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DIOPHANTINE APPROXIMATION ON CLASSICAL CURVES AND HAUSDORFF DIMENSION

V.V. BERESNEVICH AND V.I. BERNIK

Dedicated to J.P. Kubilius on the occasion of his 80-th birthday

1. Introduction. The usual inner product in $\mathbb{R}^n$ of $\mathbf{a}$ and $\mathbf{b}$ will be denoted by $\mathbf{a}\cdot\mathbf{b}$; $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ is the Euclidean norm of $\mathbf{a}$. Also, $|\mathbf{a}|_\infty = \max_{1\leqslant i\leqslant n}|a_i|$, where a_i are the coordinates of $\mathbf{a}$ in the standard basis of $\mathbb{R}^n$. The Lebesgue measure of $A\subset\mathbb{R}$ is denoted by $|A|$. Given an $\mathbf{x}\in\mathbb{R}^n$, there is a unique point $\mathbf{a}\in\mathbb{Z}^n$ such that $\mathbf{x}-\mathbf{a}\in(-1/2;1/2]^n$. This difference will be denoted by $\langle\mathbf{x}\rangle$. Throughout $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is a monotonic function. Also, $\|x\| = \operatorname{dist}(x,\mathbb{Z}) = |\langle x\rangle|$; $\#S$ is the cardinality of a set S.

We will consider two the most important types of approximation: *simultaneous and linear*. By Dirichlet's principle, for every point $\mathbf{x}\in\mathbb{R}^n$ there are infinitely many $q\in\mathbb{N}$ such that

$$|\langle q\mathbf{x}\rangle|_\infty < q^{-1/n} \tag{1}$$

and infinitely many $\mathbf{a}\in\mathbb{Z}^n$ such that

$$|\langle\mathbf{a}\cdot\mathbf{x}\rangle| < |\mathbf{a}|_\infty^{-n}\,. \tag{2}$$

The form of approximation (1) is called *simultaneous* and the other one is called *dual* or *linear*. Using the Borel-Cantelli Lemma it is a simple matter to verify that the exponents in (1) and (2) can be made smaller by some $\varepsilon>0$ only for $\mathbf{x}\in\mathbb{R}^n$ from a set of measure zero. Such points are called *very well approximable*.

The following questions then arise. First, what is the Hausdorff dimension of the set of v-approximable points, *i.e.* points $\mathbf{x}$ such that (1) (*resp.* (2)) holds infinitely often when the exponent is set to be $-v$ with $v>1/n$ (*resp.* $v>n$). Second, how does the measure of the set of $\mathbf{x}$ depend on a general approximation function ψ that is put instead of $q^{-1/n}$ (*resp.* $|\mathbf{a}|_\infty^{-n}$). Third, how the properties of almost all points in $\mathbb{R}^n$ are inherited by its proper submanifolds (curves, surfaces, etc.).

A. Khintchine [Khi24] was the first who considered a general approximation function on the right hand side of (1) with $n=1$. The Hausdorff dimension was first employed in Metric Diophantine approximation by A.V. Jarnik [Jar29] and A.S. Besicovitch [Bes34] also for $n=1$. The case of independent variables was then widely investigated in spaces of arbitrary dimension (see [BD99, Har98, Spr79] for further references).

2. Mahler's problem. The case of dependent variables has been studied since 1932 when K. Mahler suggested a classification of transcendental numbers [Mah32a] and raised a problem [Mah32b] that almost all real (complex) numbers have the same type. He

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conjectured that given any $n \in \mathbb{N}$ and any $\varepsilon > 0$, for almost all $x \in \mathbb{R}$ the point $(x, \dots, x^n)$ is not linearly $(n+\varepsilon)$-approximable (see below for the definition). Mahler himself proved this statement for $\varepsilon > 3n$. By Borel's theorem [Bor12], the conjecture holds for $n = 1$. The dependence of variables $(n > 1)$ makes the problem much more difficult. The first progress was made by J.P. Kubilius [Kub49] who proved Mahler's problem for $n = 2$. The case of $n = 3$ of Mahler's conjecture was confirmed by B. Volkmann [Vol61] and the complete solution to the problem was given by V. Sprindžuk [Spr64, Spr65] (see also [Spr69] for a comprehensive treatment and generalizations).

3. General approximation function. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function, $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{x}$ is *simultaneously ψ-approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$|\langle q\mathbf{x}\rangle|_\infty < \psi(q) \tag{3}$$

and *linearly ψ-approximable* if there are infinitely many $\mathbf{a} \in \mathbb{Z}^n$ such that

$$|\langle \mathbf{a} \cdot \mathbf{x}\rangle| < \psi(|\mathbf{a}|_\infty). \tag{4}$$

Also a point $\mathbf{x} \in \mathbb{R}^n$ is said to be *simultaneously* (*linearly*) *v-approximable*, where $v \in \mathbb{R}$, if it is simultaneously (linearly) ψ_v-approximable with $\psi_v(h) = h^{-v}$.

In 1966 A. Baker [Bak66] improved Sprindžuk's theorem by showing that for almost all points on the curve $\mathcal{V}_n = \{(x, \dots, x^n) : x \in \mathbb{R}\}$ are not linearly ψ-approximable whenever ψ is monotonic and

$$\sum_{h=1}^{\infty} \psi(h)^{1/n} < \infty. \tag{5}$$

In 1989 V. Bernik [Ber89] improved A. Baker's theorem by replacing condition (5) with the convergence of

$$\sum_{h=1}^{\infty} h^{n-1}\psi(h). \tag{6}$$

The case of divergence of (6) has been recently considered by V. Beresnevich [Ber99] who has shown that almost all points on $\mathcal{V}_n$ are linearly ψ-approximable whenever ψ is monotonic and (6) diverges.

A unifying terminology was then introduced in the book [BD99] of V.I. Bernik and M.M. Dodson. They say that a manifold $\mathcal{M}$ in $\mathbb{R}^n$ is of *Khintchine type for convergence/divergence* if the inequality (3) has finitely/infinitely many solutions whenever ψ is monotonic and the sum

$$\sum_{h=1}^{\infty} \psi(h)^n \tag{7}$$

converges/diverges. Similarly they define *Groshev type manifolds* by replacing (3) with (4) and (7) with (6). Using this terminology one reads the aforementioned theorem of Bernik as "$\mathcal{V}_n$ is of Groshev type for converges" and the one of Beresnevich tells us that $\mathcal{V}_n$ is of Groshev type for divergence.

4. Groshev type manifolds. Also much has been studied about properties of general analytic manifolds. Most of the results obtained before 1999 are reviewed in [BD99]. In 1998 D.Y. Kleinbock and G.A. Margulis [KM98] using a new approach proved that any non-degenerate manifold in $\mathbb{R}^n$ is extremal (also *strongly extremal*), *i.e* almost all points of the manifold are not very well approximable. A few years later V. Beresnevich [Ber02] confirmed their result using a completely different technique and established that non-degenerate manifolds are of Groshev type for converges. This result (as well as a multiplicative version) was independently established by Bernik, Kleinbock and Margulis [BKM01]. And very recently using a generalization of the method of [Ber99] it has been proved that non-degenerate manifolds are of Groshev type for divergence [BBKM02].

5. Khintchine type manifolds. Much less is known about Khintchine type surfaces. Manifolds satisfying quite strong conditions have been proved to be of Khintchine type for both convergence and divergence [Ber73, DRV91]. In particular, these manifolds are of high dimension and none of them is a curve. However, the parabola $\mathcal{V}_2$ has been studied much better than the other curves.

Kubilius' result that proved Mahler's conjecture for $n = 2$ was consequently improved by J.W.S. Cassels, Kubilius himself and V.I. Bernik. First, Cassels [Cas51] showed that for almost all $x \in \mathbb{R}$ the system of inequalities

$$\|qx\| < \varphi(q), \qquad \|qx^2\| < f(q)$$

has only finite number of integer solutions $q > 0$ whenever $\sum_{q=1}^{\infty} \varphi(q)f(q) < \infty$ and $f(q) \geqslant \max\{\varphi(q), q^{-1/2} \log qd(q)\}$, where $d(q)$ denotes the number of divisors of q. Then, Kubilius [Kub59] proved that almost all points of the parabola (x, x^2) are not simultaneously ψ-approximable whenever $q^{-1/2}\psi(q)$ is non-increasing and $\sum_{q=1}^{\infty} q^{-1/2}\psi(q) < \infty$. Finally, Bernik [Ber79] relaxed the convergence condition on ψ in Kubilius' theorem by proving that almost all points of the parabola are not simultaneously ψ-approximable whenever ψ is monotonic and the sum $(7)_{n=2}$ converges, thus establishing that $\mathcal{V}_2$ is of Khintchine type for convergence. The case of divergence seems to be quite difficult and still remains open.

6. Hausdorff dimension for linear approximation. Given a manifold $\mathcal{M} \subset \mathbb{R}^n$, let $\mathcal{L}_n(v, \mathcal{M})$ and $\mathcal{S}_n(v, \mathcal{M})$ be the sets of linearly and simultaneously v-approximable points on $\mathcal{M}$ respectively. As it was pointed out above, these sets have zero measure in $\mathcal{M}$ whenever $v > n$ for the linear case and $v > 1/n$ for the other one, thus leading to a natural question about their Hausdorff dimension.

A. Baker and W.M. Schmidt [BS70] proved that $\dim \mathcal{L}_n(w, \mathcal{V}_n) \geqslant (n + 1)/(w + 1)$ for all $w \geqslant n$ and created a method for obtaining lower bounds for the dimension of $\mathcal{L}_n(w, \Gamma)$ for a large spectrum of curves in $\mathbb{R}^n$. The method was subsequently developed in a number of papers, particularly in [DRV90, Ryn92]. H. Dickinson and M. Dodson [DD00] have shown that for any $C^{(1)}$ extremal manifold $\mathcal{M}$ in $\mathbb{R}^n$ one has $\dim \mathcal{L}_n(w, \mathcal{M}) \geqslant \dim \mathcal{M} - 1 + (n + 1)/(w + 1)$ for all $w \geqslant n$.

V.I. Bernik [Ber83] found the upper complementary to the one of Baker and Schmidt for $\dim \mathcal{L}_n(w, \mathcal{V}_n)$, thus getting

$$\dim \mathcal{L}_n(w, \mathcal{V}_n) = \frac{n+1}{w+1}. \tag{8}$$

Extending (8) for other (arbitrary) non-degenerate curve in $\mathbb{R}^n$ is one of the major problems in metric number theory. R. Baker [Bak78] proved that $\dim \mathcal{L}_2(w, \Gamma_2) = \frac{3}{w+1}$ for any non-degenerate planar curve Γ_2, but the general case seems to be enormously difficult. The Hausdorff dimension of $\mathcal{L}_n(w, \mathcal{M})$ was fully investigated for a class of manifolds satisfying special geometric constrains. Unfortunately this class includes no curve and still non-trivial upper bounds for $\dim \mathcal{L}_n(w, \mathcal{M})$ hasn't been obtained for non-degenerate manifolds.

7. Hausdorff dimension for simultaneous approximation. At last we would like to discuss the question of the Hausdorff dimension of simultaneously approximable points a bit more carefully. Contrary to the linear case, the dimension of $\mathcal{S}_n(v, \mathcal{M})$ behaves in a much more complicated way. It might depend on the dimension of the ambient space, approximation exponent v, the dimension of $\mathcal{M}$ and (the most surprising circumstance) on both geometric and arithmetic properties of $\mathcal{M}$.

One has to distinguish at least two cases for the exponent v. The first one is when v is close enough to the critical exponent $1/n$. There is a hope that the dimension would behave quite regular in this case (see below for conjectures). In the other case, when v is big enough, rational points that realize the approximation are forced to lie on the manifold (at least for manifolds given by algebraic equations). This leads to very different formulas for $\dim \mathcal{L}_n(v, \mathcal{M})$ for manifolds with the same geometry as some might contain no rational points at all, *e.g.* $x^n + y^n = 1$ with $n > 2$ by Fermat's last theorem. Another example is the circle $\mathcal{C}_R = \{x^2 + y^2 = R^2\}$ of radius R. The one with radius $\sqrt{3}$ contains no rational point, whereas $\mathcal{C}_1$ has plenty of them defined by the so called Pythagorean triples.

We would like to restrict our attention to planar curves, particularly to the parabola $\mathcal{V}_2 = \{(x, x^2) : x \in \mathbb{R}\}$ and the circle $x^2 + y^2 = 1$.

8. The circle and a general upper bound. Y. Melnichuk [Mel79] has found lower and upper bounds for $\dim \mathcal{S}_2(v, \mathcal{C}_1)$. Some general estimates for simultaneous Diophantine approximation were obtained by R. Baker [Bak78]. Combining these result we get

$$\frac{3(1-v)}{v+1} \leqslant \dim \mathcal{S}_2(v, \mathcal{C}_R) \leqslant \frac{3}{2(v+1)}, \qquad 1/2 \leqslant v \leqslant 1 \tag{9}$$

and

$$\frac{1}{2(v+1)} \leqslant \dim \mathcal{S}_2(v, \mathcal{C}_1) \leqslant \frac{1}{v+1}, \qquad v \geqslant 1. \tag{10}$$

Recently Dickinson and Dodson [DD01] have improved the lower bound in (10) thus getting the exact value:

$$\dim \mathcal{S}_2(v, \mathcal{C}_1) = \frac{1}{v+1} \qquad \text{for } v \geqslant 1.$$

The result relies on an explicit construction of rational points on the circle. Recently Dickinson and J. Levely [DL] have computed $\dim \mathcal{L}_n(v, \mathcal{M})$ when v is big for manifolds $\mathcal{M}$ which can be parameterized with a Monge parameterization given by polynomials with integer coefficients. Thus one might expect that similar arguments lead to the exact value of Hausdorff dimension for Diophantine approximation on other classes of algebraic curves.

Concerning the upper bound in (9) and more generally for $\dim \mathcal{S}_2(v,\Gamma)$, where Γ is a planar curve non-degenerate everywhere except a set of dimension zero, one can get

$$\dim \mathcal{S}_2(v,\Gamma) \leqslant \frac{2-v}{1+v} \qquad \text{for} \quad 1/2 < v < 1. \tag{11}$$

The estimate (11) is achieved by using the following upper bound of M. Huxly [Hux96] for the number of rational points nearby a curve $(x, f(x))$, $x \in I$, where I is a finite interval, with curvature bounded between two positive constants:

$$\#\{(p,q) \in \mathbb{Z}\times\mathbb{N} : q \leqslant Q,\ \|qf(p/q)\| < Q^{-v}\} \leqslant Q^{2-v+\varepsilon} \tag{12}$$

for all sufficiently large Q, where $\varepsilon > 0$ arbitrarily close to zero, $1/2 < v < 1$.

9. The parabola. The principle problem that leads to unsatisfactory lower bounds for $\dim \mathcal{S}_2(v,\mathcal{M})$ is absence of sufficiently good lower bounds for the number of rational points nearby $\mathcal{M}$, especially in the case when v is close to $1/n$. However this problem has recently been settled by V. Beresnevich in [Ber01] for the parabola. Moreover, there has been established that the rational points nearby the parabola are regularly distributed.

Now $\Gamma = \{(x, f(x)) : x \in I\}$, where $f(x) = x^2$ and I is an interval. It has been shown in [Ber01] that

$$\#\{(p,q) \in \mathbb{Z}\times\mathbb{N} : q \leqslant Q,\ \|qf(p/q)\| < Q^{-v},\quad p/q \in I\} \geqslant CQ^{2-v}|I| \tag{13}$$

for all sufficiently large Q, where $\varepsilon > 0$ arbitrarily close to zero, $1/2 < v < 1$ and $C > 0$ is a constant which depends on $\sup_{x\in I}|x|$ only. Moreover, the intervals of length Q^{-2+v} around the points p/q that are counted in (13) cover $\geqslant C|I|$ of the interval I. This means that the points p/q (projections of the rational points nearby the parabola on the x-axes) form a regular (or ubiquitous) system (see [Ryn92] for the definition). Therefore, on applying the Baker–Schmidt lemma [BS70] one arrive at the following lower bound:

$$\dim \mathcal{S}_2(v,\mathcal{V}_2) \geqslant \frac{2-v}{1+v}, \qquad 1/2 < v < 1. \tag{14}$$

Also it is non difficult to calculate $\dim \mathcal{S}_2(v,\mathcal{V}_2)$ for v in the range of $v > 1$ as all the rational points that may approximate some points of the parabola must lie on it in that case (see [DL]). Combining all these estimates leads to the complete exact formula for $\dim \mathcal{S}_2(v,\mathcal{V}_2)$:

$$\dim \mathcal{S}_2(v,\mathcal{V}_2) = \begin{cases} 1, & v \leqslant 1/2, \\ \dfrac{2-v}{1+v}, & 1/2 < v < 1, \\ \dfrac{1}{1+v}, & v \geqslant 1. \end{cases} \tag{15}$$

10. Open problems and conjectures.

Problem 1. Let Γ_2 be a smooth enough planar curve non-degenerate everywhere. Prove that

$$\dim \mathcal{S}_2(v,\Gamma_2) = \frac{2-v}{1+v}, \qquad 1/2 < v < 1. \tag{16}$$

The upper bound in this conjecture is (11). Thus one has to seek for the lower bound. The following is a particularly interesting special case of Problem 1:

Problem 1A. Prove that for the unit circle one has

$$\dim \mathcal{S}_2(v, \mathcal{C}_1) = \frac{2-v}{1+v}, \qquad 1/2 < v < 1. \tag{17}$$

A more general setting of Problem 1 is to consider non-degenerate curves in higher dimensions:

Problem 2. Prove that for any $n > 2$ there is an effectively computable constant $v_0(n) > 1/n$ such that for any sufficiently smooth curve Γ_n in $\mathbb{R}^n$ non-degenerate everywhere one has

$$\dim \mathcal{S}_n(v, \Gamma_n) = \frac{2 - v(n-1)}{1+v}, \qquad 1/n < v < v_0(n). \tag{18}$$

The following special case is of particular interest:

Problem 2A. Prove that for any $n > 2$ there is an effectively computable constant $v_0(n) > 1/n$ such that

$$\dim \mathcal{S}_n(v, \mathcal{V}_n) = \frac{2 - v(n-1)}{1+v}, \qquad 1/n < v < v_0(n). \tag{19}$$

Problem 3. For any $n \geqslant 3$ compute $\dim \mathcal{S}_n(v, \mathcal{M})$ for a non-degenerate manifold $\mathcal{M}$ in $\mathbb{R}^n$.

Problem 3A. The special case of Problem 3 that generalize Problem 1A is to compute $\dim \mathcal{S}_n(v, \mathbb{S}_R)$ for the sphere $\mathbb{S}_R$ of radius R, *e.g.* $R = 1$.

Problem 4. Prove that any non-degenerate manifold in $\mathbb{R}^n$ is of Khintchine type.

One has to distinguish the convergence and divergence cases in Problem 4 as it is most likely that they has to be treated with different techniques. The following special cases of Problem 4 are of interest:

Problem 4A. Prove that the parabola $\mathcal{V}_2$ is of Khintchine type for divergence.

Problem 4B. Prove that the unit circle $\mathcal{C}_1$ is of Khintchine type for convergence and for divergence.

Problem 4C. Prove that any planar curve that is sufficiently smooth and non-degenerate almost everywhere is of Khintchine type for convergence and for divergence.

Problem 4D. Prove that any non-degenerate curve in $\mathbb{R}^n$ is of Khintchine type for convergence and for divergence.

At last we would like to remind of a principle conjecture about the Hausdorff dimension in the case of linear Diophantine approximation:

Problem 5. For any $n \geqslant 3$ prove that

$$\dim \mathcal{L}_n(w, \mathcal{M}) = \dim \mathcal{M} - 1 + (n+1)/(w+1)$$

for a non-degenerate manifold $\mathcal{M}$ in $\mathbb{R}^n$.

Special cases of this problem such as curves are very welcomed.

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Department of Number Theory, Institute of Mathematics, The National Belarus Academy of Sciences, 220072, Surganova 11, Minsk, Belarus

E-mail address: beresnevich@im.bas-net.by, bernik@im.bas-net.by