Z}[t] \setminus \{0\}$.

Theorem 2 *Assume that* $c_1 c_2 \min(c_3, c_4) < 2^{-n-38} n^{-2}$ *and* $\max(c_1, c_2, c_3, c_4) \leq 1$. *Then*

$$\mu M_n(\bar{c}, Q) < \frac{1}{4}|I_1||I_2|. \tag{8}$$

To prove Theorem 2 we impose an extra condition on P. We consider only irreducible polynomials. This condition is not very restrictive and leads to an equivalent problem as shown in Sprindzuk and Bernik [8, 9].

2 Auxiliary statements

This section contains several lemmas that will be used in the proof of Theorem 2.

In what follows $\mathcal{P}_n(Q)$ denotes the class of irreducible polynomials $P(t)$ with $H(P) \leqslant Q$ such that (7) holds. Furthermore, let $\tilde{\mathcal{P}}_n(H)$ be the subclass of $\mathcal{P}_n(H)$ consisting of polynomials P with $H(P) = H$.

For each polynomial $P \in \tilde{\mathcal{P}}_n(H)$ with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, we pick a pair of roots α_i and α_j, $i \neq j$. Throughout for convenience, we shall write α_1 instead of α_i and β_1 instead of α_j. Furthermore, we order the other roots of P with respect to the distance from the roots α_1 and β_1

$$\begin{aligned} &|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant \cdots \leqslant |\alpha_1 - \alpha_n|, \\ &|\beta_1 - \beta_2| \leqslant |\beta_1 - \beta_3| \leqslant \cdots \leqslant |\beta_1 - \beta_n|. \end{aligned} \tag{9}$$

Obviously, in (9), the set $\beta_1, \beta_2, \ldots, \beta_n$ is a permutation of the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Denote

$$S(\alpha_1) = \{x \in \mathbb{R} : \ |x - \alpha_1| = \min_{1 \leqslant j \leqslant n} |x - \alpha_j|\},$$

$$S(\beta_1) = \{x \in \mathbb{R} : \ |x - \beta_1| = \min_{1 \leqslant j \leqslant n} |x - \beta_j|\}.$$

We will consider now the system of inequalities (7) for $x \in S(\alpha_1)$ and $y \in S(\beta_1)$.

Lemma 1 (see [8]) *If* $|a_n| \gg H$ *then for any* i, $1 \leqslant i \leqslant n$,

$$|\alpha_i| < c.$$

Lemma 2 *Let* $P \in \tilde{\mathcal{P}}_n(H)$ *and* $x \in S(\alpha_1)$. *Then*

$$|x - \alpha_1| \leqslant n \frac{|P(x)|}{|P'(x)|},$$

$$|x-\alpha_1| \leqslant 2^{n-1}|P(x)||P'(\alpha_1)|^{-1}, \tag{10}$$

$$|x-\alpha_1| \leqslant \min_{2\leqslant j\leqslant n}(2^{n-j}|P(x)||P'(\alpha_1)|^{-1}\prod_{k=2}^{j}|\alpha_1-\alpha_k|)^{\frac{1}{j}}.$$

The first inequality in (10) immediately follows from the identity $|P'(x)||P(x)|^{-1} = |\sum_{i=1}^{n}\frac{1}{(x-\alpha_i)}|$ and the inequalities $|x-\alpha_1| \leqslant |x-\alpha_j|$, $j = 2,\dots,n$. The remaining inequalities were proved in Sprindzuk and Bernik[8, 10].

Let $\varepsilon > 0$ be sufficiently small, and let $N = N(n) > 0$ be sufficiently large fixed numbers. Write $\varepsilon_1 = \varepsilon N^{-1}$, and $T = [\varepsilon_1]^{-1}$.

Using (9) define numbers $\rho_{1,j}$ and $\rho_{2,j}$ $(2 \leqslant j \leqslant n)$ by setting

$$\begin{array}{l} |\alpha_1-\alpha_j| = H^{-\rho_{1j}},\ \rho_{1,n} \leqslant \cdots \leqslant \rho_{12}, \\ |\beta_1-\beta_j| = H^{-\rho_{2j}},\ \rho_{2,n} \leqslant \cdots \leqslant \rho_{22}. \end{array} \tag{11}$$

By Lemma 1 the roots α_j are bounded. Then the inequalities (9) and (11) imply $\rho_{i,j} > -\frac{\varepsilon_1}{2}$.
For every polynomial there are uniquely determined integral vectors $(k_2,k_3,\dots,k_n)$ and $(l_2,l_3,\dots,l_n)$ such that the inequalities

$$\begin{array}{ll} (k_j-1)T^{-1} \leqslant \rho_{1j} < k_jT^{-1}, & 0 \leqslant k_n \leqslant \dots \leqslant k_2, \\ (l_j-1)T^{-1} \leqslant \rho_{2j} < l_jT^{-1}, & 0 \leqslant l_n \leqslant \dots \leqslant l_2 \end{array}$$

hold. Furthermore, define

$$q_i = T^{-1}\sum_{m=i+1}^{n} k_m, \quad r_i = T^{-1}\sum_{m=i+1}^{n} l_m,\ 1 \le i \le n-1.$$

Consider $\cup_{H=1}^{\infty}\tilde{\mathcal{P}}_n(H)$. Using results of Sprindzuk [8], the number of possible vectors $\overline{k} = (k_2,k_3,\dots,k_n)$ and $\overline{l} = (l_2,l_3,\dots,l_n)$ is finite.
Thus, all polynomials $P \in \tilde{\mathcal{P}}_n(H)$ corresponding to the same pair of vectors $\overline{s} = (\overline{k},\overline{l})$ can be grouped together into a class $\tilde{\mathcal{P}}_n(H,\overline{s})$.

Lemma 3 (see Bernik [10]) *Let $P \in \tilde{\mathcal{P}}_n(H,\overline{s})$. The we have*

$$\begin{array}{l} H^{1-q_1} \leqslant |P'(\alpha_1)| < H^{1-q_1+(n-1)\varepsilon_1}, \\ H^{1-r_1} \leqslant |P'(\beta_1)| < H^{1-r_1+(n-1)\varepsilon_1}, \end{array}$$

and for any k, $2 \leqslant k \leqslant n$,

$$\begin{array}{l} |P^{(k)}(\alpha_1)| \ll H^{1-q_k+k(n-1)\varepsilon_1} \\ |P^{(k)}(\beta_1)| \ll H^{1-r_k+k(n-1)\varepsilon_1}. \end{array}$$

Lemma 4 *Let $\delta, K_0, \eta_1, \eta_2 \in \mathbb{R}_+$. Furthermore, let $P_1, P_2 \in \mathbb{Z}[x]$ be two relatively prime polynomials of degree at most n with $\max(H(P_1), H(P_2)) \leqslant K$ and $K > K_0(\delta)$. Let J_1 and J_2 denote intervals with $|J_1| = K^{-\eta_1}$, $|J_2| = K^{-\eta_2}$. If there exist numbers $\tau_1, \tau_2 > 0$ such that for all $(x, y) \in J_1 \times J_2$*

$$\max(|P_1(x)|, |P_2(x)|) < K^{-\tau_1},$$
$$\max(|P_1(y)|, |P_2(y)|) < K^{-\tau_2},$$

then

$$\tau_1 + \tau_2 + 2 + 2\max(\tau_1 + 1 - \eta_1, 0) + 2\max(\tau_2 + 1 - \eta_2, 0) < 2n + \delta.$$

For the proof see Bernik [11].

Remark. Actually, a stronger result holds, namely

$$\tau_1 + \tau_2 + 2 + 2\max(\sum_{k=1}^{\infty} \tau_1 + 1 - \eta_1, 0) + 2\max(\sum_{k=1}^{\infty} \tau_2 + 1 - \eta_2, 0) < 2n + \delta.$$

When we apply Lemma 4 we will usually choose parameters τ_1, τ_2, η_1, η_2 satisfying

$$\tau_1 = k_2 T^{-1} + q_1 - 1,\ \tau_2 = l_2 T^{-1} + r_1 - 1,\ \eta_1 = k_2 T^{-1},\ \eta_2 = l_2 T^{-1}.$$

Thus, if the difference between, say, $l_2 T^1$ and r_1 is larger, then the result of Lemma 4 will be stronger. Therefore, without loss of generality, we can assume that $k_2 T^{-1} = q_1$, $l_2 T^{-1} = r_1$, and $q_j = r_j = 0$ for $j \geq 2$.

3 Proof of Theorem 2

First, we consider a special case of system (7) when $|P'(x)|$, $|P'(y)|$ are bounded below. Let us remind that $x \in S(\alpha_1)$ and $y \in S(\beta_1)$.

Proposition 1. *Let $v > \frac{1}{2}$ denote a constant and let $M_{n,1}(\bar{c}, Q)$ denote the set of solutions $(x, y) \in I_1 \times I_2$ of the system*

$$\begin{cases} |P(x)| \leq c_1 Q^{-v_1}, \\ |P(y)| \leq c_2 Q^{-v_2}, \\ Q^v < |P'(x)| < c_3 Q, \\ Q^v < |P'(y)| < c_4 Q. \end{cases} \tag{12}$$

Then

$$\mu M_{n,1}(\bar{c}, \bar{v}, Q) < \frac{1}{8}|I_1||I_2|.$$

Now estimates for $|P'(x)|$ and $|P'(y)|$ provide estimates for $|P'(\alpha_1)|$ and $|P'(\beta_1)|$.

By the first inequality in (10) for any $x \in S(\alpha_1)$ and $y \in S(\beta_1)$, we have

$$\begin{array}{l} |x-\alpha_1| < n|P(x)||P'(x)|^{-1} < c_1 n Q^{-v_1-v}, \\ |y-\beta_1| < n|P(y)||P'(y)|^{-1} < c_2 n Q^{-v_2-v}. \end{array} \tag{13}$$

The Mean Value Theorem yields

$$\begin{array}{ll} P'(x) = P'(\alpha_1) + P''(\xi_1)(x-\alpha_1) & \text{for some } \xi_1 \in (\alpha_1, x), \\ P'(y) = P'(\beta_1) + P''(\xi_2)(y-\beta_2) & \text{for some } \xi_2 \in (\beta_2, y). \end{array}$$

Obviously, we have $|P''(\xi_1)(x-\alpha_1)| \ll Q^{1-v_1-v}$, $|P''(\xi_2)(y-\beta_1)| \ll Q^{1-v_2-v}$. Thus, for sufficiently large Q we obtain

$$\begin{array}{l} \frac{3}{4}Q^v \le \frac{3}{4}|P'(x)| < |P'(\alpha_1)| < \frac{4}{3}|P'(x)| \le \frac{4}{3}c_3 Q, \\ \frac{3}{4}Q^v \le \frac{3}{4}|P'(y)| < |P'(\beta_1)| < \frac{4}{3}|P'(y)| \le \frac{4}{3}c_4 Q. \end{array} \tag{14}$$

By (14) and Lemma 2, we have

$$\begin{array}{l} |x-\alpha_1| < \frac{4}{3}n|P(x)||P'(\alpha_1)|^{-1}, \\ |y-\beta_1| < \frac{4}{3}n|P(y)||P'(\beta_1)|^{-1}. \end{array} \tag{15}$$

Let $\sigma_x(P)$, $\sigma_y(P)$ denote the sets of solutions of (15) for x and y, respectively. Let $\Pi_2(P) = \sigma_x(P) \times \sigma_y(P)$. Clearly, all solutions $(x, y) \in S(\alpha_1) \times S(\beta_1)$ of the system (12) are contained in $\Pi_2(P)$.

We introduce the intervals

$$\begin{array}{lll} \sigma_{1x}(P) & : & |x-\alpha_1| < c_5 Q^{-\gamma}|P'(\alpha_1)|^{-1}, \\ \sigma_{1y}(P) & : & |y-\beta_1| < c_5 Q^{-\gamma}|P'(\beta_1)|^{-1}, \end{array} \tag{16}$$

where values of positive constants γ and c_5 will be specified below. Assign $\Pi_3(P) = \sigma_{1x}(P) \times \sigma_{1y}(P)$.

Now we shall estimate the values of P and P' on the intervals $\sigma_{1x}(P)$ and $\sigma_{1y}(P)$. For the sake of simplicity we shall consider $P(y)$ and $P'(y)$ on $\sigma_{1y}(P)$ only. The Mean Value Theorem yields

$$\begin{array}{llll} P(y) = & P'(\beta_1)(y-\beta_1) + \frac{1}{2}P''(\xi_3)(y-\beta_1)^2 & \text{for some} & \xi_3 \in (\beta_1, y), \\ P'(y) = & P'(\beta_1) + P''(\xi_4)(y-\beta_1) & \text{for some} & \xi_4 \in (\beta_1, y). \end{array} \tag{17}$$

By (14) and (16), the second terms of $P(y)$ and $P'(y)$ may be estimated as follows

$$\begin{array}{l} |\frac{1}{2}P''(\xi_3)(y-\beta_1)^2| \ll Q^{1-2\gamma-2v}, \\ |P''(\xi_4)(y-\beta_1)| \ll Q^{1-\gamma-v}. \end{array} \tag{18}$$

From (17) and (18) we get

$$\begin{aligned} |P(y)| &< \tfrac{4}{3}c_5 Q^{-\gamma}, \\ |P'(y)| &< \tfrac{5}{3}c_4 Q. \end{aligned} \tag{19}$$

Similarly, for $P(x)$ and $P'(x)$ on interval $\sigma_{1x}(P)$ we obtain

$$\begin{aligned} |P(x)| &< \tfrac{4}{3}c_5 Q^{-\gamma}, \\ |P'(x)| &< \tfrac{5}{3}c_3 Q. \end{aligned} \tag{20}$$

Fix the vector $\bar{b} = (a_n, \ldots, a_3)$ of coefficients of $P(x)$. The polynomials $P \in \tilde{\mathcal{P}}_n(H, \bar{s})$ with the same vector $\bar{b}$ form a subclass $\mathcal{P}(\bar{b})$.

Without loss of generality, we may assume that $a_n > 0$. Otherwise multiply the polynomial by -1 which does not change the system (7). Every coefficient a_j, $(3 \leqslant j \leqslant n-1)$ may take at most (2Q+1) values. Thus we have $\#\mathcal{P}(\bar{b}) \leq Q(2Q+1)^{n-3}$. For convenience, note that $\#\mathcal{P}(\bar{b}) \leq 2^{n-1}Q^{n-2}$.

We consider two types of rectangles $\Pi_3(P)$. One type of rectangle $\Pi_3(P_1)$ with $P_1 \in \mathcal{P}(\bar{b})$ is called *inessential* if there is another rectangle $\Pi_3(P_2)$ with $P_2 \in \mathcal{P}(\bar{b})$ such that

$$\mu(\Pi_3(P_1) \cap \Pi_3(P_2)) \geqslant 0.5\, \mu(\Pi_3(P_1)). \tag{21}$$

The other type of rectangle $\Pi_3(P_1)$ and is called *essential*. It satisfies: for any $P_2 \in \mathcal{P}(\bar{b})$ different from P_1

$$\mu(\Pi_3(P_1) \cap \Pi_3(P_2)) < 0.5\, \mu(\Pi_3(P_1)).$$

The case of essential rectangles. Summing the measures of rectangles for all polynomials in $\mathcal{P}(\bar{b})$, we obtain

$$\sum_{P \in \mathcal{P}(\bar{b})} \mu\Pi_3(P) \leqslant 2|I_1| \times |I_2|. \tag{22}$$

Combining the definitions of $\sigma_{1x}(P)$, $\sigma_{1y}(P)$, $\sigma_x(P)$, $\sigma_y(P)$ (see (15),(16)), we get

$$\begin{aligned} \mu\sigma_x(P) &< \tfrac{4}{3}nc_1c_5^{-1}Q^{-v_1+\gamma}\mu\sigma_{1x}(P), \\ \mu\sigma_y(P) &< \tfrac{4}{3}nc_2c_5^{-1}Q^{-v_2+\gamma}\mu\sigma_{1y}(P). \end{aligned} \tag{23}$$

Let us estimate the measure of the union of $\Pi_2(P)$ for all polynomials selected above.

$$\sum_{P\in\mathcal{P}(\bar b)} \mu\Pi_2(P) = \sum_{P\in\mathcal{P}(\bar b)} \mu\sigma_x(P)\times\mu\sigma_y(P) <$$
$$< \sum_{P\in\mathcal{P}(\bar b)} 2n^2c_1c_2c_5^{-2}Q^{-v_1-v_2+2\gamma}\mu\sigma_{1x}(P)\times\mu\sigma_{1y}(P) =$$
$$= 2n^2c_1c_2c_5^{-2}Q^{-v_1-v_2+2\gamma}\sum_{P\in\mathcal{P}(\bar b)}\mu\Pi_3(P) <$$
$$< 4n^2c_1c_2c_5^{-2}Q^{-v_1-v_2+2\gamma}|I_1||I_2|. \quad (24)$$

Summing over $\bar b$, we get

$$\sum_{\bar b}\sum_{P\in\mathcal{P}(\bar b)}\mu\Pi_2(P) < 2^{n+1}n^2c_1c_2c_5^{-2}Q^{n-2-v_1-v_2+2\gamma}|I_1||I_2|.$$

Taking into account $v_1+v_2=n-1$, and writing $\gamma=\frac{1}{2}$, we obtain

$$\sum_{\bar b}\sum_{P\in\mathcal{P}(\bar b)}\mu\Pi_2(P) < 2^{n+1}n^2c_1c_2c_5^{-2}|I_1||I_2|. \quad (25)$$

Given $c_5^2 = 2^{n+5}n^2c_1c_2$, the estimate in (25) does not exceed $2^{-4}|I_1||I_2|$.

The case of inessential rectangles.

Define $R(t)=P_2(t)-P_1(t)=b_2t^2+b_1t+b_0$. Without loss of generality, assume $b_2\geq 0$. Obviously, $R(t)$ is not identically zero. The Conditions (19), (20), and $P_1,P_2\in\mathcal{P}(\bar b)$ imply

$$\begin{array}{rcccl} |R(x)| &=& |b_2x^2+b_1x+b_0| &<& 3c_5Q^{-\gamma},\\ |R'(x)| &=& |2b_2x+b_1| &<& 3c_3Q,\\ |R(y)| &=& |b_2y^2+b_1y+b_0| &<& 3c_5Q^{-\gamma},\\ |R'(y)| &=& |2b_2y+b_1| &<& 3c_4Q. \end{array} \quad (26)$$

Let α and β denote roots of the polynomial $R(x)$ with $\deg R=2$. By inequalities (26) for $|R(x)|$, $|R(y)|$, and Lemma 2, we can estimate

$$|x-\alpha| < 6c_5Q^{-\gamma}|R'(\alpha)|^{-1}, \quad (27)$$
$$|y-\beta| < 6c_5Q^{-\gamma}|R'(\beta)|^{-1}. \quad (28)$$

By (2), if $|\alpha-\beta|<0.08$, we arrive at a contradiction for sufficiently large Q

$$0,1 < |x-y| \leq |x-\alpha|+|y-\beta|+|\alpha-\beta| < 0,09.$$

Thus $|\alpha-\beta| \geq 0.08$ and

$$|R'(\alpha)| = |R'(\beta)| = b_2|\alpha-\beta| > 0.08b_2. \tag{29}$$

Suppose $c_4 = \min(c_3, c_4)$. Applying the Mean Value Theorem on the interval σ_{1y}, we obtain

$$R'(y) = R'(\beta) + R''(\xi_5)(y-\beta) \quad \text{for some } \xi_5 \in [\beta, y].$$

Since $|R''(\xi_5)(y-\beta)| < 24c_5Q^{1-\gamma}|R'(\beta)|^{-1}$, if $|R'(\beta)|^2 > 48c_5Q^{1-\gamma}$, then

$$|R'(\beta)| < 2|R'(y)| < 6c_4Q. \tag{30}$$

The estimate (30) follows from the inequalities (14). This implies that **the number of possible** b_2 is bounded by

$$\#b_2 < 75c_4Q. \tag{31}$$

Suppose that $I_1 = [d_1, d_2]$, $I_2 = [f_1, f_2]$, and $|I_2| \geq |I_1|$.

First let us assume that $|I_1| = |I_2| = Q^{-\mu_1}$. The point $-\frac{b_1}{2b_2}$ is the maximum of the parabola $z = b_2x^2 + b_1x + b_0$. It is easy to verify that this point lies inside the interval $[\frac{d_1+d_2}{2}, \frac{f_1+f_2}{2}]$. The conditions $x \in I_1 \subset [-\frac{1}{2}, \frac{1}{2}]$, $y \in I_2 \subset [-\frac{1}{2}, \frac{1}{2}]$ imply

$$\#b_1 \leq 2b_2Q^{-\mu_1} + 2 = 2b_2|I_1| + 2 \tag{32}$$

and $|b_1| \leq |b_2|$.

Now assume $|I_1| > |I_2|$. Divide I_2 into $m = [\frac{|I_2|}{|I_1|}]+1$ intervals J_i such that $J_i \leq |I_1|$ where $1 \leqslant j \leqslant m$. Similarly, for every pair $x \in I_1$ and $y \in J_i$ we obtain an upper bound for $\#b_1$ similar to (32). Summing (32) over j gives the following exact estimate of **the number of possible** b_1

$$\#b_1 \leq (2b_2|I_1| + 2)(|I_2||I_1|^{-1} + 1) \leq 4b_2|I_2|. \tag{33}$$

Suppose now that (26) holds for some $R_1 = b_2x^2 + b_1x + b_0$. If we take $R_2 = b_2x^2 + b_1x + b_0 + 1$ we may shift the argument by Δx, i.e.,

$$1 = R_2(x) - R_1(x) = R_1(x+\Delta x) - R_1(x) = R'(\xi_6)\Delta x \quad \text{for some } \xi_6 \in [x, x+\Delta x].$$

If $x + \Delta x \in I_1$, then $\xi \in I_1$. For a fixed pair (b_2, b_1) the estimate for the derivative in (26) can be improved, namely

$$|R'(\xi_6)| = |2b_2\xi_6 + b_1| \leq 2|b_2|\frac{1}{2} + |b_1| \leq 2|b_2|.$$

Summarizing, we conclude that

$$\Delta = |R'(\xi_6)|^{-1} \geq \frac{1}{2}|b_2|^{-1}.$$

This means that **the number of possible values of** b_0 is at most

$$\#b_0 \leqslant |I_1||\Delta|^{-1} < 2|b_2||I_1|. \tag{34}$$

By Lemma 2 and the estimates $|R'(\alpha)| > 2^{-4}b_2$, $|R'(\beta)| > 2^{-4}b_2$ from (26), we obtain

$$|x-\alpha| < 2^8 c_5 Q^{-\gamma} b_2^{-1}$$

and

$$|y-\beta| < 2^8 c_5 Q^{-\gamma} b_2^{-1}.$$

Thus, the measure of the intersection $\Pi_3(P_1) \cap \Pi_3(P_2)$ is less than $2^{18} c_5^2 b_2^{-2} Q^{-2\gamma}$. If $\gamma = \frac{1}{2}$, then the measure of the inessential rectangle is less than

$$2^{19} c_5^2 b_2^{-2} Q^{-1}. \tag{35}$$

Using the estimates for b_0, b_1, b_2 from (31), (33),(34), we may sum (35) over $(b_0,\ b_1,\ b_2)$, and get

$$\sum_{b_2}\sum_{b_1}\sum_{b_0} \mu\Pi_3(P) < 2^{29}\min(c_3,c_4)c_5^2|I_1||I_2|. \tag{36}$$

For $c_5 = 2^{n+5}n^2c_1c_2$ the estimate in (36) says

$$2^{n+34}n^2c_1c_2\min(c_3,c_4)|I_1||I_2|.$$

Given $c_1c_2\min(c_3,c_4) < 2^{-n-38}n^{-2}$, this bound is smaller than 2^{-4}. Thus, we proved that

$$\mu M_{n1}(\bar{c},Q) < \frac{1}{8}|I_1||I_2|. \tag{37}$$

□

The remaining part of the proof strongly depends on the structures of $\bar{q}$, $\bar{r}$ (they were introduced in the Auxiliary Statements) and on their relations with the degrees v_1, v_2. In all of these statements below the measure tends to zero as $Q \to \infty$. The constants c_1, c_2, c_3, c_4, and others no longer play a significant role and will be replaced by the Vinogradov symbol $\ll$ in the remaining part of the paper.

Introduce a new subclass of polynomials as follows:

$$\mathcal{P}^t = \mathcal{P}^t(\bar{q},\bar{r}) = \bigcup_{2^t \leqslant H < 2^{t+1}} \tilde{\mathcal{P}}(H,\bar{q},\bar{r}).$$

In order to proceed we need one more definition.

A polynomial $P \in \tilde{\mathcal{P}}(H, \bar{q}, \bar{r})$ is called (i_1, i_2)*–linear*, where $i_1 = 0, 1$ and $i_2 = 0, 1$, according to the ordering between $q_1 + k_2 T^{-1}$ and $v_1 + 1$, $r_1 + l_2 T^{-1}$ and $v_2 + 1$. For example, $(0, 0)$–linearity means that the following system holds:

$$\begin{aligned} q_1 + k_2 T^{-1} &< v_1 + 1, \\ r_1 + l_2 T^{-1} &< v_2 + 1. \end{aligned} \tag{38}$$

$(0, 1)$–linearity means $(<, \geqslant)$ inequalities in the system above, $(1, 1)$–linearity means $(\geqslant, \geqslant)$, and so on. The most important case are the $(1, 1)$ and $(0, 0)$– linearities. Denote

$$d_1 = q_1 + r_1, \quad d_2 = (k_2 + l_2) T^{-1}.$$

We will consider polynomials $P \in \mathcal{P}^t$ such that $H \asymp Q$. The main differences between 0– and 1–linearity will be finding proper estimates of the differences $|x - \alpha_1|$ and $|y - \beta_1|$ when applying Lemma 2. We use the first estimate in (13) for 0 –linearity and the second estimate in (13) for 1–linearity.

Proposition 2. *Let $M_{n,2}(\bar{c}, \bar{v}, Q)$ denote the set of $(x, y) \in I_1 \times I_2$ such that the system of inequalities*

$$\begin{cases} |P(x)| \ll Q^{-v_1}, \\ |P(y)| \ll Q^{-v_2} \end{cases} \tag{39}$$

holds for $(1, 1)$–linearity. Then

$$\mu M_{n,2}(\bar{c}, \bar{v}, Q) < \frac{1}{32} |I_1||I_2|. \tag{40}$$

Proof.

$(1, 1)$–linearity implies $d_1 + d_2 \geq n + 1$. By Lemmas 2 and 3,

$$\begin{cases} |x - \alpha_1| \ll Q^{-\frac{v_1+1}{2} + \frac{q_1}{2} + (n-1)\varepsilon_1}, \\ |y - \beta_1| \ll Q^{-\frac{v_2+1}{2} + \frac{r_1}{2} + (n-1)\varepsilon_1}. \end{cases} \tag{41}$$

Suppose $\rho_1 = \frac{v_1 - q_2 + 1}{2}$. Let us divide the interval I_1 into equal subintervals I_i, where $|I_i| = Q^{-\rho_1 + \varepsilon}$. Similarly, suppose $\rho_2 = \frac{v_2 - r_2 + 1}{2}$ and divide I_2 into equal subintervals I_j, where $|I_j| = Q^{-\rho_2 + \varepsilon}$.

Then the number of rectangles $I_i \times I_j$ does not exceed

$$c(n) Q^{\frac{1}{2}(v_1 + v_2 + 2) - q_2 - r_2 - 2\varepsilon} |I_1||I_2| = c(n) Q^{\frac{1}{2}(n+1) - q_2 - r_2 - 2\varepsilon} |I_1||I_2|. \tag{42}$$

Choose rectangles $I_i \times I_j$ that contain not more than one solution P of system (39). From (41) and (42) it follows that the measure of the solution set of (39) does not exceed

$$c(n)Q^{-2\varepsilon+2(n-1)\varepsilon_1}|I_1||I_2| < \frac{1}{64}|I_1||I_2|. \tag{43}$$

Let us show that the case where (39) holds for at least two polynomials leads to a contradiction. Using a Taylor expansion on I_i and I_j, we obtain

$$P_1(x) = P'(\alpha_1)(x-\alpha_1) + \frac{1}{2}P''(\alpha_1)(x-\alpha_1)^2 + \sum_{j=3}^{n}(j!)^{-1}P^{(j)}(\alpha_1)(x-\alpha_1)^j,$$

$$P_1(y) = P'(\beta_1)(y-\beta_1) + \frac{1}{2}P''(\beta_1)(y-\beta_1)^2 + \sum_{j=3}^{n}(j!)^{-1}P^{(j)}(\beta_1)(y-\beta_1)^j.$$

Similarly we obtain an expansion for P_2. The above estimates of $|x-\alpha_1|$, $|y-\beta_1|$, and the estimates for the derivatives that follow from Lemma 3 lead to the following inequalities:

$$\begin{cases} |P_1(x)| \ll Q^{-v_1+(n-1)\varepsilon_1+2\varepsilon}, \\ |P_1(y)| \ll Q^{-v_2+(n-1)\varepsilon_1+2\varepsilon}, \\ |P_2(x)| \ll Q^{-v_1+(n-1)\varepsilon_1+2\varepsilon}, \\ |P_2(y)| \ll Q^{-v_2+(n-1)\varepsilon_1+2\varepsilon}. \end{cases} \tag{44}$$

Since P_1 and P_2 are irreducible they have no common roots. Thus, we can apply Lemma 4 to obtain

$$\tau_1 + 1 = v_1 - (n-1)\varepsilon_1 - 2\varepsilon, \;\; 2(\tau_1 + 1 - \eta_1) = v_1 + 1 + q_2 + 2(n-1)\varepsilon_1 - 4\varepsilon,$$

$$\tau_2 + 1 = v_2 - (n-1)\varepsilon_1 - 2\varepsilon, \;\; 2(\tau_2 + 1 - \eta_2) = v_2 + 1 + r_2 + 2(n-1)\varepsilon_1 - 4\varepsilon,$$

and in the left side of the inequality in Lemma 4 we get

$$2v_1 + 2v_2 + 4 - 12\varepsilon - 6(n-1)\varepsilon_1 = 2n + 2 - 12\varepsilon - 6(n-1)\varepsilon_1.$$

The right-hand side of this inequality then becomes $2n+\delta$. Given ε, ε_1, we obtain a contradiction to Lemma 4 when $\delta < 0.5$. $\square$

Now let consider the case of $(0,0)-$ linearity. Suppose that $n + 0.1 < d_1 + d_2 < n+1$, namely

$$\begin{cases} q_1 + k_2 T^{-1} \leqslant v_1 + 1, \\ r_1 + l_2 T^{-1} \leqslant v_2 + 1, \\ d_1 + d_2 > n + 0.1. \end{cases} \tag{45}$$

Proposition 3. *Let $M_{n,3}(\bar{c}, \bar{v}, Q)$ denote the set of $(x, y) \in I_1 \times I_2$ such that (39) holds together with (45). Then*

$$\mu M_{n,3}(\bar{c}, \bar{v}, Q) < \frac{1}{32}|I_1||I_2|. \tag{46}$$

Proposition 3 can be proved in a similar manner. When (45) holds the first estimate is sharper then the second one in (13).

Again divide the rectangle $I_1 \times I_2$ into equal rectangles $I_i \times I_j$, where $|I_i| = Q^{-\rho_3+\varepsilon}$, $|I_j| = Q^{-\rho_4+\varepsilon}$ and $\rho_3 = k_2 T^{-1}$, $\rho_4 = l_2 T^{-1}$. Then the number of rectangles $I_i \times I_j$ does not exceed

$$c(n) Q^{(k_2+l_2)T^{-1}-2\varepsilon}|I_1||I_2|. \tag{47}$$

Again choose rectangles $I_i \times I_j$ such that there are no solutions or there is at most one solution P of the system (39) with an extra condition (45). By Lemma 2, we have for fixed a polynomial $P(t)$

$$\begin{cases} |x - \alpha_1| \ll Q^{-v_1-1+q_1+(n-1)\varepsilon_1}, \\ |y - \beta_1| \ll Q^{-v_2-1+r_1+(n-1)\varepsilon_1}. \end{cases}$$

Their product gives us an upper estimate for the measure of $\{(x, y) : x \in S(\alpha_1),\ y \in S(\beta_1)\}$. Multiplying it by (47), we get the following upper estimate for the measure of the solution set:

$$c(n) Q^{-v_1-v_2-2+(k_2+l_2)T^{-1}+q_1+r_1-2\varepsilon+2(n-1)\varepsilon}|I_1||I_2| \ll Q^{-\varepsilon}|I_1||I_2| < \frac{1}{32}|I_1||I_2|.$$

Assume that there are at least two solutions in the rectangle $I_1 \times I_2$. Again using a Taylor expansion of P and estimating its summands from above we obtain

$$\begin{cases} |P_1(x)| \ll Q^{1-q_1-k_2T^{-1}+(n-1)\varepsilon_1-\varepsilon}, \\ |P_1(y)| \ll Q^{1-r_1-l_2T^{-1}+(n-1)\varepsilon_1-\varepsilon}, \\ |P_2(x)| \ll Q^{1-q_1-k_2T^{-1}+(n-1)\varepsilon_1-\varepsilon}, \\ |P_2(y)| \ll Q^{1-r_1-l_2T^{-1}+(n-1)\varepsilon_1-\varepsilon}. \end{cases} \tag{48}$$

By Lemma 4 for

$$\tau_1 + 1 = q_1 + k_2 T^{-1} - (n-1)\varepsilon_1 - \varepsilon, \quad 2(\tau_1 + 1 - \eta_1) = 2q_1 - 2(n-1)\varepsilon_1 - 2\varepsilon,$$

$$\tau_2 + 1 = r_1 + l_2 T^{-1} - (n-1)\varepsilon_1 - \varepsilon, \quad 2(\tau_2 + 1 - \eta_2) = 2r_1 - 2(n-1)\varepsilon_1 - 2\varepsilon,$$

we get the following left-hand side for the inequality in Lemma 4

$$3q_1 + k_2 T^{-1} + 3r_1 + l_2 T^{-1} - 6(n-1)\varepsilon_1 - 6\varepsilon. \tag{49}$$

But $k_2T^{-1} \leqslant q_1$, $l_2T^{-1} \leqslant r_1$, and (45) implies that the expression in (49) is at least

$$2(d_1+d_2) - 6\varepsilon - 6(n-1)\varepsilon_1 \geqslant 2(v_1+v_2) + 3.6 - 6\varepsilon - 6(n-1)\varepsilon_1 =$$

$$= 2n + 0.2 - 6\varepsilon - 6(n-1)\varepsilon_1.$$

Given ε, ε_1, we obtain a contradiction to Lemma 4 when $\delta < 0.1$.

Now let us consider the case of $(0,0)$-linearity for

$$n - 0.3 < d_1 + d_2 \leq n + 0.1 \tag{50}$$

Proposition 4. *Let $M_{n,4}(\bar{c}, \bar{v}, Q)$ denote the set of $(x, y) \in I_1 \times I_2$ such that (38), (39) hold together with* (50). *Then*

$$\mu M_{n,4}(\bar{c}, \bar{v}, Q) < \frac{1}{32}|I_1||I_2|. \tag{51}$$

Proof.

Let us divide the rectangle $I_1 \times I_2$ into equal rectangles $I_i \times I_j$, where $|I_i| = Q^{-k_2T^{-1}-\gamma_1}$, $|I_j| = Q^{-l_2T^{-1}-\gamma_1}$ for some $\gamma_1 > 0$ that will be specified below. Let us choose those rectangles where the system (39) has at least $c(n)Q^{\theta_1}$ solutions in polynomials $P(t)$ for some $\theta_1 \geq 0$. Estimate the measure of $A_1 = \{(x, y) \ : \ (x, y) \in I_i \times J_j\}$, which satisfies (39).

$$\mu A_1 \ll Q^{-v_1-1+q_1-v_2-1+r_1+k_2T^{-1}+l_2T^{-1}+2\gamma_1+\theta_1}|I_1| \times |I_2| \ll$$

$$\ll Q^{\theta_1-n-1+d_1+d_2+2\gamma_1}|I_1||I_2|.$$

When

$$\theta_1 < n + 1 - d_1 - d_2 - 2\gamma_1$$

the statement of Proposition 4 can be easily verified.

Consider now the opposite inequality

$$\theta_1 \geq u_1 = n + 1 - d_1 - d_2 - 2\gamma_1. \tag{52}$$

By (50), $\theta_1 > 0$ for $\gamma_1 \leq 0.4$.

Similarly to (48), estimate $P_l(t)$, $l = 1, 2$, in $I_i \times J_j$. We obtain

$$|P_l(x)| \ll Q^{1-q_1-k_2T^{-1}-\gamma_1+(n-1)\varepsilon_1}, \tag{53}$$

$$|P_l(y)| \ll Q^{1-r_1-l_2T^{-1}-\gamma_1+(n-1)\varepsilon_1}. \tag{54}$$

Apply Lemma 4 to $P_1(t)$ and $P_2(t)$ with following parameters

$$\begin{aligned}\tau_1 + 1 &= q_1 + k_2 T^{-1} + \gamma_1 - (n-1)\varepsilon_1,\\ 2(\tau_1 + 1 - \eta_1) &= 2q_1 - 2(n-1)\varepsilon_1,\\ \tau_2 + 1 &= r_1 + l_2 T^{-1} + \gamma_1 - (n-1)\varepsilon_1,\\ 2(\tau_2 + 1 - \eta_2) &= 2r_1 - 2(n-1)\varepsilon_1.\end{aligned}$$

By Lemma 4 and (50), the inequality

$$2(d_1 + d_2) + 0.8 - 6(n-1)\varepsilon_1 < 2n + \delta \tag{55}$$

leads to a contradiction. □

Consider now the next case when

$$n - 0.55 < d_1 + d_2 \le n - 0.3. \tag{56}$$

Proposition 5. *Let $M_{n,5}(\bar{c}, \bar{v}, Q)$ denote the set of $(x, y) \in I_1 \times I_2$ such that (38), (39) hold together with* (56). *Then*

$$\mu M_{n,5}(\bar{c}, \bar{v}, Q) < \frac{1}{32}|I_1||I_2|. \tag{57}$$

Proof.

The proof of Proposition 5 is similar to the proof of Proposition 4. Let us divide the rectangle $I_1 \times I_2$ into equal rectangles $I_i \times I_j$, where $|I_i| = Q^{-k_2 T^{-1} - \gamma_2}$, $|I_j| = Q^{-l_2 T^{-1} - \gamma_2}$ for some $\gamma_2 > 0$. Similarly, we introduce a constant $\theta_2 \ge 0$ and a set A_2. When $\theta_2 < n + 1 - d_1 - d_2 - 2\gamma_2$ holds, then Proposition 4 can be easily proved. So consider

$$\theta_2 \ge u_2 = n + 1 - d_1 - d_2 - 2\gamma_2. \tag{58}$$

By (56), we can choose $\gamma_2 = 0.6$ in (58). Similarly to (53), estimate $P_l(t)$, $l = 1, 2$ in newly constructed rectangles $I_i \times J_j$. Applying Lemma 4, we obtain an inequality similar to (55)

$$2(d_1 + d_2) + 1.2 - 6(n-1)\varepsilon_1 < 2n + \delta.$$

Since (56) and $\delta < 0.05$, the inequality leads to a contradiction. □

Let

$$2 < d_1 + d_2 \le n - 0.55. \tag{59}$$

Proposition 6. *Let $M_{n,6}(\bar{c}, \bar{v}, Q)$ denote the set of $(x, y) \in I_1 \times I_2$ such that (38), (39) hold together with* (59). *Then*

$$\mu M_{n,5}(\bar{c}, \bar{v}, Q) < \frac{1}{32}|I_1||I_2|. \tag{60}$$

Proof.

The start of the proof is similar to the proofs of Propositions 4 and 5. We divide the rectangle $I_1 \times I_2$ into equal rectangles $I_i \times I_j$, where $|I_i| = Q^{-k_2T^{-1}}$, $|I_j| = Q^{-l_2T^{-1}}$. Similarly, we introduce the constant $\theta_3 \geq 0$ and the set A_3. When $\theta_3 < n+1-d_1-d_2$ holds the proof of Proposition 6 is obvious. Consider now

$$\theta_3 \geq u_3 = n+1-d_1-d_2 \geq 1.45. \tag{61}$$

We can rewrite u_3 as

$$u_3 = [u_3] + \{u_3\}, \quad [u_3] \geq 1.$$

Expanding $P_l(t)$ and $P_l'(t)$ on intervals I_j and J_i into a Taylor series and estimating its terms above, we obtain

$$\begin{cases} |P(x)| \ll Q^{1-q_1-k_2T^{-1}}, \\ |P'(x)| \ll Q^{1-q_1}, \\ |P(y)| \ll Q^{1-r_1-l_2T^{-1}}, \\ |P'(y)| \ll Q^{1-r_1}. \end{cases} \tag{62}$$

Since there are at most $c(n)Q^{[u_3]+\{u_3\}}$ polynomials $P(t)$ that belong to $I_j \times J_i$, then, by Dirichlet's principle, there are at least $K = c(n)Q^{\{u_1\}}$ polynomials with equal coefficients of t^n, t^{n-1}, ..., $t^{n-[u_3]+1}$.

Now we construct further polynomials with degree at most $n-[u_3]$

$$R_{j-1}(t) = P_j(t) - P_1(t) \quad j = 2, \dots, K.$$

By (62) for $R_i(f)$, $i = 1, \dots, K-1$, we have

$$\begin{cases} |R_i(x)| \ll Q^{1-q_1-k_2T^{-1}+(n-1)\varepsilon_1}, \\ |R'(x)| \ll Q^{1-q_1}, \\ |R_i(y)| \ll Q^{1-r_1-l_2T^{-1}+(n-1)\varepsilon_1}, \\ |R'(y)| \ll Q^{1-r_1}, \\ \deg R_i \leq n-[u_3] = d_1+d_2+\{u_3\}-1. \end{cases} \tag{63}$$

We apply Lemma 4 to the two polynomials $R_{s_1}(t)$ and $R_{s_2}(t)$. This results in a contradiction when $\{u_3\} \leq 0.7$.

Thus assume that $\{u_3\} > 0.7$. Again we divide the rectangle $I_1 \times I_2$ into equal rectangles $I_i \times I_j$, where $|I_i| = Q^{-k_2T^{-1}-\gamma_3}$, $|I_j| = Q^{-l_2T^{-1}-\gamma_3}$ for some $\gamma_3 > 0$ such that $2\gamma_3 \leq \{u_3\}$. If the number of polynomials in these rectangles is $c(n)Q^{\theta_3}$ and $\theta_3 < u_3 = n + 1 - d_1 - d_2 - 2\gamma_3$ then Proposition 6 can be easily proved. When

$$\theta_3 \geq u_3 = n + 1 - d_1 - d_2 - 2\gamma_3 = [u_3] + \{u_3\}n - 2\gamma_3$$

one can obtain (63) with an approximation of $|R_i(x)|$ and $|R_i(y)|$ of the type $1 - q_1 - k_2T^{-1} - \gamma_3 + (n-1)\varepsilon_1$ and $1 - r_1 - l_2T^{-1} - \gamma_3 + (n-1)\varepsilon_1$ respectively. Applying Lemma 4 to the pair of coprime polynomials, we get

$$2(d_1 + d_2) - 6(n-1)\varepsilon_1 + 2\gamma_3 < 2(d_1 + d_2) - 2 + 2\{u_4\} + \delta$$

that leads to a contradiction for $\gamma_3 = \frac{\{u_4\}}{2}$ and $\delta = 0.1$. □

Let us show how the theorem can be proved for the cases of $(1, 0)$ and $(0, 1)$-linearity. Since both proofs are absolutely similar we will demonstrate the method for $(1, 0)$-linearity only.

Proposition 7. *Let $M_{n,7}(\bar{c}, \bar{v}, Q)$ denote the set of $(x, y) \in I_1 \times I_2$ such that (39) hold together with*

$$\begin{cases} q_1 + k_2T^{-1} > v_1 + 1, \\ r_1 + l_2T^{-1} \leq v_2 + 1. \end{cases} \tag{64}$$

Then

$$\mu M_{n,7}(\bar{c}, \bar{v}, Q) < \frac{1}{32}|I_1||I_2|.$$

Proof.

Again divide the rectangle $I_1 \times I_2$ into rectangles $I_i \times I_j$, where $|I_i| = Q^{-\frac{v_1 - q_2 + 1}{2} + \varepsilon}$, $|I_j| = Q^{-l_2T^{-1} + \varepsilon}$. We replace the second inequality in (64) by

$$v_2 + 0.5 < r_1 + l_2T^{-1} \leq v_2 + 1. \tag{65}$$

Consider the rectangles $I_i \times I_j$ which contain no more than one polynomial $P(t)$. Fix such a polynomial $P(t)$. Then the solution of (39) belongs to the rectangle

$$\begin{cases} |x - \alpha| \ll Q^{-\frac{v_1 + 1 - q_2}{2}}, \\ |y - \beta| \ll Q^{-v_2 - 1 + r_1}. \end{cases} \tag{66}$$

Multiplying the estimates (66), we sum them over all rectangles $I_i \times I_j$. Thus we get the estimate of the kind $c(n)Q^{-\varepsilon}|I_1||I_2|$ that proves Proposition 7. If there are at least two polynomials such that belong to $I_i \times I_j$, then we expand them into Taylor series. We get

$$|P_i(x)| \ll Q^{-v_1 + (n-1)\varepsilon_1 + 2\varepsilon},$$

$$|P_i(y)| \ll Q^{1-r_1-l_2T^{-1}}.$$

Apply Lemma 4 with

$$\tau_1 = v_1 + 1 - 2\varepsilon - (n-1)\varepsilon_1,$$

$$2(\tau_1 + 1 - \eta_1) = v_1 + 1 + q_2 - 2\varepsilon - 2(n-1)\varepsilon_1,$$
$$\tau_2 + 1 = r_1 + l_2T^{-1} - \varepsilon,$$
$$2(\tau_2 + 1 - \eta_2) = 2r_1.$$

Then,

$$2v_1 + 2 + l_2T^{-1} + 3r_1 + q_2 - 3(n-1)\varepsilon_1 - 4\varepsilon < 2n + \delta. \tag{67}$$

However, by (65), we have $l_2T^{-1} + 3r_1 > 2v_2 + 1$, and the left side in (67) is larger than $2n + 1 - 5\varepsilon$. Thus, for $\delta < 0.5$ we arrive at a contradiction.

The final part of the proof is similar to the proof of the $(0,0)$–linearity. We omit the above estimate in (65) until we can use Dirichlet's principle, which results in polynomials of lower degree. □

The case $r_1 < \frac{1}{2}$ and $r_1 < \frac{1}{2}$ is considered in Proposition 1. It remains to consider polynomials such that

$$1 \leq d_1 + d_2 \leq 2 \tag{68}$$

holds. Here as in Proposition 1 we can pass to first degree polynomials which lead to a contradiction with (3) or to the second degree polynomials. For this case Theorem 1 was proved in Proposition 1.

Combining the results of all Propositions, we finally get

$$\mu M_n(\bar{c}, \bar{v}, Q) \leqslant \sum_{j=1}^{7} \mu M_{n,j}(\bar{c}, \bar{v}, Q) \leqslant \frac{1}{4}|I_1||I_2|,$$

concluding the proof of Theorem 2. ■

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Institute of Mathematics
Surganova str. 11
220072, Minsk,
Belarus
E-mail: bernik@im.bas-net.by
olga_kukso@tut.by

Universität Bielefeld
Fakultät für Mathematik
Postfach 10 01 31
33501 Bielefeld
Germany
E-mail: goetze@math.uni-bielefeld.de