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On Palais universal G-spaces and isovariant absolute extensors

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Abstract. We develop the theory of isovariant absolute extensors which were earlier introduced by R. Palais. The existence of injective objects of the isovariant category is proved and their properties are studied.

Bibliography: 23 items.

Keywords: classifying G-spaces, isovariant absolute extensor.

§ 1. Introduction

A productive tendency in the study of spaces with action of a compact group G is their examination as generalized principal G-bundles. This approach is justified by existing equivalence between categories of free G-spaces and principal G-bundles (which is reflected in the fact that the orbit projection of a free G-space is a principal G-bundle, and the total space of a principal G-bundle is canonically supplied with a free action).

When passing to arbitrary actions, we shall follow the principle of naturalness expressed in an attempt to preserve in the category of generalized bundles (being constructed) as high a number of characteristic properties of principal G-bundles as possible (among them the possibility of their homotopy classification with the help of universal principal G-bundles). If the situation with objects of such a category is perfectly clear — these are G-spaces, then the choice of morphisms is ambiguous since beyond the class of free G-spaces the equivariant maps begin to behave differently: points of an orbit are glued by its action (this is not the case for isovariant maps). This observation leads to the alternative between the category of metric G-spaces and equivariant maps (briefly, EQUIV-TOP) and the category of metric G-spaces and isovariant maps (briefly, ISOV-TOP).

However, in terms of declared positions the category EQUIV-TOP possesses an essential deficiency: here the classification of G-spaces is not possible in any reasonable sense, since all morphisms with an equivariant contractible space as a range, are G-homotopic to a constant map. It counts in favour of the isovariant category that the theorem on covering homotopy of G-spaces — an analogue of the corresponding fact for principal G-bundles, holds. To confirm finally the validity of the choice of the last mentioned category we subject the *injective objects of the category* ISOV-TOP or the *isovariant absolute extensors* (briefly, the Isov-AE-space) to detailed analysis.[1]

[1]The definition of such an object is available in the paper [1], 2.6, of R. Palais, where the term 'universal G-space' was used.

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If such an object exists, then its orbit brings about the classification of all G-spaces, and the category ISOV-TOP is homotopical [2]. It is precisely this object that should be considered as a universal generalized principal G-bundle. However, the partial results established in Palais's problem on the existence of isovariant absolute extensors deal with the finite-dimensional case under some restrictions on orbit types [1], 2.6, and without such restrictions [3]; the methods produced in the process of establishing them fail in the general infinite-dimensional case.

From Theorem 3.2 proved below the complete solution of Palais's problem follows: isovariant absolute extensors exist without any restrictions on dimension and orbit types. For example, the countable power $\mathbb{J} \rightleftharpoons (\operatorname{Con}\mathbb{T})^\omega$ of the metric cone $\operatorname{Con}\mathbb{T}$ over a discrete union $\mathbb{T}$ of all homogeneous spaces $G/H \in G$-ANE is appropriate: $\mathbb{J} \in$ Isov-AE. Therefore, there appears the possibility of developing an adequate theory of extensors of the category ISOV-TOP, where the principal objects of examination are isovariant absolute neighbourhood extensors, Isov-ANE-spaces.

It turns out that the isovariant absolute extensor constructed in Theorem 3.2 possesses the extra property of extension: each partial equivariant (not necessarily isovariant) map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{X} \in$ Isov-ANE has a neighbourhood extension $\psi\colon \mathbb{U} \to \mathbb{X}$ isovariant on the complement (Theorem 3.3). This combined property of extension in its turn implies several important relations between injective objects of the isovariant and (more studied) equivariant categories. We single out among them the following properties:

- each Isov-A[N]E-space is an Equiv-A[N]E-space (Theorem 3.5);
- each G-map $\varphi\colon \mathbb{Z} \to \mathbb{X} \in$ Isov-ANE is approximated arbitrarily closely by an isovariant map $\psi\colon \mathbb{Z} \to \mathbb{X}$, and each G-homotopy $F\colon \mathbb{Z} \times I \to \mathbb{X}$ between isovariant maps can be transformed into an isovariant homotopy between the same maps (Theorem 8.4).

This implies the important conclusion that the properties of equivariant and isovariant homotopy equivalences for Isov-ANE-spaces are equivalent (Theorem 8.1).

In what follows the theory of isovariant extensors developed for the compact Lie group G is applied to the investigation of the equivariant homotopy category EQUIV-HOMOT, the objects of which are Equiv-ANE-spaces and the morphisms — G-homotopy classes of equivariant maps. We substantiate the thesis that this category needs to be studied jointly with the arising isovariant homotopy category ISOV-HOMOT, the objects of which are Isov-ANE-spaces and the morphisms are isovariant homotopy classes of isovariant maps. As established, the forgetful functor

$$\mathfrak{F}\colon \text{ISOV-HOMOT} \to \text{EQUIV-HOMOT}$$

is an equivalence of categories, the inverse of which will be Borel's homotopy functor

$$[\mathrm{E}]\colon \text{EQUIV-HOMOT} \to \text{ISOV-HOMOT}$$

constructed in the paper. Therefore, the study of the G-homotopy type of Equiv-ANE-spaces is completely and sufficiently constructively reduced to the investigation of the isovariant homotopy type of Isov-ANE-spaces. In so doing the volume of information about G-homotopy type is not reduced and invoking new objects, isovariant extensors, permits us to succeed in the solution of a number of problems.

For example, it is possible to define in a natural way the endofunctor of passing to the bundle orbit of the given type $\mathscr{F} \subset \mathrm{Conj}_G$ in the category EQUIV-HOMOT (which is sufficiently complicated to make without previous preparation, as, for example, even nice spaces have a bad local structure of orbit bundle of a given type).

The constructed equivalence of homotopy categories admits an essential refinement related to the spaces with a fixed orbit type $\mathscr{F} \subset \mathrm{Conj}_G$, for which purpose we develop (as a preliminary) the theory of extensors of full subcategories: $\mathrm{ISOV}_{\mathscr{F}}$-TOP $\subset$ ISOV-TOP and $\mathrm{EQUIV}_{\mathscr{F}}$-TOP $\subset$ EQUIV-TOP, the objects of which are metric G-spaces with the orbit type $\mathscr{F}$. The absolute neighbourhood extensors of these categories are connected by the following relation: $\mathrm{Isov}_{\mathscr{F}}$-ANE $\subset$ $\mathrm{Equiv}_{\mathscr{F}}$-ANE (property (1), § 6), the class of $\mathrm{Equiv}_{\mathscr{F}}$-ANE-space coincides with Equiv-ANE-spaces, which have the orbit type $\mathscr{F}$ (Theorem 6.1). It turns out that in the absolute case the forgetful functor $\mathfrak{F}\colon$ $\mathrm{ISOV}_{\mathscr{F}}$- HOMOT $\to$ $\mathrm{EQUIV}_{\mathscr{F}}$- HOMOT is an equivalence of full subcategories ISOV- HOMOT and EQUIV- HOMOT, the objects of which are metric $\mathrm{Isov}_{\mathscr{F}}$-ANE-spaces and $\mathrm{Equiv}_{\mathscr{F}}$-ANE-spaces, respectively.

A number of equivariant homotopy invariants are closely connected with $\mathscr{F}$-classifying spaces in the sense of [4], [5] (differently, terminal objects of the category $\mathrm{EQUIV}_{\mathscr{F}}$- HOMOT). As an example, the generalized cohomology of an acting group G from [6], [7] is Bredon's cohomology of $\mathscr{F}$-classifying spaces. We take a new glance at such spaces with the help of the *concentration effect* based on a simple observation: each $\mathscr{F}$-orbit bundle of an isovariant absolute extensor is an $\mathscr{F}$-classifying space. Therefore, the various $\mathscr{F}$-classifying spaces are situated in a unique isovariant absolute extensor, the orbit space of which turns out to be partitioned into classifying spaces for appropriate principal bundles. This method permits one to perform effective calculations and to find in specific cases the generalized cohomology of the group G in an explicit form.

The present paper owes its appearance to one more circumstance. Investigating in [8] the orbit space $\exp(S^1)/S^1$ of the hyperspace $\exp(S^1)$ of all nonvoid compact subsets with the natural action of the circle S^1, Toruńczyk and West made the important observation that some bundles of orbit types are endowed with the structure of an Eilenberg-MacLain complex. Unfortunately, their Lemma 1 is wrong: each map $f\colon \exp(S^1) \to (0,1]$ such that inequality $d_H(A,B) < f(A)$ holds, where d_H is the Hausdorff metric, implies that the embedding $(S^1)_B \subset (S^1)_A$ of stabilizers is discontinuous at any point A with nontrivial stabilizer. This circumstance destroys the most part of the paper. West communicated to us that, nevertheless, some results can be saved (this is not true for Lemma 9), provided that suitable changes in Lemma 1 are made. Later similar structures turned up in studies of the topology of the Banach-Mazur compactum [9], [10] (we point out that the paper [11] written at the same time contains a similar error in Lemma 5 as in [8]).

Are all these observations a curious demonstration of properties of concrete spaces or embodiments of a general regularity? One aim of this paper is to give an answer to this intriguing question. It turns out that it is precisely the absolute isovariant extensors that are responsible for similar phenomena: for each Isov_{S^1}-AE-space $\mathbb{W}$ the orbit space $W_{\mathscr{D}}$ of discrete orbit types is an Eilenberg-MacLain complex $\mathrm{K}(\mathbb{Q}, 2)$ (see Theorem 12.1).

We return to [8] once more and note that in a forthcoming paper we will show the universality (in the sense of Palais) of the S^1-space $\exp(S^1)$ (as well as the universality of the $\mathrm{O}(2)$-space of convex bodies in $\mathbb{R}^2$). Hence all results of the paper of Toruńczyk and West remain true in spite of the gaps that were revealed.

§ 2. Preliminary facts and results

In what follows we shall assume all spaces (all maps) to be metric (continuous, respectively), if they do not arise as a result of some constructions or if the opposite is not claimed; all acting groups are assumed to be compact Lie groups.

We present the basic notions of the theory of G-spaces [12]. An action of a compact group G on a space $\mathbb{X}$ is a continuous map μ from the product $G\times\mathbb{X}$ into $\mathbb{X}$ satisfying the following properties:

- $\mu(g,\mu(h,x))=\mu(g\cdot h,x)$;
- $\mu(e,x)=x$ for all $x\in\mathbb{X}$, $g,h\in G$ (here e is the unity of the group G).

As a rule, $\mu(g,x)$ will be written as $g\cdot x$ or just gx. A space $\mathbb{X}$ with an action of the group G is called a *G-space*. The map $f\colon\mathbb{X}\to\mathbb{Y}$ of G-spaces is called a *G-map* or an *equivariant map* if $f(g\cdot x)=g\cdot f(x)$ for all $x\in\mathbb{X}$, $g\in G$.

The subset $\{g\cdot x\mid g\in G\}=G\cdot x$ is called the *orbit* $G(x)$ of the point $x\in\mathbb{X}$ which turns out to be closed. The natural map $\pi=\pi_{\mathbb{X}}\colon\mathbb{X}\to\mathbb{X}/G$, $x\mapsto G(x)$, of the space $\mathbb{X}$ into the space $\mathbb{X}/G$ of the quotient partition is said to be the *orbit projection*. We call the space of the quotient partition, equipped with the quotient topology induced by π, the *orbit space*. We will denote it by $X\rightleftharpoons\mathbb{X}/G$, provided that no confusion occurs. The subset A is called *invariant* or a *G-subset* if $\pi^{-1}\pi(A)=G\cdot A$.

For each point $x\in\mathbb{X}$ the subset $G_x=\{g\in G\mid g\cdot x=x\}$ is a closed subgroup of G and is called a *stabilizer of x*. For each closed subgroup $H<G$ let us consider the following subsets of $\mathbb{X}$:

$$\mathbb{X}^H=\{x\in\mathbb{X}\mid H\cdot x=x\}=\{x\in\mathbb{X}\mid H\subset G_x\}$$

(*the set of H-fixed points*),

$$\mathbb{X}_H=\{x\in\mathbb{X}\mid H=G_x\},\qquad \mathbb{X}_{(H)}=\{x\in\mathbb{X}\mid H\text{ conjugates with }G_x\}.$$

Let $\mathscr{F}$ be a family of orbit types (that is, a subset of the set Conj_G of conjugate classes of closed subgroups of G). Then the set $\mathbb{X}_{\mathscr{F}}\rightleftharpoons\{x\mid(G_x)\in\mathscr{F}\}\subset\mathbb{X}$ is called the *$\mathscr{F}$-orbit bundle of* $\mathbb{X}$. We say that a G-space $\mathbb{X}$ is of *orbit type $\mathscr{F}$* or simply is a G-*$\mathscr{F}$-space* if $\mathbb{X}=\mathbb{X}_{\mathscr{F}}$.

Note that metric G-orbit spaces of the type $\mathscr{F}$ and G-maps between them form a category, which is denoted by $G_{\mathscr{F}}$-TOP or EQUIV$_{\mathscr{F}}$-TOP, if it is clear which group G we are talking about. We will freely use the symbol 'G-' or 'Equiv-' meaning *equivariant*. If '$***$' is any notion from nonequivariant topology, then 'G-$***$' or 'Equiv-$***$' means the corresponding equivariant analogue.

The equivariant map $f\colon\mathbb{X}\to\mathbb{Y}$ is called *isovariant* if f preserves stabilizers, that is, $G_x=G_{f(x)}$ for every $x\in\mathbb{X}$. The following useful fact — the *equimorphism criterion* — is well-known (see [12], p. 77, Proposition 10 and [5], 8.1.3).

Theorem 2.1. *The isovariant map $f\colon \mathbb{X} \to \mathbb{Y}$ is an equimorphism if and only if f induces a homeomorphism of orbit spaces X and Y.*

The category formed by metric G-spaces of orbit type $\mathscr{F}$ and isovariant maps is denoted by ISOV$_{\mathscr{F}}$-TOP (it is always clear about the group G in question).

We introduce several concepts related to extension of G-maps in the category $\mathscr{C}$ coinciding with ISOV$_{\mathscr{F}}$-TOP or with EQUIV$_{\mathscr{F}}$-TOP. A space $\mathbb{X}$ with an action of compact group G which is an object of the category $\mathscr{C}$ is called an *absolute neighbourhood $\mathscr{C}$-extensor* (denoted by $\mathbb{X} \in \mathscr{C}$-ANE) if for each morphism $\varphi\colon \mathbb{A} \to \mathbb{X}$ from $\mathscr{C}$, defined on a closed G-subset $\mathbb{A} \subset \mathbb{Z}$ of a G-space $\mathbb{Z}$ and called a *partial $\mathscr{C}$-morphism*, can be extended into some G-neighbourhood $\mathbb{U} \subset \mathbb{Z}$ of $\mathbb{A}$ to a morphism $\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X} \in \mathscr{C}$. If it is always possible to make $\mathbb{U}$ equal to $\mathbb{Z}$, then $\mathbb{X}$ is called an *absolute $\mathscr{C}$-extensor*, $\mathbb{X} \in \mathscr{C}$-AE. If the acting group G is trivial (that is, all spaces are considered without actions), then this notion is transformed into the notion of absolute [neighbourhood] extensors for metric spaces, A[N]E (see [14], [15]).

If $\mathscr{C}$ coincides with the category EQUIV-TOP (ISOV-TOP), then absolute [neighbourhood] $\mathscr{C}$-extensors will be called equivariant [neighbourhood] extensors (isovariant [neighbourhood] extensors) or briefly — Equiv-A[N]E- or Equiv$_G$-A[N]E-spaces (briefly as Isov-A[N]E- or Isov$_G$-A[N]E-spaces). In what follows we will denote injective objects of the category EQUIV$_{\mathscr{F}}$-TOP (ISOV$_{\mathscr{F}}$-TOP) by Equiv$_{\mathscr{F}}$-A[N]E (Isov$_{\mathscr{F}}$-A[N]E).

A metric G-space $\mathbb{X}$ is called an *equivariant absolute neighbourhood retract*, $\mathbb{X} \in$ G-ANR, if for each closed G-embedding $\mathbb{X}$ into a metric G-space $\mathbb{Z}$ there exists a neighbourhood G-retraction $r\colon \mathbb{U} \to \mathbb{X}$, $r \circ r = r$. If it is always possible to choose a G-retraction r defined on $\mathbb{U} = \mathbb{Z}$, then $\mathbb{X}$ is called an *equivariant absolute retract*, $\mathbb{X} \in$ G-AR. Since each metric G-space $\mathbb{X}$ can be closedly G-embedded into a G-AE-space (see [13]), we have G-A[N]E $\equiv$ G-A[N]R.

The *slice map* is, by definition, a G-map $\alpha\colon \mathbb{X} \to G/H$, $H < G$, into a homogeneous space. The well-known Slice Theorem (see, for instance, [12]) asserts that each orbit has a neighbourhood $\mathbb{U}$ which admits a slice map $\varphi\colon \mathbb{U} \to G/G_x \cong_G G(x)$ which is the identity on this orbit. This fact is equivalent to the fact that each homogeneous space G/H belongs to the class G-ANE.

If a $\mathscr{C}$-homotopy $H\colon \mathbb{X} \times [0,1] \to \mathbb{Y}$ joins morphisms $f, g\colon \mathbb{X} \to \mathbb{Y}$ of the category $\mathscr{C}$, then it will be briefly written as $f \simeq_{\mathrm{Equiv}} g$, provided that $\mathscr{C}$ = EQUIV-TOP; $f \simeq_{\mathrm{Isov}} g$, provided that $\mathscr{C}$ = ISOV-TOP. The theorem on extension of $\mathscr{C}$-homotopy formulated below has a proof coinciding in outline with the proof of the classic Borsuk Theorem on homotopy.

Theorem 2.2. *Let $\mathbb{X} \in \mathscr{C}$-ANE and $\mathbb{Z} \in \mathscr{C}$. If a $\mathscr{C}$-homotopy $H\colon \mathbb{A} \times [0,1] \to \mathbb{X}$ joins partial $\mathscr{C}$-maps $f, g\colon \mathbb{A} \to \mathbb{X}$, and f admits an extension to a $\mathscr{C}$-map $\widehat{f}\colon \mathbb{Z} \to \mathbb{X}$, then there exists an extension of the $\mathscr{C}$-homotopy H to a $\mathscr{C}$-homotopy $\widehat{H}\colon \mathbb{Z} \times [0,1] \to \mathbb{X}$ connecting $\widehat{f}$ with some $\mathscr{C}$-map $\widehat{g}\colon \mathbb{Z} \to \mathbb{X}$.*

A $\mathscr{C}$-map $f\colon \mathbb{X} \to \mathbb{Y}$ is called a *$\mathscr{C}$-homotopy equivalence* if there exists a $\mathscr{C}$-map $g\colon \mathbb{Y} \to \mathbb{X}$ such that $g \circ f$ and $f \circ g$ are $\mathscr{C}$-homotopic to $\mathrm{Id}_{\mathbb{X}}$ and $\mathrm{Id}_{\mathbb{Y}}$, respectively. It is easily seen that if f is an Isov-homotopy equivalence (an Equiv-homotopy equivalence), then $f_H\colon \mathbb{X}_H \to \mathbb{Y}_H$ ($f^H\colon \mathbb{X}^H \to \mathbb{Y}^H$) is a homotopy equivalence for each subgroup $H < G$.

Let $\mathfrak{R}$ be the set of all irreducible orthogonal representations of G (including the trivial representation), and $\mathbb{R}_\varrho$ and $\mathbb{D}_\varrho$ the space $\varrho \in \mathfrak{R}$ of the representation and its unit ball. An *equivariant Hilbert cube* $\mathbb{Q}$ for a compact metric group G is $\bigl(\prod\{\mathbb{D}_\varrho \mid \varrho \in \mathfrak{R}\}\bigr)^\omega$. An *equivariant Hilbert space* $\mathbb{L}_2$ is

$$\biggl\{(v_\varrho) \in \biggl(\bigoplus_{\varrho\in\mathfrak{R}} \mathbb{R}_\varrho\biggr)^\omega \biggm| \Sigma\|v_\varrho\|^2 < \infty\biggr\}.$$

Let us consider two G-maps $\mathbb{C} \xrightarrow{g} \mathbb{A} \xleftarrow{f} \mathbb{B}$. We define the fibrewise product of G-spaces $\mathbb{C}$ and $\mathbb{B}$ respectively of the maps g and f to be the G-subset $\{(c,b) \mid g(c) = f(b)\} \subset \mathbb{C}\times\mathbb{B}$ and denote it by $\mathbb{C}_g \times_f \mathbb{B}$. The projections $\mathbb{D} = \mathbb{C}_g \times_f \mathbb{B}$ onto the factors $\mathbb{C}$ and $\mathbb{B}$ generate the G-maps $\check{f}\colon \mathbb{D} \to \mathbb{C}$ and $\check{g}\colon \mathbb{D} \to \mathbb{B}$. The map $\check{f}$ is called the *map parallel to* f, and $\check{g}$ is the *map parallel to* g. If the group G is trivial, then the action in the fibrewise product is absent.

The most important example of the fibrewise product in the theory of topological transformation groups is provided by isovariant maps. If $\pi_{\mathbb{X}}\colon \mathbb{X} \to X$ is an orbit projection and $\varphi\colon Y \to X$ is a continuous map into an orbit space, then the fibrewise product $\mathbb{Y} \rightleftharpoons Y_\varphi \times_{\pi_{\mathbb{X}}} \mathbb{X}$ is endowed with the action of the group G in the natural manner: $g\cdot(y,x) = (y, g\cdot x)$. It is customary to call the fibrewise product $\mathbb{Y}$ by the G-space *induced from* $\mathbb{X}$ *by the map* $\varphi\colon Y \to X$, and it is denoted by $\varphi^*(\mathbb{X})$.

It is easily seen that the orbit space $\mathbb{Y} = \varphi^*(\mathbb{X})$ coincides with Y, the orbit projection $\pi_{\mathbb{Y}}\colon \mathbb{Y} \to Y$ is parallel to $\pi_{\mathbb{X}}$ and the map $\check{\varphi}\colon \mathbb{Y} \to \mathbb{X}$, which is parallel to φ, is isovariant. It turns out that the converse fact (the proof of which easily follows from the equimorphism criterion) also holds.

Proposition 2.3. *Let $h\colon \mathbb{Y} \to \mathbb{X}$ be an isovariant map, $\widetilde{h}\colon Y \to X$ the map of orbit spaces generated by h. Then $\mathbb{Y}$ is the fibrewise product $(\widetilde{h})^*(\mathbb{X}) = Y_{\widetilde{h}} \times_{\pi_{\mathbb{X}}} \mathbb{X}$. Moreover, h and $\widetilde{h}$ and also the orbit projections $\pi_{\mathbb{Y}}$ and $\pi_{\mathbb{X}}$ are parallel.*

§ 3. The existence of isovariant absolute extensors

We say that a G-space $\mathbb{X}$ *has an action complexity highly competitive with an action complexity of a G-space* $\mathbb{Y}$ (briefly, $\mathbb{Y} \leqslant \mathbb{X}$) if there exists an isovariant map from $\mathbb{Y}$ into $\mathbb{X}$. It is easy to establish that $\mathbb{Y} \leqslant \mathbb{X}$ if and only if one of the following properties holds: either $\mathbb{Y}$ admits a closed G-embedding into $L \times \mathbb{X}$ where L is a linear normed space, or the G-space $\mathbb{Y}$ is induced from $\mathbb{X}$ by some map $f\colon Y \to X$ of its orbit spaces.

Definition 3.1. Each G-space $\mathbb{X}$ with the greatest action complexity (that is, each G-space $\mathbb{Y}$ admits an isovariant map into $\mathbb{X}$) is called an Isov-*generating* space.

An example of Isov-generating space is the countable power $\mathbb{J} \rightleftharpoons (\operatorname{Con}\mathbb{T})^\omega$ of the metric cone $\operatorname{Con}\mathbb{T}$ over a discrete union $\mathbb{T}$ of all homogeneous spaces $G/H \in G$-ANE (see [16]). The following theorem describes the Isov-AE-spaces among products of G-spaces.

Theorem 3.2. *Let* Equiv-AE-*space* $\mathbb{X}_i$ *be an* Isov-*generating space for all* $i \geqslant 1$. *Then* $\prod\{\mathbb{X}_i \mid i \geqslant 1\} \in$ Isov-AE.

Since $\mathbb{J}$ is an Isov-generating space, and also is an Equiv-AE-space, and, moreover, $\mathbb{J} \cong \mathbb{J}^\omega$, it follows by Theorem 3.2 that $\mathbb{J} \in$ Isov-AE. We will prove Theorem 3.2 in the following refined form, which enables us to endow the Isov-AE-spaces with several additional properties.

Theorem 3.3. *Let an* Equiv-AE-*space* $\mathbb{X}_i$ *be an* Isov-*generating space for every* $i \geqslant 1$. *Then for each partial G-map* $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{X} \rightleftharpoons \prod\{\mathbb{X}_i \mid i \geqslant 1\}$ *there exists a G-map* $\psi\colon \mathbb{Z} \to \mathbb{X}$ *extending* φ *such that* $\psi\restriction_{\mathbb{Z}\setminus\mathbb{A}}$ *is an isovariant map* (*that is*, ψ *is isovariant on the complement*).

Proof. Since the class of the Equiv-AE-spaces is closed with respect to countable products, we have $\mathbb{X} \in$ Equiv-AE. Therefore there exists an equivariant map $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$ extending φ. Consequently, for the proof it is sufficient to find a G-map $\psi\colon \mathbb{Z} \to \mathbb{X}$ extending $\widehat{\varphi}\restriction_{\mathbb{A}}$ such that $\psi\restriction_{\mathbb{Z}\setminus\mathbb{A}}$ is an isovariant map. Since $\mathbb{A} \subset \mathbb{X}$ is closed, it is possible to choose a sequence of invariant neighbourhoods $\mathbb{Z} = \mathbb{U}_0 \Supset \mathbb{U}_1 \Supset \cdots$ such that $\bigcap\{\mathbb{U}_i \mid i \geqslant 0\} = \mathbb{A}$.[2] Let $\chi_i\colon \mathbb{Z} \to [0,1]$, $i \geqslant 1$, be G-functions such that $\chi_i^{-1}(0) \supset \mathbb{Z}\setminus\mathbb{U}_i$ and $\chi_i^{-1}(1) \supset \mathbb{U}_{i+1}$.

We represent the map $\widehat{\varphi}$ in the coordinate form $\prod\widehat{\varphi}_i$, where $\widehat{\varphi}_i\colon \mathbb{Z} \to \mathbb{X}_i$ is an equivariant map, and fix an isovariant map $e_i\colon \mathbb{Z} \to \mathbb{X}_i$ (which exists since $\mathbb{X}_i$ is an Isov-generating space). Let $H_i\colon \mathbb{Z}\times I \to \mathbb{X}_i$ be an Equiv-homotopy joining e_i with $\widehat{\varphi}_i$ (which exists since $\mathbb{X}_i \in$ Equiv-AE). It is clear that for $n \geqslant 1$ the formula

$$\xi_n = \widehat{\varphi}_1 \times \cdots \times \widehat{\varphi}_{n-1} \times H_n(z, \chi_{n-1}(z)) \times e_{n+1} \times \cdots$$

defines the map $\xi_n\colon \mathbb{Z} \to \mathbb{X}$ isovariant outside $\mathbb{U}_{n-1}$. Then the map $\psi_n\colon \mathbb{Z} \to \mathbb{X}$, coinciding with ξ_i on $\mathbb{U}_{i-1}\setminus\mathbb{U}_i$ for $i \leqslant n$ and coinciding with $\widehat{\varphi} = \widehat{\varphi}_1 \times \widehat{\varphi}_2 \times \widehat{\varphi}_3 \times \cdots$ on $\mathbb{U}_n$, is continuous and isovariant outside $\mathbb{U}_n$. Since the sequence $\{\psi_n\}$ uniformly converges, its limit $\psi\colon \mathbb{Z} \to \mathbb{X}$ is a continuous map coinciding with φ on $\mathbb{A}$. Since almost all the $\{\psi_i\}$ outside $\mathbb{U}_n$ coincide with ψ_{n+1}, the map ψ is isovariant on the complement to $\mathbb{A}$.

We fix a closed topological embedding $j\colon X \hookrightarrow L$ of the orbit space of an arbitrary G-space $\mathbb{X}$ into some linear normed space L [15]. Since $\mathbb{J} = (\mathrm{Con}\,\mathbb{T})^\omega$ is an Isov-generating space [16], there exists an isovariant map $f\colon \mathbb{X} \to \mathbb{J}$. It is evident that the product $(j \circ p) \times f$ is a closed topological G-embedding of $\mathbb{X}$ into the G-space $\mathbb{Y} \rightleftharpoons L \times \mathbb{J}$, which is easily seen to belong to Isov-AE. Hence we have proved the following result.

Theorem 3.4. *Each G-space admits a closed G-embedding into an* Isov-AE-*space* $L \times \mathbb{J}$.

Other examples of Isov-AE-spaces are the equivariant Hilbert space $\mathbb{L}_2$ and the space $\mathrm{C}(G, \mathbb{R}^1)$ of all continuous functions (with the metric of uniform convergence), the regular representation of the group G.

Since $L \times \mathbb{J} \in$ Equiv-AE, one can easily deduce by Theorem 3.3 and Theorem 3.4 the important relation between injective objects of isovariant and equivariant categories.

Theorem 3.5. *For each* Isov-A[N]E-*space* $\mathbb{X}$ *Theorem* 3.3 *holds and therefore each* Isov-A[N]E-*space is an* Equiv-A[N]E-*space.*

[2]We shall say that the embedding $A \subset B$ is *strong* and write $A \Subset B$ if $\mathrm{Cl}\,A \subset \mathrm{Int}\,B$.

We say that a G-space $\mathbb{X}$ is *locally* Isov-*contractible at a point* $x \in \mathbb{X}$ if for each $\varepsilon > 0$ there exists an equivariant homotopy $H\colon \mathbb{U} \times [0,1] \to \mathbb{X}$ of some neighbourhood $\mathbb{U} \ni x$ such that

(1) $H_t\colon \mathbb{U} \to \mathbb{X}$ is an isovariant map for all $t < 1$;
(2) $H_0 = \mathrm{Id}$ and H_1 maps $\mathbb{U}$ into the orbit $G(x)$;
(3) $\operatorname{diam} H(x,[0,1]) \leqslant \varepsilon$.

If properties (1)–(2) hold for some point $x \in \mathbb{X}^G$ and $\mathbb{U} = \mathbb{X}$, we say that the G-space $\mathbb{X}$ is Isov-*contractible*; the G-space $\mathbb{X}$ is called *locally* Isov-*contractible* (briefly, $\mathbb{X} \in$ Isov-LC), if $\mathbb{X}$ is locally Isov-contractible at each point $x \in \mathbb{X}$.

It easily follows by Theorem 3.3 that each Isov-AE-space is Isov-contractible. The local variant of this fact is a little more complicated to establish.

Theorem 3.6. *Each* Isov-ANE-*space* $\mathbb{Y}$ *is an* Isov-LC-*space.*

Proof. Using the concrete form of the Isov-AE-space $\mathbb{J}$ it is easy to prove that $L \times \mathbb{J} \in$ Isov-LC for each linear normed spaces L. Next we should apply Theorem 3.4 and invoke the closedness of the class of Isov-LC-spaces under isovariant retracting.

If $\mathbb{X} \in$ Equiv-AE, then the orbit bundle $\mathbb{X}_{\leqslant(H)} \rightleftharpoons \{x \in \mathbb{X} \mid G_x$ conjugates with a subgroup from $H\}$ is not an AE-space in general. However, the isovariant extensors have a completely different behaviour.

Theorem 3.7. *If $H < G$ is a subgroup of a metric compact group G, then $\mathbb{X} \in$* Isov_G-A[N]E *implies $\mathbb{Y} \in$* Isov_H-A[N]E *where $\mathbb{Y}$ coincides with $\mathbb{X}$ or with one of its G-subspaces: $\mathbb{X}_{\leqslant(H)}$ or $\mathbb{X}_{\leqslant H} \rightleftharpoons \{x \in \mathbb{X} \mid G_x \subset H\}$.*

Proof. Each partial H-isovariant map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{Y}$ generates the partial G-map

$$G \times_H \mathbb{Z} \hookleftarrow G \times_H \mathbb{A} \xrightarrow{\Phi} \mathbb{X}, \quad \text{given by the formula } \Phi([g,a]) = g \cdot \varphi(a).$$

In general, Φ is not an isovariant map since $H_x = G_x \cap H$ and the stabilizers H_x and G_x are different. In view of Theorem 3.3, $\mathbb{X} \in \mathrm{Isov}_G$-AE implies that there exists a global G-extension $\widehat{\Phi}\colon G \times_H \mathbb{Z} \to \mathbb{X}$ of the G-map Φ, which is G-isovariant on the complement.

Next we will understand *orbit considerations* as the comparison of G-stabilizers of points and their images under an isovariant map. Since the G-stabilizer of each point u from $\{e\} \times_H (\mathbb{Z} \setminus \mathbb{A})$ is preserved under $\widehat{\Phi}$, we have $G_{\widehat{\Phi}(u)} \subset H$. Therefore $\widehat{\Phi}(G \times_H (\mathbb{Z} \setminus \mathbb{A})) \subset \mathbb{X}_{\leqslant(H)}$ and $\widehat{\Phi}(\{e\} \times_H (\mathbb{Z} \setminus \mathbb{A})) \subset \mathbb{X}_{\leqslant H}$.

It is evident that the restriction of $\widehat{\Phi}$ to $\{e\} \times_H \mathbb{Z} \cong_H \mathbb{Z}$ is the desired global H-isovariant extension $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{Y}$ onto $\mathbb{Z}$ of the map φ. Hence $\mathbb{Y} \in \mathrm{Isov}_H$-AE. The local variant of the theorem, $\mathbb{X} \in \mathrm{Isov}_H$-ANE, is proved analogously.

It is easily established that the metric Isov-A[N]E-space $\mathbb{X}$ is a disjoint union of A[N]E-spaces $X_H, H < G$. It turns out that the family $\{\mathbb{X}_H \mid H < G\}$ in the aggregate also possesses an extensor property.

Theorem 3.8. *Let* $\mathbb{X}$ *be a metric* Isov-A[N]E-*space. Then*:
(a) $X_H \in$ A[N]E *for each subgroup* $H < G$;
(b) *the family* $\{\mathbb{X}_H \mid H < G\}$ *possesses the property*[3] equi-LAE, *in particular, it is equipotentionally locally contractible.*

Proof. (a) For each partial map $W \hookleftarrow A \xrightarrow{\varphi} \mathbb{X}_H$ we take into consideration the partial G-map $G/H \times W \hookleftarrow G/H \times A \xrightarrow{\Phi} \mathbb{X}$ given by the formula $\Phi(gH, a) = g \cdot a$. If $\mathbb{X} \in$ Isov-AE, then there exists a global G-extension $\widehat{\Phi}\colon G/H \times W \to \mathbb{X}$ of the G-map Φ, which is G-isovariant on the complement. Then the restriction of $\widehat{\Phi}$ to $(e \cdot H) \times A$ is the desired extension of the map φ. It turns out that the local variant is proved analogously.

(b) First we introduce a preliminary definition. We say that the *partial map* $W \hookleftarrow A \xrightarrow{\varphi} \mathbb{X}$ into G-space $\mathbb{X}$ *converges to* $x_0 \in \mathbb{X}$, provided that there exists a point $w_0 \in A$ such that $W \setminus \{w_0\}$ is the discrete union $\coprod\{W_i \mid i \geqslant 1\}$ of clopen (in W) subspaces $\{W_i\}$ for which $\operatorname{dist}(w_0, W_i) < 2^{-i}$, $\varphi(A_i) \subset \mathbb{X}_{H_i}$, where $A_i \rightleftharpoons W_i \cap A$, and $\varphi(w_0) = x_0$. It is clear that W is obtained from the discrete union $\coprod\{W_i\}$ by adding a point; it follows by continuity of φ that $\{\varphi(A_i)\} \to \{x_0\}$.

Assuming that the property equi-LAE fails at the point $x_0 \in \mathbb{X}$, we easily construct a partial map $W \hookleftarrow A \xrightarrow{\varphi} \mathbb{X}$ converging to x_0 which cannot be extended to a neighbourhood U of the point w_0 in such a manner that

(4) $W_i \cap U$ is transformed into $\mathbb{X}_{H_i}$.

Let us consider an isovariant map $\varphi'\colon A \times G \to \mathbb{X}$ given by the formula $\varphi'(a, g) = g \cdot \varphi(a)$, and also the equivalence relation on $W \times G$

$$(w_1, g_1) \approx (w_2, g_2) \iff \begin{cases} w_1 = w_2 \text{ and } g_2 \cdot g_1^{-1} \text{ lies in } H_i & \text{if } w_1 \in W_i, \\ g_2 \cdot g_1^{-1} \text{ lies in } G_{x_0} & \text{if } w_1 = w_2 = w_0. \end{cases}$$

It is clear that this relation is preserved by the action of the group G, the quotient space $\mathbb{W} \rightleftharpoons (W \times G)/{\approx}$ is a metric G-space and the quotient space $\mathbb{A} \rightleftharpoons (A \times G)/{\approx}$ is a closed invariant space of $\mathbb{W}$.

It is easily seen that the formula $\psi([a, g]) = \varphi'(w, g)$ correctly defines the continuous partial isovariant map $\mathbb{W} \hookleftarrow \mathbb{A} \xrightarrow{\psi} \mathbb{X}$. If $\mathbb{X} \in$ Isov-AE, then there exists a global isovariant extension $\widehat{\psi}\colon \mathbb{W} \to \mathbb{X}$ of ψ. Note that $(W \times \{e\})/{\approx}$ (here e is the unity of the group) naturally lies in $\mathbb{W}$ and is homeomorphic to W. Since $\widehat{\psi}$ is isovariant, we have $\widehat{\psi}((W_i \times \{e\})/{\approx}) \subset X_{H_i}$. Restricting $\widehat{\psi}$ to $(W \times \{e\})/{\approx}$, we obtain the continuous extension of φ onto the whole of W satisfying (4),— a contradiction.

The local variant of the theorem is proved analogously.

We cite without proof several examples indicating the degree of distinction of these two classes:

(5) for each family $\mathscr{F} \subset \operatorname{Orb}_G$ there exists an Equiv-ANE-space with $\operatorname{Orb}(\mathbb{X}) = \mathscr{F}$ and contractible $\mathbb{X}^H$ for each $(H) \in \mathscr{F}$;
(6) the family $\mathscr{F} \subset \operatorname{Orb}_G$ is an orbit type of some Isov-ANE-space if and only if $\mathscr{F}$ is closed in Orb_G;
(7) if $\operatorname{Con} \mathbb{X} \in$ Isov-AE and $\mathbb{X}^G \in$ AE, then $\mathbb{X} \in$ Isov-AE.

[3] That is, for each neighbourhood U of the point $x \in \mathbb{X}$ there exists a neighbourhood V such that for each partial map $Z \hookleftarrow A \xrightarrow{\varphi} V \cap \mathbb{X}_H$ there exists its extension $\widehat{\varphi}\colon Z \to U \cap \mathbb{X}_H$.

§ 4. Isovariant extensors and twisted products

A G-space admitting a slice map is useful for representing in the form of a twisted product. Recall the corresponding definitions: let G be a compact Lie group, $\mathbb{S}$ a metric H-space where $H < G$. Consider the diagonal action of the group H on the product $G \times \mathbb{S}$ defined by the formula $h \cdot (g, y) \rightleftharpoons (g \cdot h^{-1}, h \cdot y)$, and we denote the element $H \cdot (g, y) = \{(g \cdot h^{-1}, h \cdot y) \mid h \in H\}$ of the orbit space $(G \times \mathbb{S})/H$ by $[g, y]$. It turns out that by the formula $g_1 \cdot [g, y] = [g_1 \cdot g, y]$, where $g, g_1 \in G$ and $y \in \mathbb{S}$ the continuous action of the group G on the orbit space $(G \times \mathbb{S})/H$ is correctly defined; it is called the *twisted product* and is denoted by $G \times_H \mathbb{S}$.

If $f \colon \mathbb{X} \to G/H$ is a slice map of the G-space, then $\mathbb{X}$ can be represented as the twisted product $G \times_H \mathbb{S}$ where $\mathbb{S} = f^{-1}([H])$ is an H-slice. For instance, this and other properties of the twisted product are presented in [12]. We take a closer examination of the interrelation of the isovariant extensor properties of the spaces $\mathbb{X}$ and $\mathbb{S}$ mentioned above.

Theorem 4.1. *If* $\mathbb{S} \in \mathrm{Isov}_H$-A[N]E, *then* $G \times_H \mathbb{S} \in \mathrm{Isov}_G$-ANE.

Proof. We examine only the case $\mathbb{S} \in \mathrm{Isov}_H$-AE, for which purpose we consider the composite χ of the partial G-isovariant map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} G \times_H \mathbb{S}$ and the slice map $G \times_H \mathbb{S} \xrightarrow{\psi} G/H$, $\psi[g, s] = [g \cdot H]$. Since $G/H \in G$-ANE, there exists an extension of the composite $\chi = \psi \circ \varphi$ to a G-map $\widehat{\chi} \colon \mathbb{U} \to G/H$, defined on a neighbourhood $\mathbb{U} \supset \mathbb{A}$.

The preimages $\mathbb{U}_0 = \widehat{\chi}^{-1}[H] \supset \mathbb{A}_0 = \chi^{-1}[H]$ are H-spaces and, in view of the condition $\mathbb{S} \in \mathrm{Isov}_H$-AE, the partial H-map $\mathbb{U}_0 \hookleftarrow \mathbb{A}_0 \overset{\varphi \rightleftharpoons \chi\restriction}{\longrightarrow} \{e\} \times_H \mathbb{S} \cong_H \mathbb{S}$ has a global H-extension $v \colon \mathbb{U}_0 \to \mathbb{S}$ which is H-isovariant on the complement. Since $G \times_H \mathbb{U}_0 = \mathbb{U}$, the desired neighbourhood G-isovariant extension $\widehat{\varphi} \colon \mathbb{U} \to G \times_H \mathbb{S}$ is constructed according to the formula $\widehat{\varphi}([g, z]) = [g, v(z)]$, where $[g, z]$ is an arbitrary point of $\mathbb{U} = G \times_H \mathbb{U}_0$. The correctness of $\widehat{\varphi}$ easily follows from the fact that v is equivariant. Therefore $G \times_H \mathbb{S} \in \mathrm{Isov}_G$-ANE.

The local variant of the theorem ($\mathbb{S} \in \mathrm{Isov}_H$-ANE) is established analogously.

We examine the inheritance of isovariant extensor properties when passing from the twisted product $G \times_H \mathbb{S}$ to its slice $\mathbb{S} \rightleftharpoons f^{-1}([H])$.

Theorem 4.2. *Let H be a closed subgroup of the compact Abelian Lie group G. If* $G \times_H \mathbb{S} \in \mathrm{Isov}_G$-ANE, *then the H-space $\mathbb{S}$ is an* Isov_H-ANE.

Proof. In parallel with the action of the group H on G we consider the right regular action of H on G. Thus the continuous action of the Lie group $H \times H$ on G given by the formula $(h, h') \cdot g = h \cdot g \cdot (h')^{-1}$ occurs. Since H is the $(H \times H)$-orbit of the point e and $H \times H$ is a Lie group, H is an $\mathrm{Equiv}_{H \times H}$-ANE-space and therefore there exists an $(H \times H)$-retraction $r \colon U \to H$ of some $(H \times H)$-neighbourhood $U \subset G$. Hence $r(h \cdot g \cdot h') = h \cdot r(g) \cdot h'$ for all $h, h' \in H$. It is easy to check that the $(H \times H)$-stabilizers of all points $g \in G$ coincide with H. Therefore the retraction r is $(H \times H)$-isovariant.

Since $G \times_H \mathbb{S} \in \mathrm{Isov}_G$-ANE, we have by Theorem 3.7 that $G \times_H \mathbb{S} \in \mathrm{Isov}_H$-ANE and, for this reason, its open H-subset $U \times_H \mathbb{S}$ is an Isov_H-ANE.

Let us consider the twisted product $R \colon U \times_H \mathbb{S} \to H \times_H \mathbb{S} \equiv \mathbb{S}$, $[u, s] \mapsto r(u) \cdot s$, of the $(H \times H)$-isovariant retraction r and $\mathrm{Id}_{\mathbb{S}}$, which is correctly defined and is an

H-isovariant retraction of the H-space $U \times_H \mathbb{S}$ onto $\mathbb{S}$. Since $U \times_H \mathbb{S} \in \mathrm{Isov}_H$-ANE, $\mathbb{S} \in \mathrm{Isov}_H$-ANE as an image of an H-isovariant retraction.

Corollary 4.3. *If the compact Abelian Lie group G acts on the space $\mathbb{X} \in \mathrm{Isov}_G$-ANE and $\varphi\colon \mathbb{U} \to G/H$ is a slice at the point x, then $\mathbb{S}_\varphi = \varphi^{-1}([H]) \in \mathrm{Isov}_H$-ANE.*

Proof. Since $\mathbb{U} \cong_G G \times_H \mathbb{S}_\varphi \in \mathrm{Isov}_G$-ANE, we have $\mathbb{S}_\varphi \in \mathrm{Isov}_H$-ANE in view of Theorem 4.2.

In completion of this section we examine preserving isovariant extensor properties when passing to orbit spaces.

Theorem 4.4. *If $\mathbb{W} \in \mathrm{Isov}_G$-AE, H is a normal subgroup of G (this is written as $H \lhd G$ for short), then the G/H-space $\mathbb{W}/H \in \mathrm{Isov}_{G/H}$-AE.*

Since the proof of the theorem is based on the method of extending the action of transformation groups, we briefly recall its essence (see [16]). A diagram $\mathscr{D}$ of the form $\mathbb{X} \xrightarrow{p} X \xhookrightarrow{i} Y$, in which the G-space $\mathbb{X}$ has an orbit type $\mathscr{F} \subset \mathrm{Orb}_G$, $p\colon \mathbb{X} \to X$ is an orbit projection and i is a closed topological embedding of the orbit space X into the metric space Y, is called *$\mathscr{F}$-admissible.* We say that the *problem of extending the action* (*PEA* for short) is *solvable for an $\mathscr{F}$-admissible diagram $\mathscr{D}$* if there exists an equivariant embedding $j\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ into the metric G-space $\mathbb{Y}$ of orbit type $\mathscr{F}$ (called an *$\mathscr{F}$-solution* of the problem of extending the action for the given diagram) covering i, that is, the embedding $\widetilde{j}\colon X \hookrightarrow p(\mathbb{Y})$ of orbit spaces inducing j exactly coincides with i (from the definition it follows that the embedding j is closed and $p(\mathbb{Y}) = Y$). We say that the *problem of extending the action is $\mathscr{F}$-solvable* if there exists an $\mathscr{F}$-solution of the PEA for each $\mathscr{F}$-admissible diagram $\mathscr{D}$.

Let $\mathbb{W}$ be an Isov-AE-space. Then using the fact that $\mathbb{W}_{\mathscr{F}} \in \mathrm{Isov}_{\mathscr{F}}$-AE and the line of reasoning from [16], we can prove that

(1) the problem of extending the action is $\mathscr{F}$-solvable if and only if the orbit space $W_{\mathscr{F}} \in$ AE.

If the family of orbit types $\mathscr{F}$ coincides with Orb_G, then in all definitions mentioned above the specification $\mathscr{F}$ is omitted. It was established in [16], [17] that the problem of extending the action is solvable. Moreover,

(2) if $H \lhd G$, then for each diagram of the form $\mathbb{X} \xrightarrow{p} \mathbb{X}/H \xhookrightarrow{i} \mathbb{Z}$ in which $p\colon \mathbb{X} \to \mathbb{X}/H$ is an H-orbit projection and i a closed topological (G/H)-embedding of the orbit space $\mathbb{X}/H$ into the metric (G/H)-space $\mathbb{Z}$, there exists a G-embedding $j\colon \mathbb{X} \hookrightarrow \overline{\mathbb{Z}}$ into the metric G-space $\overline{\mathbb{Z}}$ inducing i.

Proof of Theorem 4.4. Let us consider the partial (G/H)-map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{W}/H$. Let $\varphi^*(\mathbb{W})$ be a fibrewise product of the H-orbit projection $\pi\colon \mathbb{W} \to \mathbb{W}/H$ and the H-map $\varphi\colon \mathbb{A} \to \mathbb{W}/H$. In view of (2), there exists a G-embedding $j\colon \varphi^*(\mathbb{W}) \hookrightarrow \overline{\mathbb{Z}}$ into the metric G-space $\overline{\mathbb{Z}}$ inducing the embedding $\varphi^*(\mathbb{W})/H \hookrightarrow \overline{\mathbb{Z}}/H$ which identically coincides with the embedding $\mathbb{A} \hookrightarrow \mathbb{Z}$. The arising partial G-map $\overline{\mathbb{Z}} \hookleftarrow \varphi^*(\mathbb{W}) \xrightarrow{\check{\varphi}} \mathbb{W} \in$ Isov-AE parallel to φ has a global G-extension $\chi\colon \overline{\mathbb{Z}} \to \mathbb{W}$, which is G-isovariant on the complement. It is easily seen that the (G/H)-map $\widehat{\varphi}\colon \mathbb{Z} = \overline{\mathbb{Z}}/H \to \mathbb{W}/H$ induced by χ (that is, $\widehat{\varphi}(z) = H \cdot \chi(z')$ where $H \cdot z' = z$) is a (G/H)-extension of φ and is G-isovariant on the complement.

§ 5. Homotopy dense embeddings

The equivariant embedding $e\colon \mathbb{X}_0 \hookrightarrow \mathbb{X}$ is called *G-homotopy dense* if there exists a G-homotopy $H\colon \mathbb{X} \times I \to \mathbb{X}$ such that $H_0 = \mathrm{Id}$ and $\operatorname{Im} H_t \subset \mathbb{X}_0$ for every $t > 0$. It is easily seen that this definition is equivalent to the possibility of extending the partial G-map $\mathbb{X} \times I \hookleftarrow \mathbb{X} \times \{0\} \xrightarrow{\mathrm{Id}} \mathbb{X}$ to a G-map $\varphi\colon \mathbb{U} \to \mathbb{X}$ given on a G-neighbourhood such that $\varphi(\mathbb{U} \setminus (\mathbb{X} \times \{0\})) \subset \mathbb{X}_0$. Note that a G-homotopy dense embedding is a G-homotopy equivalence.

Theorem 5.1. *Let G be a compact Lie group, $\mathbb{X} \in \mathrm{Isov}_G$-ANE. Then for each subgroup $H < G$*

(a) *$\mathbb{X}_{\leqslant H} = \{x \in \mathbb{X} \mid G_x \subset H\}$ is H-homotopy densely contained in $\mathbb{X}_{\leqslant (H)} = \{x \in \mathbb{X} \mid G_x$ conjugates with a subgroup of $H\}$;*

(b) *the embedding $\mathbb{X}_H \hookrightarrow \mathbb{X}^H$ is homotopy dense.*

Proof. The partial H-map $\mathbb{Z} \rightleftharpoons \mathbb{A} \times I \hookleftarrow \mathbb{A} \times \{0\} \xrightarrow{\varphi} \mathbb{X}_{\leqslant (H)}$ and the map $Z \rightleftharpoons A \times I \hookleftarrow A \times \{0\} \xrightarrow{\psi} \mathbb{X}^H$ generate the partial G-map $G \times_H \mathbb{Z} \hookleftarrow G \times_H \mathbb{A} \xrightarrow{\Phi} \mathbb{X}_{\leqslant (H)}$ given by the formula $\Phi([g,a]) = g \cdot \varphi(a)$, and the map $G/H \times Z \hookleftarrow G/H \times A \xrightarrow{\Psi} \mathbb{X}_{\geqslant (H)}$ given by the formula $\Psi(gH, a) = g \cdot \psi(a)$, respectively.

Since $\mathbb{X} \in \mathrm{Isov}$-ANE, there exists a G-extension $\widehat{\Phi}$ of the map Φ and a G-extension $\widehat{\Psi}$ of the map Ψ onto some G-neighbourhoods $\mathbb{U}$ and $\mathbb{V}$, respectively, which are G-isovariant on the complement. As in the proof of Theorem 3.7, we apply orbit considerations: since the G-stabilizer of each point from $\{e\} \times_H \mathbb{Z}$ is contained in H and the G-stabilizer of each point from $eH \times Z$ equals H, it follows that $\Phi\bigl((\mathbb{U} \setminus \mathbb{A}) \cap (\{e\} \times_H \mathbb{Z})\bigr) \subset \mathbb{X}_{\leqslant H}$ and $\Psi\bigl((\mathbb{V} \setminus \mathbb{A}) \cap (eH \times Z)\bigr) \subset \mathbb{X}_H$. It is evident that the restriction of $\widehat{\Phi}$ to $\mathbb{U} \cap (\{e\} \times_H \mathbb{Z})$ and the restriction of $\widehat{\Psi}$ to $\mathbb{V} \cap (eH \times Z)$ are the desired neighbourhood extensions.

Theorem 5.2. *Let G be a compact Lie group, $\mathbb{X} \in \mathrm{Isov}_G$-ANE. Then the map $\mathbb{X}^K \hookrightarrow \mathbb{X}^H$ is Z-embedding for each subgroup $H < K$.*

Proof. In view of Theorem 5.1, (b) the composite of the embeddings $\mathbb{X}_H \hookrightarrow \mathbb{X}^H \setminus \mathbb{X}^K$ and $\mathbb{X}^H \setminus \mathbb{X}^K \hookrightarrow \mathbb{X}^H$ coincides with the homotopy dense embedding $\mathbb{X}_H \hookrightarrow \mathbb{X}^H$. Therefore the embedding $\mathbb{X}^H \setminus \mathbb{X}^K \hookrightarrow \mathbb{X}^H$ is homotopy dense.

§ 6. Extensors of the category $\mathbf{ISOV}_{\mathscr{F}}$-TOP, $\mathscr{F} \subset \mathrm{Conj}_G$

Let $\mathbb{W}$ be an Isov-AE-space. It follows from Theorem 3.5 that each partial G-map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{W}_{\mathscr{F}}$ given on a closed subset of the G-$\mathscr{F}$-space $\mathbb{Z}$ can be extended onto some G-neighbourhood $\mathbb{U} \subset \mathbb{Z}$ of the set $\mathbb{A}$ to a G-map $\widehat{\varphi}\colon \mathbb{U} \to \mathbb{W}$ isovariant on the complement $\mathbb{Z} \setminus \mathbb{A}$. Therefore $\widehat{\varphi}(\mathbb{U}) \subset \mathbb{W}_{\mathscr{F}}$, that is, $\mathbb{W}_{\mathscr{F}} \in \mathrm{Equiv}_{\mathscr{F}}$-ANE.

It is established analogously to Theorem 3.4 that each G-space $\mathbb{X} \in \mathrm{Isov}_{\mathscr{F}}$-ANE may be considered as a closed G-subset in the G-space $L \times \mathbb{W}_{\mathscr{F}}$ where L is a linear normed space. Therefore $\mathbb{X}$ is an isovariant neighbourhood retract of $L \times \mathbb{W}_{\mathscr{F}} \in \mathrm{Equiv}_{\mathscr{F}}$-ANE, that is,

(1) each $\mathrm{Isov}_{\mathscr{F}}$-ANE-space $\mathbb{X}$ is an $\mathrm{Equiv}_{\mathscr{F}}$-ANE-space.

Let us prove the fact, which together with (1) permits us to assert that

(2) $\mathrm{Isov}_{\mathscr{F}}$-ANE $\subset \mathrm{Equiv}_{\mathscr{F}}$-ANE $\subset$ Equiv-ANE.

Theorem 6.1. *If* $\mathbb{X}$ *is an* $\mathrm{Equiv}_{\mathscr{F}}$-ANE-*space, then* $\mathbb{X}$ *is an* Equiv-ANE-*space.*

The proof of the theorem is established with the help of the so-called *majorized problem of extending the group action.* Recall that the solution $s\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ of the PEA for an admissible diagram $\mathscr{D} = (\mathbb{X} \xrightarrow{p} X \overset{i}{\hookrightarrow} Y)$ *majorizes* the solution $s_1\colon \mathbb{X} \hookrightarrow \mathbb{Y}_1$ of the PEA for $\mathscr{D}$ if there exists a G-map $h\colon \mathbb{Y} \to \mathbb{Y}_1$ such that $h \circ s = s_1$, $h\restriction_{\mathbb{X}} = \mathrm{Id}_{\mathbb{X}}$ and $\widetilde{h} = \mathrm{Id}_Y$.

Let us take a (not necessarily $\mathscr{F}$-) solution $s\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ of the PEA for the $\mathscr{F}$-admissible diagram $\mathscr{D}$ (the existence of such a solution is established in [16]). In [18] the key fact for establishing the relation $\mathrm{Equiv}_{\mathscr{F}}$-ANE $\subset$ Equiv-ANE was proved:

(3) if G is a compact Lie group, then there exists an $\mathscr{F}'$-solution $s_1\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ of the PEA for $\mathscr{D}$ majorized by s, $s \geqslant s_1$ (here $\mathscr{F}' \rightleftharpoons \mathscr{F} \cup \{(G)\}$).

Proof of Theorem 6.1. Assume that the G-space $\mathbb{X} \in \mathrm{Equiv}_{\mathscr{F}}$-ANE is closedly contained in a G-space $\mathbb{Y}$. In view of (3), $\mathbb{X}$ is closedly contained in a G-space $\mathbb{Y}_1$ of the orbit type $\mathscr{F}' = \mathscr{F} \cup \{(G)\}$, for which there is a G-map $h\colon \mathbb{Y} \to \mathbb{Y}_1$ with $h\restriction_{\mathbb{X}} = \mathrm{Id}_{\mathbb{X}}$. Since $\mathbb{X}$ is an $\mathrm{Equiv}_{\mathscr{F}}$-ANE-space, there exists a neighbourhood G-retraction $\mathbb{Y}_1 \hookleftarrow \mathbb{V} \xrightarrow{r} \mathbb{X}$. Hence $\mathbb{Y} \hookleftarrow \mathbb{U} \rightleftharpoons h^{-1}(\mathbb{V}) \xrightarrow{r \circ h} \mathbb{X}$ is the desired neighbourhood G-retraction, that is, $\mathbb{X} \in$ G-ANR $\equiv$ G-ANE.

The following fact is proved analogously to the case $\mathscr{F} = \mathrm{Conj}_G$ (see Theorems 3.8 and 5.1).

Theorem 6.2. *Let* $\mathscr{F} \subset \mathrm{Conj}_G$ *and* $H < G$, $(H) \in \mathscr{F}$. *If*

(a) $\mathbb{X}$ *is an* $\mathrm{Isov}_{\mathscr{F}}$-AE-*space, then* $\mathbb{X}_H$ *is contractible and the family* $\{\mathbb{X}_H \mid (H) \in \mathscr{F}\}$ *possesses the property* equi-LAE;

(b) $\mathbb{X} \in \mathrm{Isov}_{\mathscr{F}}$-ANE, *then the embedding* $e\colon \mathbb{X}_H \hookrightarrow \mathbb{X}^H$ *is homotopy dense.*

Let $\mathbb{W}$ be an Isov-AE-space and $\mathscr{F} \subset \mathscr{C} \subset \mathrm{Orb}_G$. It is clear that the embedding $W_{\mathscr{F}} \hookrightarrow W_{\mathscr{C}}$ of orbit spaces is a homotopy equivalence, provided that the embedding $\mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$ is an equivariant homotopy equivalence. Let us find conditions when the latter is possible.

Theorem 6.3. *The following conditions are pairwisely equivalent*:

(a) $\mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$ *is an equivariant homotopy equivalence*;

(b) $\mathbb{W}_{\mathscr{F}} \in \mathrm{Equiv}_{\mathscr{C}}$-AE;

(c) *for each* G-$\mathscr{C}$-*space* $\mathbb{Z}$ *and its closed* G-$\mathscr{F}$-*subspace* $\mathbb{Y} \subset \mathbb{Z}$ *there exists a* G-*map* $h\colon \mathbb{Z} \to \overline{\mathbb{Z}}$ *into a* G-$\mathscr{F}$-*space* $\overline{\mathbb{Z}}$ *which* G-*homeomorphically maps* $\mathbb{Y}$ *onto a closed* G-*subspace* $\overline{\mathbb{Y}} \subset \overline{\mathbb{Z}}$;

(d) *there exists a* G-*map* $h\colon \mathbb{W}_{\mathscr{C}} \to \overline{\mathbb{Z}}$ *into a* G-$\mathscr{F}$-*space* $\overline{\mathbb{Z}}$ *which* G-*homeomorphically maps* $\mathbb{W}_{\mathscr{F}}$ *onto a closed* G-*subspace* $\overline{\mathbb{Y}} \subset \overline{\mathbb{Z}}$.

Proof. Since the implication $(a) \Leftarrow (b)$ is easy, we dwell on the inverse implication $(a) \Rightarrow (b)$. Let $\theta\colon \mathbb{W}_{\mathscr{C}} \to \mathbb{W}_{\mathscr{F}}$ be a map G-homotopy inverse to the embedding $\mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$. In what follows we consider the partial G-map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{W}_{\mathscr{F}}$, $\mathrm{Orb}\,\mathbb{Z} \subset \mathscr{C}$, which, in view of $\mathbb{W}_{\mathscr{C}} \in \mathrm{Isov}_{\mathscr{C}}$-AE, has a G-extension $\chi\colon \mathbb{Z} \to \mathbb{W}_{\mathscr{C}}$ isovariant on the complement. Thus the composite $\eta \rightleftharpoons \theta \circ \chi\colon \mathbb{Z} \to \mathbb{W}_{\mathscr{F}}$ is such that $\eta\restriction_{\mathbb{A}} \simeq_G \varphi$. It follows by Theorem 2.2 (Borsuk Theorem on homotopy) that the map φ has a G-extension $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{W}_{\mathscr{F}}$.

The implication $(b) \Rightarrow (c)$. Consider a partial G-map

$$\mathbb{Z} \hookleftarrow \mathbb{Y} \xrightarrow{\alpha} \mathbb{W}_{\mathscr{F}} \in \mathrm{Isov}_{\mathscr{F}}\text{-AE},$$

where α is an isovariant map. Since $\mathbb{W}_{\mathscr{F}} \in \mathrm{Equiv}_{\mathscr{C}}$-AE and $\operatorname{Orb} \mathbb{Z} \subset \mathscr{C}$, there exists an equivariant extension $\widehat{\alpha}\colon \mathbb{Z} \to \mathbb{W}_{\mathscr{F}}$. It is possible to canonically represent the map $\widehat{\alpha}$ as the composite of $h\colon \mathbb{Z} \to \overline{\mathbb{Z}}$ and $\gamma\colon \overline{\mathbb{Z}} \to \mathbb{W}_{\mathscr{F}}$, where h is a G-map identical on the orbit spaces and γ is an isovariant map. Since α is isovariant, $h\restriction_{\mathbb{Y}}\colon \mathbb{Y} \to h(\mathbb{Y})$ is a G-homeomorphism. It follows that $\mathbb{Y}$ is naturally contained in $\overline{\mathbb{Z}}$.

Since the implication $(c) \Rightarrow (d)$ is trivial, we examine the remaining implication $(d) \Rightarrow (b)$. Consider a partial G-map $\mathbb{W}_{\mathscr{C}} \hookleftarrow \mathbb{W}_{\mathscr{F}} \xrightarrow{\mathrm{Id}} \mathbb{W}_{\mathscr{F}}$. By hypothesis there exists a G-map $h\colon \mathbb{W}_{\mathscr{C}} \to \overline{\mathbb{Z}}$ into a G-$\mathscr{F}$-space $\overline{\mathbb{Z}}$ which G-homeomorphically maps $\mathbb{W}_{\mathscr{F}}$ onto a closed G-subspace $\overline{\mathbb{Y}} \subset \overline{\mathbb{Z}}$.

Since $\mathbb{W}_{\mathscr{C}} \in \mathrm{Isov}_{\mathscr{C}}$-AE, the partial G-map $\overline{\mathbb{Z}} \hookleftarrow \overline{\mathbb{Y}} \xrightarrow{h^{-1}} \mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$ admits an equivariant extension $g\colon \overline{\mathbb{Z}} \to \mathbb{W}_{\mathscr{C}}$ isovariant on the complement $\overline{\mathbb{Z}} \setminus \overline{\mathbb{Y}}$. Since $\operatorname{Orb} \overline{\mathbb{Z}} \subset \mathscr{F}$, it follows by orbit considerations that $g(\overline{\mathbb{Z}} \setminus \overline{\mathbb{Y}}) \subset \mathbb{W}_{\mathscr{F}}$. Therefore the composite $\mathbb{W}_{\mathscr{C}} \xrightarrow{h} \overline{\mathbb{Z}} \xrightarrow{g} \mathbb{W}_{\mathscr{F}}$ is the desired G-retraction.

Now we examine in more detail the important special case of $\mathscr{F} = (H)$ and $\mathscr{C} = \{(K) \mid K < H\}$. We say that a closed G-subspace $\mathbb{Y} \subset \mathbb{X}$ is a *strong isovariant deformation G-retract of* $\mathbb{X}$ if there exists an equivariant homotopy $H\colon \mathbb{X} \times [0,1] \to \mathbb{X}$ such that $H = \mathrm{Id}$ on $\mathbb{Y} \times [0,1] \cup \mathbb{X} \times \{0\}$; H_t is an isovariant map for $t < 1$, and H_1 is a G-retraction of $\mathbb{X}$ onto $\mathbb{Y}$.

Theorem 6.4. *Let $H \lhd G$ be a normal subgroup of a compact Lie group and $\mathbb{W} \in$* Isov_G-AE. *Then the G-embedding $\mathbb{W}_H \hookrightarrow \mathbb{W}_{\leqslant H}$ is a strong isovariant deformation G-retraction.*

Proof. We consider the closed G-embedding $\mathbb{W}_{(H)} \hookrightarrow \mathbb{W}_{\leqslant(H)}/H$ and an H-orbit projection, the G-map $h\colon \mathbb{W}_{\leqslant(H)} \to \mathbb{W}_{\leqslant(H)}/H$. Since h on $\mathbb{W}_{(H)}$ is identical, condition (d) holds. In view of Theorem 6.3, $\mathbb{W}_{(H)} \in \mathrm{Equiv}_{\leqslant(H)}$-AE, which implies the G-retraction $r\colon \mathbb{W}_{\leqslant(H)} \to \mathbb{W}_{(H)}$.

We now construct a strong isovariant G-deformation $\widehat{F}$ of $\mathbb{W}_{\leqslant(H)}$ onto $\mathbb{W}_{(H)}$. To do this, we consider the partial G-map

$$\mathbb{W}_{\leqslant(H)} \times [0,1] \hookleftarrow \mathbb{W}_{\leqslant(H)} \times \{0,1\} \cup \mathbb{W}_{(H)} \times [0,1] \xrightarrow{F} \mathbb{W}_{\leqslant(H)}$$

where the G-map F coincides with r on $\mathbb{W}_{\leqslant(H)} \times \{1\}$ and with the identity map on the remaining part. Since $\mathbb{W} \in$ Isov-AE, the G-map F has a global G-extension $\widehat{F}\colon \mathbb{X}_{\leqslant(H)} \times I \to \mathbb{W}$ isovariant on the complement. We again invoke the orbit considerations and obtain that the target space of $\widehat{F}$ is contained in $\mathbb{W}_{\leqslant(H)}$.

The remark that the equalities $\mathbb{W}_{(H)} = \mathbb{W}_H$ and $\mathbb{W}_{\leqslant(H)} = \mathbb{W}_{\leqslant H}$ hold for a normal subgroup $H \lhd G$ completes the proof.

If $H \lhd G$ and $(H) \in \mathscr{F} \subset \mathscr{C} \subset \{(K) \mid K < H\}$, then the embedding $i\colon \mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$ is not in general a strong isovariant deformation G-retraction. However, we can assert that

(4) $i\colon \mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$ is an equivariant homotopy equivalence.

To establish this, we consider the G-retraction $r\colon \mathbb{W}_{\leqslant(H)} \to \mathbb{W}_{(H)}$ constructed in Theorem 6.4. It follows from this theorem that the restriction $r\restriction_{\mathbb{W}_{\mathscr{C}}}\colon \mathbb{W}_{\mathscr{C}} \to \mathbb{W}_{\mathscr{F}}$ and the embedding $\mathbb{W}_{\mathscr{F}} \hookrightarrow \mathbb{W}_{\mathscr{C}}$ are equivariantly homotopy reciprocally inverse. From (4) and Theorem 6.3 it follows that in this case $\mathbb{W}_{\mathscr{F}} \in \mathrm{Equiv}_{\mathscr{C}}$-AE.

By Theorem 6.4 it follows that for each normal subgroup $H \triangleleft G$ the embedding $\mathbb{W}_H \hookrightarrow \mathbb{W}_{\leqslant H}$ is a G-homotopy equivalence, and therefore the corresponding embedding of orbit spaces is a homotopy equivalence. In this connection there arises the very plausible conjecture that in general this is the case.

Conjecture 6.5. The embedding $W_{(H)} \hookrightarrow W_{\leqslant(H)}$ of orbit spaces is a homotopy equivalence for each subgroup $H < G$.

Unfortunately, this hypothesis fails, generally speaking, for subgroups which are not normal. But Theorem 6.4 admits the following generalization.

Theorem 6.6. *Let $\mathbb{W} \in \mathrm{Isov}_G$-AE, $\mathrm{N}(H) < G$ be a normalizer of a subgroup $H < G$ of the compact Lie group G. Then the embedding $\varphi\colon \mathbb{W}_H \hookrightarrow \mathbb{W}_{\leqslant H}$ is a strong isovariant $\mathrm{N}(H)$-deformation retraction, and the embedding*[4] *$\mathbb{W}_{(H)} \hookrightarrow G \times_{\mathrm{N}(H)} \mathbb{W}_{\leqslant H}$ is a strong isovariant G-deformation retraction.*

Proof. In view of Theorem 3.7, $\mathbb{W}$ is an $\mathrm{Isov}_{\mathrm{N}(H)}$-AE-space. Since the subgroup $H < G$ is normal in $\mathrm{N}(H)$, the required property of the embedding φ easily follows from Theorem 6.4 applied for $\mathrm{N}(H)$. Since $\mathbb{W}_{(H)}$ coincides with $G \times_{\mathrm{N}(H)} \mathbb{W}_H$, the required property of the embedding $\mathbb{W}_{(H)} \hookrightarrow G \times_{\mathrm{N}(H)} \mathbb{W}_{\leqslant H}$ easily follows.

From the above theorem it follows that the G-embedding $\mathbb{W}_{(H)} \hookrightarrow \mathbb{W}_{\leqslant(H)}$ and the natural G-map $G \times_{\mathrm{N}(H)} \mathbb{W}_{\leqslant H} \to \mathbb{W}_{\leqslant(H)}$ are equivalent in the category $\mathrm{EQUIV}_{\mathscr{F}}$-HOMOT. Consequently, Conjecture 6.5 is equivalent to the hypothesis that the map $\mathbb{W}_{\leqslant H}/\mathrm{N}(H) \to W_{\leqslant(H)}$ of orbit spaces is a homotopy equivalence.

Now we shall show that Conjecture 6.5 fails for some dihedral subgroup $\mathrm{D}_n < G \rightleftharpoons \mathrm{O}(2)$:

(5) if $\mathbb{W} \in \mathrm{Isov}_{\mathrm{O}(2)}$-AE, then the spaces $W_{(\mathrm{D}_n)}$ and $W_{\leqslant(\mathrm{D}_n)}$ are not homotopy equivalent.

First we note that D_n coincides with its Weyl group, and therefore $\mathbb{W}_{\mathrm{D}_n} \cong W_{\mathrm{D}_n}$. Since $\mathbb{W}_{\mathrm{D}_n} \in$ AE, we obtain, assuming the opposite to (4), $W_{\leqslant(\mathrm{D}_n)} \in$ AE for each n. Hence $W_{<(G)}$ is an AE-space as a growing sequence of open subsets $W_{\leqslant(\mathrm{D}_1)} \subset W_{\leqslant(\mathrm{D}_2)} \subset \dots \subset W$. However, this orbit space by Theorem 13.1 is not contractible — a contradiction.

§ 7. Universal principal bundles and isovariant extensors

Let $\mathbb{W}$ be an Isov-AE-space. We show that the orbit space W turns out to be partitioned into classifying spaces $W_{(H)}$, $(H) \in \mathrm{Orb}_G$, for appropriate principal bundles.

Consider the free $\mathrm{W}(H)$-space $\mathbb{W}_H$, where $\mathrm{W}(H) = \mathrm{N}(H)/H$ is the Weyl group, which is an AE-space in view of Theorem 3.8. Therefore the restriction of the

[4] Recall that $\mathbb{W}_{(H)}$ is naturally identified with $G \times_{\mathrm{N}(H)} \mathbb{W}_H$.

orbit projection $p\restriction\colon \mathbb{W}_H \to W_{(H)}$ is a universal principal $\mathrm{W}(H)$-bundle. Next we consider the exact sequence of homotopy groups corresponding to $p\restriction$:

$$\cdots \to \pi_{i+1}(\mathbb{W}_H) \xrightarrow{p_*} \pi_{i+1}(W_{(H)}) \xrightarrow{\partial} \pi_i(\mathrm{W}(H)) \xrightarrow{e_*} \pi_i(\mathbb{W}_H) \to \cdots .$$

Since all $\pi_i(\mathbb{W}_H)$ are equal to 0,

(1) $\pi_{i+1}(W_{(H)}) = \pi_i(\mathrm{W}(H))$ for all $i \geqslant 1$ and $\pi_1(W_{(H)}) = 0$ for the connected space $\mathrm{W}(H)$.

If $H < G$ is a normal subgroup, then the Weyl group $\mathrm{W}(H)$ coincides with G/H, $\mathbb{W}_H = \mathbb{W}_{(H)}$ is a free G/H-space, and therefore the orbit projection $p\colon \mathbb{W}_{(H)} \to W_{(H)}$ is a universal principal G/H-bundle with $\pi_{i+1}(W_{(H)}) = \pi_i(G/H)$ for all $i \geqslant 1$ and $\pi_1(W_{(H)}) = 0$ for the connected space G/H.

Consider a further example of a universal principal bundle generated by the Isov-AE-space $\mathbb{W}$. In view of Theorem 3.7, $\mathbb{W}_{\leqslant H}, H < G$, is an Isov_H-AE-space and hence it is an H-contractible space. Therefore the orbit space $\mathbb{W}_{\leqslant H}/H$ is also contractible. If $H \lhd G$ is a normal subgroup, then $\mathbb{W}_{\leqslant H}/H$ is a free G/H-space, and therefore

(2) the G/H-orbit projection $p\colon \mathbb{W}_{\leqslant H}/H \to W_{\leqslant H}$ represents a universal principal G/H-bundle.

Examining the exact sequence of homotopy groups corresponding to this bundle, we come to the conclusion that

(3) $\pi_{i+1}(W_{\leqslant H}) = \pi_i(G/H)$ for all $i \geqslant 1$ and $\pi_1(W_{\leqslant H}) = 0$ if G/H is connected.

Now let $K \lhd G$ and let $H \lhd G$ be normal subgroups and $\mathbb{W}$ an Isov_G-AE-space as before. If $K < H$, then one can consider the natural G-embedding $e\colon \mathbb{W}_{\leqslant K} \hookrightarrow \mathbb{W}_{\leqslant H}$ of G-spaces and the induced embedding $\widetilde{e}\colon W_{\leqslant K} \hookrightarrow W_{\leqslant H}$ of orbit spaces.

Theorem 7.1. *The homomorphism*

$$e_{i+1}\colon \pi_{i+1}(W_{\leqslant K}) \to \pi_{i+1}(W_{\leqslant H})$$

of homotopy groups generated by $\widetilde{e}$ is naturally isomorphic to the homomorphism $p_i\colon \pi_i(G/K) \to \pi_i(G/H)$ of homotopy groups generated by the quotient map $p\colon G/K \to G/H$.

Proof. The embedding $e\colon \mathbb{W}_{\leqslant K} \hookrightarrow \mathbb{W}_{\leqslant H}$ generates in natural manner the map $\alpha\colon \mathbb{W}_{\leqslant K}/K \to \mathbb{W}_{\leqslant H}/H$ of free G/K- and G/H-spaces, which in its turn generates a morphism of universal principal bundles:

$$\begin{tikzcd}
\mathbb{W}_{\leqslant K}/K \arrow[r, "\alpha"] \arrow[d] & \mathbb{W}_{\leqslant H}/H \arrow[d] \\
W_{\leqslant K} \arrow[r, hook, "e"] & W_{\leqslant H}.
\end{tikzcd}$$

Write down the morphism of exact sequences of homotopy groups of bundles generated by α:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \pi_{i+1}(\mathbb{W}_{\leqslant K}/K) \arrow[r] \arrow[d, "\alpha_{i+1}"'] & \pi_{i+1}(W_{\leqslant K}) \arrow[r] \arrow[d, "e_{i+1}"'] & \pi_i(G/K) \arrow[r] \arrow[d, "p_i"'] & \pi_i(\mathbb{W}_{\leqslant K}/K) \arrow[r] \arrow[d, "\alpha_i"'] & \cdots \arrow[d] \\
\cdots \arrow[r] & \pi_{i+1}(\mathbb{W}_{\leqslant H}/H) \arrow[r] & \pi_{i+1}(W_{\leqslant H}) \arrow[r] & \pi_i(G/H) \arrow[r] & \pi_i(\mathbb{W}_{\leqslant H}/H) \arrow[r] & \cdots
\end{tikzcd}$$

In the part of this diagram presented above only the four central groups are nontrivial. Therefore the homomorphism e_{i+1} is isomorphic to the homomorphism $p_i\colon \pi_i(G/K) \to \pi_i(G/H)$ of homotopy groups generated by the natural map $p\colon G/K \to G/H$.

§ 8. Equivalence of isovariant and equivariant homotopy equivalences

It is evident that each isovariant homotopy equivalence is an equivariant homotopy equivalence. The converse is not true since spaces of the same equivariant homotopy type may possess distinct orbit types. However, the inverse of the implication is still possible under some additional conditions, and it is the content of the following important theorem.

Theorem 8.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be* Isov-ANE*-spaces. If the isovariant map $f\colon \mathbb{X} \to \mathbb{Y}$ is an equivariant homotopy equivalence*, *then f is an isovariant homotopy equivalence.*

It follows that two Isov-ANE-spaces have the same equivariant homotopy type if and only if they have the same isovariant homotopy type.

The proof of the above result will be based on Theorem 8.4 which in its turn needs some preliminary facts. First we establish the theorem on isovariant extension of G-homotopy, which is an essential complement of Theorem 2.2 on extension of $\mathscr{C}$-homotopies.

Theorem 8.2. *Let $\mathbb{X}$ be an* Isov-ANE*-space and $H\colon \mathbb{A} \times [0,1] \to \mathbb{X}$ an equivariant ε-homotopy that joins the partial G-maps $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{f,g} \mathbb{X}$, and suppose that f has an extension to the G-map $\widehat{f}\colon \mathbb{Z} \to \mathbb{X}$ (here $\varepsilon \in \operatorname{cov}\mathbb{X}$). Then there exists a G-extension of the homotopy H to an ε-homotopy $\widehat{H}$:* $\operatorname{Cyl}\mathbb{Z} \rightleftharpoons \mathbb{Z} \times [0,1] \to \mathbb{X}$ *isovariant on the complement* $\operatorname{Cyl}\mathbb{Z} \setminus \mathrm{M}(\mathbb{Z},\mathbb{A}) = (\mathbb{Z} \setminus \mathbb{A}) \times (0,1]$ *and joining $\widehat{f}$ with some G-map $\widehat{H}(z,1)$ (here* $\mathrm{M}(\mathbb{Z},\mathbb{A}) \rightleftharpoons \mathbb{Z} \times \{0\} \cup \mathbb{A} \times [0,1]$*).*

Proof. It is easily seen that the map $\Phi\colon\ \mathrm{M}(\mathbb{Z},\mathbb{A}) \to \mathbb{X}$ given by the formula

$$\Phi(z,t) = \begin{cases} f(z) & \text{if } (z,t) \in \mathbb{Z} \times \{0\}, \\ H(z,t) & \text{if } (z,t) \in \mathbb{A} \times I, \end{cases}$$

is a continuous G-map. Since by hypothesis $\mathbb{X} \in$ Isov-ANE and $\mathrm{M}(\mathbb{Z},\mathbb{A}) \subset \mathbb{Z} \times I$ — a closed G-subspace, there exists a neighbourhood G-extension $\widehat{\Phi}\colon \mathbb{V} \to \mathbb{X}$ of the map Φ which is isovariant on the complement to $\mathrm{M}(\mathbb{Z},\mathbb{A})$.

It is easy to construct a continuous G-function $\beta\colon \mathbb{Z} \to (0,1]$ for which

(1) $\beta = 1$ on the set $\mathbb{A}$;

(2) $\{(z,t) \mid 0 \leqslant t \leqslant \beta(z)\} \subset \mathbb{V}$;

(3) $\{\widehat{\Phi}(z,[0,\beta(z)]) \mid z \in \mathbb{Z}\} \prec \varepsilon$.

The desired G-extension $\widehat{H}\colon \mathbb{Z} \times [0,1] \to \mathbb{X}$ of the homotopy H is defined by the formula $\widehat{H}(z,t) = \widehat{\Phi}(z, t \cdot \beta(z))$.

Let us apply the established fact to approximation of equivariant maps by isovariant ones.

Theorem 8.3. *Let* $\mathbb{X}$ *be an* Isov-ANE-*space and* $\varphi\colon \mathbb{Z} \to \mathbb{X}$ *a* G*-map. Then for each closed subset* $\mathbb{A} \subset \mathbb{Z}$ *and each cover* $\varepsilon \in \operatorname{cov} \mathbb{X}$ *there exists a* G*-map* $\psi\colon \mathbb{Z} \to \mathbb{X}$ *isovariant on the complement to* $\mathbb{A}$ *such that* $\varphi = \psi$ *on* $\mathbb{A}$ *and* $\operatorname{dist}(\varphi, \psi) \prec \varepsilon$.

Proof. We consider the partial G-map

$$\operatorname{Cyl} \mathbb{Z} \hookleftarrow \mathrm{M}(\mathbb{Z}, \mathbb{A}) \xrightarrow{\Phi} \mathbb{X},$$

given as $\Phi = \varphi$ on $\mathbb{Z} \times \{0\}$ and $\Phi_t = \varphi$, $t \in I$, on $\mathbb{A} \times \{t\}$. Since $\mathbb{X} \in$ Isov-ANE, it follows in view of Theorem 8.2 that the map Φ admits a G-extension to an ε-homotopy $\widehat{\Phi}\colon \operatorname{Cyl} \mathbb{Z} \to \mathbb{X}$ which is isovariant on the complement $\operatorname{Cyl} \mathbb{Z} \setminus \mathrm{M}(\mathbb{Z}, \mathbb{A})$. Then $\widehat{\Phi}(z, 1)\colon \mathbb{Z} \to \mathbb{X}$ is the desired G-map.

Putting in Theorem 8.3 $\mathbb{A} = \varnothing$, we obtain that

(4) each G-map $\varphi\colon \mathbb{Z} \to \mathbb{X} \in$ Isov-ANE is joined by a G-homotopy $F\colon \mathbb{Z} \times I \to \mathbb{X}$ with an isovariant map $\psi\colon \mathbb{Z} \to \mathbb{X}$ such that the homotopy F is isovariant on $\mathbb{Z} \times (0, 1)$.

Putting in Theorem 8.3 $\mathbb{A} = \mathbb{P} \times \{0, 1\}$, we obtain the following result.

Theorem 8.4. *Let* $\mathbb{X}$ *be an* Isov-ANE-*space. If the* G*-homotopy* $\varphi\colon \mathbb{Z} = \mathbb{P} \times I \to \mathbb{X}$ *joins two isovariant maps* $f\colon \mathbb{P} \to \mathbb{X}$ *and* $h\colon \mathbb{P} \to \mathbb{X}$, *then* f *and* h *are isovariantly homotopic.*

Proof of Theorem 8.1. Let us suppose that $h\colon \mathbb{Y} \to \mathbb{X}$ is G-homotopy inverse to f. In view of (4), one can consider that h is an isovariant map. In what follows one must apply Theorem 8.4.

§ 9. Properties of softness and the Borel functor

To define the *Borel functor* $\mathrm{E}_{\mathscr{F}}\colon$ EQUIV-TOP $\to$ ISOV$_{\mathscr{F}}$-TOP, we fix a family $\mathscr{F} \subset \mathrm{Conj}_G$ and an Isov-AE-space $\mathbb{W}$. We set $\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \rightleftharpoons \{(w, x) \mid G_w \subset G_x, (G_w) \in \mathscr{F}\}$ and associate with the G-map $f\colon \mathbb{X} \to \mathbb{Y}$ an isovariant map $\mathrm{E}_f\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathrm{E}_{\mathscr{F}}(\mathbb{Y})$ defined by the formula[5] $\mathrm{E}_f(w, x) = (w, f(x))$. In the case of $\mathscr{F} = \mathrm{Conj}_G$, let us denote for brevity by $\mathrm{E}(\mathbb{X})$ the set $\mathrm{E}_{\mathscr{F}}(\mathbb{X}) = \{(w, x) \mid G_w \subset G_x\}$ and by E the Borel functor.

One of the basic properties of the Borel functor deals with its softness, for which purpose we first give the necessary definitions. We consider the square commutative diagram $\mathscr{D}$

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\varphi"] \arrow[d, phantom, "\cap"] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{Z} \arrow[r, "\psi"] & \mathbb{Y} \end{tikzcd}$$

in the category $\mathscr{C}$ coinciding with the category ISOV$_{\mathscr{F}}$-TOP or with EQUIV$_{\mathscr{F}}$-TOP. We say that the *morphism* φ *is a partial lifting of the morphism* ψ *with respect to* f.

[5] A similar construction for the single-element family $\mathscr{F} = \{e\}$ was suggested by A. Borel.

We say that the *problem of extending the partial lifting for $\mathscr{D}$ is globally* [*locally*] *solvable* if there exists a morphism $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$ $[\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X}$, where $\mathbb{U} \subset \mathbb{Z}$ is a neighbourhood of $\mathbb{A}]$ extending φ and such that $f \circ \widehat{\varphi} = \psi$ $[f \circ \widehat{\varphi} = \psi \restriction_{\mathbb{U}}]$. Following the terminology chosen above we say that $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$ $[\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X}]$ is a global [local] lifting of ψ with respect to f.

Definition 9.1. The morphism $f\colon \mathbb{X} \to \mathbb{Y}$ of the category $\mathscr{C}$ is called: *$\mathscr{C}$-soft* [*locally $\mathscr{C}$-soft*] if for each square commutative diagram $\mathscr{D}$ in the category $\mathscr{C}$ the problem of extension of the partial lifting is globally [locally] solvable; the *$\mathscr{C}$-Hurewiczs bundle* if for each square commutative diagram $\mathscr{D}$ in the category $\mathscr{C}$, where $\mathbb{Z} = \operatorname{Cyl}\mathbb{P}$ and $\mathbb{A} = \mathrm{M}(\mathbb{P}, \mathbb{Q})$, the problem of extension of the partial lifting is globally solvable.

From the result formulated below it follows that the natural projection $p = p_{\mathbb{X}}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}$, $p(w,x) = x$, is an equivariant homotopy equivalence under some conditions.

Theorem 9.2. *If $\mathbb{X}$ is a* G-$\mathscr{F}$*-space, then* $p\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}$ *is an* $\mathrm{Equiv}_{\mathscr{F}}$*-soft map.*

Proof. Consider an admissible (for verification of $\mathrm{Equiv}_{\mathscr{F}}$-softness of the map p) square commutative diagram in the category EQUIV$_{\mathscr{F}}$-TOP

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\varphi} & \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \\ \cap & & \big\downarrow p \\ \mathbb{Z} & \xrightarrow{\psi} & \mathbb{Y}. \end{array}$$

Since $\mathbb{W} \in \mathrm{Isov}_{\mathscr{F}}$-AE, the partial map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\chi} \mathbb{W}$, where $\chi \rightleftharpoons \mathrm{q} \circ \varphi$ and $\mathrm{q} = \mathrm{q}_{\mathbb{W}}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{W}$ is the projection onto the first factor, admits a G-extension $\widehat{\chi}\colon \mathbb{Z} \to \mathbb{W}$ isovariant on the complement. For this reason, we have $G_{\widehat{\chi}(z)} = G_z \subset G_{\psi(z)}$ for all $z \notin \mathbb{A}$. Taking into consideration this remark, it is easy to check that the desired equivariant map $\widehat{\varphi}\colon \mathbb{Z} \to \mathrm{E}_{\mathscr{F}}(\mathbb{X})$ is correctly defined by the formula $\widehat{\varphi}(z) = (\widehat{\chi}(z), \psi(z))$.

It follows that $\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \in \mathrm{Equiv}_{\mathscr{F}}$-ANE, provided that the space $\mathbb{X}$ is[6] $\mathrm{Equiv}_{\mathscr{F}}$-ANE. However, the considerably more precise fact is true.

Theorem 9.3. *If the space $\mathbb{X}$ is* $\mathrm{Equiv}_{\mathscr{F}}$-ANE, *then*

(a) *the isovariant map $q = q_{\mathbb{X}}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{W}_{\mathscr{F}}$, $q(w,z) = w$, is simultaneously locally* $\mathrm{Isov}_{\mathscr{F}}$*-soft and the Hurewicz* $\mathrm{Isov}_{\mathscr{F}}$*-bundle*;

(b) $\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \in \mathrm{Isov}_{\mathscr{F}}$-ANE.

Proof. Consider an admissible (for verification of isovariant local softness of q) square commutative diagram in the category ISOV$_{\mathscr{F}}$-TOP

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\varphi} & \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \\ \cap & & \big\downarrow p \\ \mathbb{Z} & \xrightarrow{\psi} & \mathbb{W}_{\mathscr{F}}. \end{array}$$

[6] It follows from Theorem 6.1 that $\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \in$ Equiv-ANE.

Since $\mathbb{X} \in \mathrm{Equiv}_{\mathscr{F}}$-ANE and $\mathrm{Orb}\,\mathbb{Z} \subset \mathscr{F}$, the partial map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\chi} \mathbb{X}$, where $\chi \rightleftharpoons \mathrm{p}_{\mathbb{X}} \circ \varphi$ and $\mathrm{p}_{\mathbb{X}}\colon \mathrm{E}(\mathbb{X}) \to \mathbb{X}$ is the natural projection, admits a G-extension $\widehat{\chi}\colon \mathbb{Z} \to \mathbb{X}$. Since ψ is isovariant, $G_{\psi(z)} = G_z \subset G_{\widehat{\chi}(z)}$ for all $z \in \mathbb{Z}$. The desired equivariant map $\widehat{\varphi}\colon \mathbb{Z} \to \mathrm{E}_{\mathscr{F}}(\mathbb{X})$ is correctly defined by the formula $\widehat{\varphi}(z) = (\psi(z), \widehat{\chi}(z))$. Similar reasoning with the use of Borsuk's Theorem 2.2 on extension of equivariant homotopy shows that q is the Hurewicz $\mathrm{Isov}_{\mathscr{F}}$-bundle — property (a) is established.

Since $\mathbb{W}_{\mathscr{F}} \in \mathrm{Isov}_{\mathscr{F}}$-ANE, it easily follows from (a) that $\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \in \mathrm{Isov}_{\mathscr{F}}$-ANE.

Theorem 9.4. *If* $\mathbb{X}$ *is* $\mathrm{Equiv}_{\mathscr{F}}$-ANE, *then the induced map* $\widetilde{q}\colon (\mathrm{E}_{\mathscr{F}}(\mathbb{X}))/G \to W_{\mathscr{F}}$ *of orbit spaces is locally soft and simultaneously the Hurewicz bundle, and the preimage* $\widetilde{q}^{-1}([w])$ *is homeomorphic to* $\mathbb{X}^H$, *where* $H = G_w$.

This theorem follows from Theorem 9.3 and the fact established below.

Theorem 9.5. *If an isovariant map* $f\colon \mathbb{X} \to \mathbb{Y}$ *is isovariantly locally soft* (*is the Hurewicz* Isov*-bundle*), *then the map* $\widetilde{f}\colon X \to Y$ *of orbit spaces induced by* f *is locally soft* (*is the Hurewicz* Isov*-bundle*).

Proof. Consider the admissible (for verification of necessary softness of f) square commutative diagram in the category TOP placed at the left and the corresponding commutative square in the category ISOV-TOP placed at the right:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & X \\
\cap & & \downarrow{\scriptstyle \widetilde{f}} \\
Z & \xrightarrow{\psi} & Y,
\end{array}
\qquad\qquad
\begin{array}{ccc}
\varphi^*(\mathbb{X}) & \xrightarrow{\check{\varphi}} & \mathbb{X} \\
\cap & & \downarrow{\scriptstyle f} \\
\psi^*(\mathbb{Y}) & \xrightarrow{\check{\psi}} & \mathbb{Y}.
\end{array}
$$

If f is isovariantly soft, then there exists an isovariant extension $\chi\colon \psi^*(\mathbb{Y}) \to \mathbb{X}$ of the map $\check{\varphi}$ such that $f \circ \chi = \check{\psi}$. Going over to the orbit spaces, we obtain the desired extension $\widetilde{\chi}\colon Z \to X$. Analogously one can examine all the remaining cases.

§ 10. The equivalence of equivariant and isovariant homotopy categories

Consider the category EQUIV-HOMOT whose objects are Equiv-ANE-spaces and whose morphisms are G-homotopy classes of equivariant maps, and the category ISOV-HOMOT whose objects are Isov-ANE-spaces and whose morphisms are isovariant homotopy classes of isovariant maps. Since the Borel functor $\mathrm{E}_{\mathscr{F}}$ transforms the equivariant homotopy $f_t\colon \mathbb{X} \to \mathbb{Y}$ into an isovariant one $\mathrm{E}_{f_t}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathrm{E}_{\mathscr{F}}(\mathbb{Y})$, it can be considered as a *homotopy Borel functor* $[\mathrm{E}_{\mathscr{F}}]$ from the category EQUIV-HOMOT into the category $\mathrm{ISOV}_{\mathscr{F}}$-HOMOT, which is the full subcategory of ISOV-HOMOT generated by $\mathrm{Isov}_{\mathscr{F}}$-ANE-spaces. As an evident remark we point out that the composite of the homotopy Borel functor

$$[\mathrm{E}]\colon \text{EQUIV-HOMOT} \to \text{ISOV-HOMOT}$$

and the isovariant functor

$$[\mathrm{R}_{\mathscr{F}}]\colon \text{ISOV-HOMOT} \to \mathrm{ISOV}_{\mathscr{F}}\text{-HOMOT}$$

of transition to the bundle of $\mathscr{F}$-orbits coincides with the homotopy Borel functor $[\mathrm{E}_{\mathscr{F}}]\colon$ EQUIV-HOMOT $\to$ ISOV$_{\mathscr{F}}$-HOMOT.

Let us show that the homotopy Borel functor realizes the equivalence of the categories EQUIV-HOMOT and ISOV-HOMOT. Therefore the investigation of the G-homotopy type of Equiv-ANE-spaces will be completely reduced to the investigation of the isovariant homotopy type of Isov-ANE-spaces. In its turn this enables us to obtain the new information on the equivariant homotopy type of Equiv-ANE-spaces by means of a deeper examination of Isov-ANE-spaces.

Consider the full subcategory

$$\text{EQUIV}_{\mathscr{F}}\text{-HOMOT} \subset \text{EQUIV-HOMOT},$$

generated by the Equiv$_{\mathscr{F}}$-ANE-spaces. Since by Theorem 6.1 Isov$_{\mathscr{F}}$-ANE $\subset$ Equiv-ANE, the forgetful functor $\mathfrak{F}\colon$ ISOV$_{\mathscr{F}}$-HOMOT $\to$ EQUIV$_{\mathscr{F}}$-HOMOT can be correctly introduced into consideration.

Theorem 10.1. *The categories* EQUIV-HOMOT *and* ISOV-HOMOT *are equivalent, and the homotopy Borel functor*

$$[\mathrm{E}]\colon \text{EQUIV-HOMOT} \to \text{ISOV-HOMOT}$$

is an equivalence. Moreover, for each $\mathscr{F} \subset \mathrm{Conj}_G$ *the homotopy Borel functor* $[\mathrm{E}_{\mathscr{F}}]\colon$ EQUIV$_{\mathscr{F}}$-HOMOT $\to$ ISOV$_{\mathscr{F}}$-HOMOT *generated by* $\mathscr{F} \subset \mathrm{Conj}_G$ *is also an equivalence of categories, the inverse to which is the forgetful functor* $\mathfrak{F}\colon$ ISOV$_{\mathscr{F}}$-HOMOT $\to$ EQUIV$_{\mathscr{F}}$-HOMOT.

Proof. By definition of an equivalence of categories [19] it is necessary to construct a morphism (or a natural transformation) of the functor $\mathfrak{F} \circ [\mathrm{E}_{\mathscr{F}}]$ into the identity functor

$$\mathrm{Id}\colon \text{EQUIV}_{\mathscr{F}}\text{-HOMOT} \to \text{EQUIV}_{\mathscr{F}}\text{-HOMOT}$$

and a morphism of the functor $[\mathrm{E}_{\mathscr{F}}] \circ \mathfrak{F}$ into the identity functor

$$\mathrm{Id}\colon \text{ISOV}_{\mathscr{F}}\text{-HOMOT} \to \text{ISOV}_{\mathscr{F}}\text{-HOMOT},$$

components of which consist of G-homotopy and isovariant homotopy equivalences, respectively.

The first morphism is given by a collection of maps $p_{\mathbb{X}}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}$, which are G-homotopy equivalences by Theorem 9.2. It is clear that all corresponding square diagrams are exactly commutative:

$$\begin{tikzcd}
\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \arrow[r, "p_{\mathbb{X}}"] \arrow[d, "\mathrm{E}_f"] & \mathbb{X} \arrow[d, "f"] \\
\mathrm{E}_{\mathscr{F}}(\mathbb{X}) \arrow[r, "p_{\mathbb{Y}}"] & \mathbb{Y}.
\end{tikzcd}$$

The second morphism of the functors is given by a collection of isovariant homotopy equivalences $\psi(\mathbb{X})\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}$, for which all corresponding square diagrams $\mathscr{D}$ are commutative in the category ISOV$_{\mathscr{F}}$-HOMOT. For this it is sufficient to set $\psi(\mathbb{X})$ equal to an isovariant map from $\mathrm{E}_{\mathscr{F}}(\mathbb{X})$ into $\mathbb{X}$ which is equivariantly homotopic to $p_{\mathbb{X}}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}$ (such a map exists in view of Theorem 8.3 because $\mathbb{X}$ and $\mathrm{E}_{\mathscr{F}}(\mathbb{X})$ are $\mathrm{Isov}_{\mathscr{F}}$-ANE-spaces). Since $p_{\mathbb{X}}$ is an equivariant homotopy equivalence, $\psi(\mathbb{X})$ is an isovariant homotopy equivalence. In view of Theorem 8.4, the following square diagram

$$\begin{tikzcd} \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \arrow[r, "\psi(\mathbb{X})"] \arrow[d, "\mathrm{E}_f"] & \mathbb{X} \arrow[d, "f"] \\ \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \arrow[r, "\psi(\mathbb{Y})"] & \mathbb{Y} \end{tikzcd}$$

is commutative in the category $\mathrm{ISOV}_{\mathscr{F}}$-HOMOT.

§ 11. The equivariant homotopy functor of transition to an $\mathscr{F}$-orbit bundle

The homotopy Borel functor

$$[\mathrm{E}_{\mathscr{F}}]\colon \text{EQUIV-HOMOT} \to \mathrm{ISOV}_{\mathscr{F}}\text{-HOMOT}$$

can be involved in the composite of three functors

$$\text{EQUIV-HOMOT} \xrightarrow{E} \text{ISOV-HOMOT} \xrightarrow{R_{\mathscr{F}}} \mathrm{ISOV}_{\mathscr{F}}\text{-HOMOT}$$
$$\xrightarrow{[E]_{\mathscr{F}}^{-1}=\mathfrak{F}} \mathrm{EQUIV}_{\mathscr{F}}\text{-HOMOT},$$

among which the first and the last functors are equivalences. It is clear that the arising functor $\mathfrak{R}_{\mathscr{F}}\colon$ EQUIV-HOMOT $\to$ $\mathrm{EQUIV}_{\mathscr{F}}$-HOMOT is conjugated to the isovariant functor $R_{\mathscr{F}}$ in the sense of the category theory and, for this reason, it is reasonable to perceive the functor $\mathfrak{R}_{\mathscr{F}}$ as an *equivariant homotopy functor of transition to an orbit bundle of the given type* $\mathscr{F} \subset \mathrm{Conj}_G$.

Note that there is a morphism Ψ of the functor $\mathfrak{R}_{\mathscr{F}}$ into the identity functor Id: EQUIV-HOMOT $\to$ EQUIV-HOMOT, the components $\{\psi(\mathbb{X})\colon \mathfrak{R}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}\}$ of which are the maps $p_{\mathbb{X}}\colon \mathrm{E}_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}$. In view of Theorem 9.2, they are $\mathrm{Equiv}_{\mathscr{F}}$-soft maps and therefore *weak* $\mathrm{Equiv}_{\mathscr{F}}$*-homotopy equivalences.*[7]

It turns out that this property of the functor $\mathfrak{R}_{\mathscr{F}}$ is characteristic. We give a formally more general definition of an equivariant homotopy functor of transition to an orbit bundle of given type and show that this functor reduces to $\mathfrak{R}_{\mathscr{F}}$.

Definition 11.1. We say that a functor

$$\mathfrak{R}'_{\mathscr{F}}\colon \text{EQUIV-HOMOT} \to \mathrm{EQUIV}_{\mathscr{F}}\text{-HOMOT}$$

is an equivariant homotopy functor of transition to an $\mathscr{F}$-orbit bundle if there exists a morphism Ψ' of the functor $\mathfrak{R}'_{\mathscr{F}}$ into the identity functor

$$\mathrm{Id}\colon \text{EQUIV-HOMOT} \to \text{EQUIV-HOMOT},$$

the components $\{\psi'(\mathbb{X})\colon \mathfrak{R}'_{\mathscr{F}}(\mathbb{X}) \to \mathbb{X}\}$ of which are weak $\mathrm{Equiv}_{\mathscr{F}}$-homotopy equivalences.

[7] A G-map $f\colon \mathbb{X} \to \mathbb{Y}$ is called a *weak* $\mathrm{Equiv}_{\mathscr{F}}$*-homotopy equivalence* if the map $f^H\colon \mathbb{X}^H \to \mathbb{Y}^H$ of H-fixed point sets is a homotopy equivalence for each $(H) \in \mathscr{F}$ with $\mathbb{X}^H \neq \varnothing$.

Theorem 11.2. *Let*

$$\mathfrak{R}_{\mathscr{F}}\colon \text{EQUIV-HOMOT} \to \text{EQUIV}_{\mathscr{F}}\text{-HOMOT},$$
$$\mathfrak{R}'_{\mathscr{F}}\colon \text{EQUIV-HOMOT} \to \text{EQUIV}_{\mathscr{F}}\text{-HOMOT}$$

be equivariant homotopy functors of transition to an $\mathscr{F}$-orbit bundle. Then there exists a morphism Θ of the functor $\mathfrak{R}'_{\mathscr{F}}$ into the functor $\mathfrak{R}_{\mathscr{F}}$ possessing the following properties:

(a) *all its components $\{\theta(\mathbb{X})\colon \mathfrak{R}'_{\mathscr{F}}(\mathbb{X}) \to \mathfrak{R}_{\mathscr{F}}(\mathbb{X})\}$ are* Equiv-*homotopy equivalences*;

(b) $\Psi \circ \Theta = \Psi'$.

We begin the proof of the theorem by several necessary definitions and facts. In [20] the following characterization of weak $\text{Equiv}_{\mathscr{F}}$-homotopy equivalences was found: a G-map $f\colon \mathbb{X} \to \mathbb{Y}$ of Equiv-ANE-spaces is a weak G-$\mathscr{F}$-homotopy equivalence if and only if f is *homotopy* $\text{EQUIV}_{\mathscr{F}}$*-soft*, that is, for each square commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\varphi"] \arrow[d, phantom, "\cap"] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{Z} \arrow[r, "\psi"] & \mathbb{Y}, \end{tikzcd}$$

where $\mathbb{Z}$ and its closed subset $\mathbb{A}$ are objects of the category $\text{EQUIV}_{\mathscr{F}}$-TOP, there exists a G-map $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$ such that $\widehat{\varphi} = \text{ext}(\varphi)$ and $f \circ \widehat{\varphi} \simeq_G \psi$. It easily follows that

(1) if f is a weak G-$\mathscr{F}$-homotopy equivalence, then for each G-$\mathscr{F}$-space $\mathbb{Z}$ the G-map f induces a bijection of equivariant homotopy classes of maps $[\mathbb{Z}, \mathbb{X}]$ and $[\mathbb{Z}, \mathbb{Y}]$.

In [20] it was also found that a G-morphism $f\colon \mathbb{X} \to \mathbb{Y}$ of $\text{Equiv}_{\mathscr{F}}$-ANE-spaces is a G-homotopy equivalence if and only if f is homotopy $\text{EQUIV}_{\mathscr{F}}$-soft.

Proof of Theorem 11.2. In view of the theorem on characterization of weak $\text{Equiv}_{\mathscr{F}}$-homotopy equivalences, there exists a G-map $\theta(\mathbb{X})\colon \mathfrak{R}'_{\mathscr{F}}(\mathbb{X}) \to \mathfrak{R}_{\mathscr{F}}(\mathbb{X})$ for which $\psi(\mathbb{X}) \circ \theta(\mathbb{X}) = \psi'(\mathbb{X})$. Since $\theta(\mathbb{X})$ is a weak $\text{Equiv}_{\mathscr{F}}$-homotopy equivalence of $\text{Equiv}_{\mathscr{F}}$-ANE-spaces, $\theta(\mathbb{X})$ is an Equiv-homotopy equivalence by [20].

We now verify that the collection $\{\theta(\mathbb{X})\}$ defines a morphism of the functor $\mathfrak{R}'_{\mathscr{F}}$ into the functor $\mathfrak{R}_{\mathscr{F}}$. To do this, it is necessary to establish the homotopy commutativity of the central square diagram

$$\begin{tikzcd} \mathbb{X} \arrow[d, "f"'] & \mathfrak{R}'_{\mathscr{F}}(\mathbb{X}) \arrow[l, "\psi'(\mathbb{X})"'] \arrow[r, "\theta(\mathbb{X})"] \arrow[d, "\mathfrak{R}'(f)"'] & \mathfrak{R}_{\mathscr{F}}(\mathbb{X}) \arrow[r, "\psi(\mathbb{X})"] \arrow[d, "\mathfrak{R}(f)"'] & \mathbb{X}, \arrow[d, "f"] \\ \mathbb{Y} & \mathfrak{R}'_{\mathscr{F}}(\mathbb{Y}) \arrow[l, "\psi'(\mathbb{Y})"'] \arrow[r, "\theta(\mathbb{Y})"] & \mathfrak{R}_{\mathscr{F}}(\mathbb{Y}) \arrow[r, "\psi(\mathbb{Y})"] & \mathbb{Y}, \end{tikzcd}$$

that is, to show that $\alpha \rightleftharpoons \mathfrak{R}(f) \circ \theta(\mathbb{X})$ and $\beta \rightleftharpoons \theta(\mathbb{Y}) \circ \mathfrak{R}'(f)$ are G-homotopic. For this purpose, first we prove that $\psi(\mathbb{Y}) \circ \alpha \simeq_G \psi(\mathbb{Y}) \circ \beta$, and since by (1)

$\psi(\mathbb{Y})$ is a homotopy $\mathrm{EQUIV}_{\mathscr{F}}$-soft map (one can also use the $\mathrm{EQUIV}_{\mathscr{F}}$-softness of $\psi(\mathbb{Y}) = p_{\mathbb{X}}$), we have $\alpha \simeq_G \beta$. In view of commutativity of the right square, we have

$$\psi(\mathbb{Y}) \circ \alpha = \psi(\mathbb{Y}) \circ \mathfrak{R}(f) \circ \theta(\mathbb{X}) \simeq_G f \circ \bigl(\psi(\mathbb{X}) \circ \theta(\mathbb{X})\bigr),$$

and in view of Theorem 11.2, (b), we have

$$\psi(\mathbb{Y}) \circ \beta = \psi(\mathbb{Y}) \circ \theta(\mathbb{Y}) \circ \mathfrak{R}'(f) \simeq_G \psi'(\mathbb{Y}) \circ \mathfrak{R}'(f), \qquad f \circ (\psi(\mathbb{X}) \circ \theta(\mathbb{X})) \simeq_G f \circ \psi'(\mathbb{X}).$$

Finally, by the commutativity of the left square, we have $\psi'(\mathbb{Y}) \circ \mathfrak{R}'(f) \simeq_G f \circ \psi'(\mathbb{X})$: the equality $\alpha \rightleftharpoons \mathfrak{R}(f) \circ \theta(\mathbb{X}) \simeq_G \beta \rightleftharpoons \theta(\mathbb{Y}) \circ \mathfrak{R}'(f)$ is proved.

§ 12. The structure of the Eilenberg-MacLane complex in the orbit space of an $\mathrm{Isov}_{\mathrm{S}^1}$-AE-space

Recall that all proper closed subgroups of the group $G = \mathrm{SO}(2) = \{e^{i\varphi} \mid \varphi \in \mathbb{R}^1\}$ are cyclic subgroups $\mathbb{Z}_n$, $n \geqslant 1$. It is clear that the following assertions hold for $\mathscr{D}_n \rightleftharpoons \{H < G \mid H < \mathbb{Z}_n\}$:

- $\mathscr{D}_n \subset \mathscr{D}_m$ if and only if $\mathbb{Z}_n < \mathbb{Z}_m$ (or m is divided by n);
- $\mathscr{D} \rightleftharpoons \bigcup\{\mathscr{D}_n \mid n \geqslant 1\}$ consists of all discrete subgroups of $\mathrm{SO}(2)$.

If $\mathbb{W} \in \mathrm{Isov}_G$-AE, then $\mathbb{W}_n \rightleftharpoons \mathbb{W}_{\mathscr{D}_n} = \mathbb{W}_{\leqslant \mathbb{Z}_n} \in \mathrm{Isov}_{\mathbb{Z}_n}$-AE by Theorem 3.7. From the results of § 7 it follows that $p\colon \mathbb{W}_n/\mathbb{Z}_n \to W_n \rightleftharpoons \mathbb{W}_n/G$ is a principal $\mathrm{SO}(2)/\mathbb{Z}_n$-bundle. If $\mathscr{D}_n \subset \mathscr{D}_m$, then by Theorem 7.1 the homomorphisms $(e_{nm})_*\colon \pi_{i+1}(W_n) \to \pi_{i+1}(W_m)$ of homotopy groups generated by the natural embedding $\widetilde{e}_{nm}\colon W_n \hookrightarrow W_m$ of orbit spaces are isomorphic to the homomorphisms $\pi_i(\mathrm{SO}(2)/\mathbb{Z}_n) \to \pi_i(\mathrm{SO}(2)/\mathbb{Z}_m)$ of homotopy groups generated by the natural map $\mathrm{SO}(2)/\mathbb{Z}_n \to \mathrm{SO}(2)/\mathbb{Z}_m$. Therefore the homomorphism $(e_{nm})_2\colon \pi_2(W_n) = \mathbb{Z} \to \pi_2(W_m) = \mathbb{Z}$ is multiplication by integer m/n, all the remaining homomorphisms $(e_{nm})_k$ are zero.

We calculate the homotopy groups of the space $W_{<\mathrm{SO}(2)}$ coinciding with $\bigcup\{W_n \mid n \geqslant 1\}$. For this purpose, let us partially order the set of natural numbers with the help of divisibility. Further for each $k \geqslant 1$ we consider an inductive system indexed by elements of this ordered set consisting of groups $\Gamma_s^k = \pi_k(W_s)$ and homomorphisms $\Gamma_s^k \to \Gamma_t^k$ which are multiplications by integer t/s. The equality for direct limits of groups is known (see, for instance, [21]):

$$\varinjlim\{\Gamma_s^k \mid s\} = \begin{cases} \mathbb{Q} \text{ is the group of rationals} & \text{if } k = 2, \\ 0 & \text{if } k \neq 2. \end{cases}$$

Since each W_n is open in $W_{<\mathrm{SO}(2)}$ and the k-dimensional sphere is compact, it follows that

$$\pi_k(W_{<\mathrm{SO}(2)}) = \pi_k\Bigl(\bigcup\{W_n \mid n \geqslant 1\}\Bigr) \text{ coincides with } \varinjlim\{\Gamma_s^k \mid s\}.$$

From the above calculations it follows that $\pi_k(W_{<\mathrm{SO}(2)}) = \mathbb{Q}$ only for $k = 2$, in all other cases it is zero. Consequently, we establish

Theorem 12.1. *Let* $\mathbb{W}$ *be an* Isov_{S^1}-AE-*space and let the family* $\mathscr{D}$ *consist of all discrete orbit types of the group* S^1. *Then the orbit space* $W_{\mathscr{D}}$ *is an Eilenberg-MacLane complex* $\mathrm{K}(\mathbb{Q}, 2)$ *and the Čech cohomology ring* $\check{H}^*(W_{\mathscr{D}}; \mathbb{Q})$ *coincides with the polynomial ring* $\mathbb{Q}[x]$ *of the variable* x *of degree* 2.

The latter assertion of the theorem follows from [21], Ch. 2: the cohomology ring $\check{H}^*(\mathrm{K}(\mathbb{Q},2);\mathbb{Q})$ coincides with $\mathbb{Q}[x], \deg x = 2$.

§ 13. The orbit space of an $\mathbf{Isov_{O(2)}}$-AE-space

First we note that the family of discrete subgroups of the orthogonal group $\mathrm{O}(2)$ consists of the cyclic subgroups $\mathbb{Z}_n < \mathrm{SO}(2)$, $n \geqslant 2$, and the dihedral subgroups $\mathrm{D}_n < \mathrm{O}(2)$, $n \geqslant 2$. We shall equally denote by $\mathscr{D}$ the family of all discrete orbit types in the group $G \rightleftharpoons \mathrm{O}(2)$ as well as in its subgroup $\mathrm{SO}(2)$.

Restricting the group action from the $\mathrm{O}(2)$-space $\mathbb{W} \in \mathrm{Isov}_{\mathrm{O}(2)}$-AE to the subgroup $\mathrm{SO}(2)$, we obtain an $\mathrm{SO}(2)$-space which will be denoted by $\mathbb{V}$. In view of Theorem 3.7, the $\mathrm{SO}(2)$-space $\mathbb{V}$ is an Isov-$\mathrm{AE}_{\mathrm{SO}(2)}$-space and the $\mathbb{Z}_n$-space $\mathbb{V}_n \rightleftharpoons \mathbb{V}_{\leqslant \mathbb{Z}_n}$ an $\mathrm{Isov}_{\mathbb{Z}_n}$-AE-space. Since $\mathbb{V}_{\mathscr{D}} \equiv \mathbb{W}_{\mathscr{D}}$ and $\mathrm{SO}(2)$ is a normal subgroup of $\mathrm{O}(2)$, it follows that

(1) the orbit spaces $\mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2)$ and $\mathbb{W}_{\mathscr{D}}/\mathrm{SO}(2)$ coincide, and, for this reason, $\mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2)$ is naturally endowed with action of the group

$$\mathbb{Z}_2 = \mathrm{O}(2)/\mathrm{SO}(2)\colon \quad (g \cdot \mathrm{SO}(2)) \cdot (\mathrm{SO}(2) \cdot w) = \mathrm{SO}(2) \cdot (g \cdot w).$$

We consider on $\mathbb{V}$ a new action of the group $\mathrm{SO}(2)$ given by the formula $e^{i\varphi} * w = e^{-i\varphi} \cdot w$, $e^{i\varphi} \in \mathrm{SO}(2)$, and denote the arising $\mathrm{SO}(2)$-space by $\mathbb{V}^*$. Since for each $v \in \mathbb{V}$ and $H < \mathrm{SO}(2)$ the orbits $H \cdot v$ and $H * v$ coincide, the orbit space $\mathbb{V}/H$ is naturally identified with the orbit space $\mathbb{V}^*/H$. In particular, $\mathbb{V}^*_{\mathscr{D}}/\mathrm{SO}(2) \equiv \mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2)$.

Note that for an improper motion $s \in \mathrm{O}(2)\setminus\mathrm{SO}(2)$ the equality $s\cdot(e^{i\varphi}\cdot v) = e^{-i\varphi}\cdot(s\cdot v)$ is valid for all $e^{i\varphi} \in \mathrm{SO}(2)$, $v \in \mathbb{W}$. Therefore the map $\sigma\colon \mathbb{V} \to \mathbb{V}^*$, $\sigma(v) = s\cdot v$, is an $\mathrm{SO}(2)$-homeomorphism. Since $\mathbb{V}$ and $\mathbb{V}^*$ are equimorphic, $\mathbb{V}^* \in \mathrm{Isov}_{\mathrm{SO}(2)}$-AE.

It is easily seen that the induced map

$$\widetilde{\sigma}\colon \mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2) \to \mathbb{V}^*_{\mathscr{D}}/\mathrm{SO}(2)$$

of orbit spaces represents the involution, which specifies the action of the group $\mathbb{Z}_2$ on $\mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2)$ introduced above in (1). It is clear that $(\bigcup\{\mathbb{W}_{\mathrm{D_n}} \mid n \geqslant 2\})/\mathrm{SO}(2)$ will be the fixed point set of this involution. In the proof of the following assertion we repeat the reasoning from [10].

Theorem 13.1. *The group $\check{H}^4(W_{\mathscr{D}};\mathbb{Q})$ of (Čech) cohomologies of the orbit spaces $W_{\mathscr{D}} = \mathbb{W}_{\mathscr{D}}/\mathrm{O}(2)$ with coefficients in the field $\mathbb{Q}$ of rational numbers is nontrivial, and therefore $W_{\mathscr{D}}$ is not contractible.*

Proof. In view of Theorem 12.1, $Y \rightleftharpoons \mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2)$ is $\mathrm{K}(\mathbb{Q},2)$ and its cohomology ring $\check{H}^*(Y;\mathbb{Q})$ is $\mathbb{Q}[x], \deg x = 2$.

By Smith theory [12] it is known that the images of the homomorphisms

$$\mathrm{Id} + \widetilde{\sigma}^*\colon \check{H}^*(Y;\mathbb{Q}) \to \check{H}^*(Y;\mathbb{Q}) \quad \text{and} \quad p^*\colon \check{H}^*(Y/\mathbb{Z}_2;\mathbb{Q}) \to \check{H}^*(Y;\mathbb{Q})$$

coincide (here $p\colon Y = \mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2) \to Y/\mathbb{Z}_2 = \mathbb{V}_{\mathscr{D}}/\mathrm{O}(2)$ is an orbit projection and $\widetilde{\sigma}^*\colon \check{H}^*(Y;\mathbb{Q}) \to \check{H}^*(Y;\mathbb{Q})$ is a homomorphism of cohomologies induced by $\widetilde{\sigma}\colon Y \to Y$). Let us show that for the generating $x \in \check{H}^2(Y;\mathbb{Q}) = \mathbb{Q}$ the element $x^2 + (\widetilde{\sigma})^*(x^2) \in \check{H}^4(Y;\mathbb{Q})$ is nonzero, and therefore the ring $\check{H}^4(W_{\mathscr{D}};\mathbb{Q})$ is nontrivial.

Indeed, if $(\widetilde{\sigma})^*(x) = \mu \cdot x$, where $\mu \in \mathbb{Q}$, then $(\widetilde{\sigma}^*(x))^2 = \mu^2 \cdot x^2$. In view of the commutativity of multiplication in cohomologies with the induced homomorphism $\widetilde{\sigma}^*$, we have also

$$(\widetilde{\sigma})^*(x^2) = (\widetilde{\sigma}^*(x))^2 = \mu^2 \cdot x^2 \text{ and therefore } x^2 + (\widetilde{\sigma})^*(x^2) = (1+\mu^2)\cdot x^2 \neq 0.$$

We strengthen this result and show that in fact $\check{H}^*(W_{\mathscr{D}};\mathbb{Q})$ is the ring of the polynomials $\mathbb{Q}[y]$ of the variable y of degree 4. In view of Theorem 3.7, the $\mathrm{SO}(2)$-space $\mathbb{V}_n \rightleftharpoons \mathbb{V}_{\leqslant \mathbb{Z}_n}$ is an $\mathrm{Isov}_{\mathbb{Z}_n}$-AE-space, and the orbit projection $p\colon \mathbb{V}_n/\mathbb{Z}_n \to \mathbb{V}_n/\mathrm{SO}(2)$ is by the property (2) from § 7 a universal principal $\mathrm{SO}(2)/\mathbb{Z}_n$-bundle.

The restriction of the $\mathrm{SO}(2)$-homeomorphism $\sigma\restriction\colon \mathbb{V} \to \mathbb{V}^* \rightleftharpoons \mathbb{V}^*_{\leqslant \mathbb{Z}_n}$ generates the $\mathrm{SO}(2)/\mathbb{Z}_n$-map $\sigma_n\colon \mathbb{V}_n/\mathbb{Z}_n \to \mathbb{V}^*_n/\mathbb{Z}_n$ and induces the map $\widetilde{\sigma}_n\colon \mathbb{V}_n/\mathrm{SO}(2) \to \mathbb{V}^*_n/\mathrm{SO}(2)$ of orbit spaces. Since $\mathbb{V}_n/H = \mathbb{V}^*_n/H$ for each subgroup $H < \mathrm{SO}(2)$, further we will look at the homeomorphisms σ_n and $\widetilde{\sigma}_n$ as autohomeomorphisms.

We consider the left square commutative diagram

$$\begin{array}{ccc} \mathbb{V}_n/\mathbb{Z}_n & \xrightarrow{\sigma_n} & \mathbb{V}_n/\mathbb{Z}_n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ V_n/\mathrm{SO}(2) & \xrightarrow{\widetilde{\sigma}_n} & \mathbb{V}_n/\mathrm{SO}(2), \end{array} \qquad \begin{array}{ccc} V_n/\mathrm{SO}(2) & \xrightarrow{\widetilde{\sigma}_n} & \mathbb{V}_n/\mathrm{SO}(2) \\ {\scriptstyle e_{nm}}\downarrow & & \downarrow{\scriptstyle e_{nm}} \\ \mathbb{V}_m/\mathrm{SO}(2) & \xrightarrow{\widetilde{\sigma}_m} & \mathbb{V}_m/\mathrm{SO}(2), \end{array}$$

in which $p\colon \mathbb{V}_n/\mathbb{Z}_n \to \mathbb{V}_n/\mathrm{SO}(2)$ is an automorphism of the locally trivial bundle, and write the morphism of exact sequences of homotopy groups of bundles generated by σ_n and $\widetilde{\sigma}_n$. Note that the homomorphism $\varphi\colon \pi_i(\mathrm{SO}(2)/\mathbb{Z}_n) \to \pi_i(\mathrm{SO}(2)/\mathbb{Z}_n)$ induced by the map of the fibres $\pi^{-1}(*)$ and $\pi^{-1}(\widetilde{\sigma}_n(*))$ is nontrivial only in dimension $i=1$, where it coincides with $-\mathrm{Id}\colon \mathbb{Z} \to \mathbb{Z}$. Since the spaces $\mathbb{V}_n/\mathbb{Z}_n$ and $\mathbb{V}^*_n/\mathbb{Z}_n$ are contractible, we come to the conclusion by applying further the reasoning from the proof of Theorem 7.1 that the induced homomorphism

$$(\widetilde{\sigma}_n)_*\colon \pi_i(\mathbb{V}_n/\mathrm{SO}(2)) \to \pi_i(\mathbb{V}_n/\mathrm{SO}(2))$$

is nontrivial only in dimension $i=2$, where it is multiplication of $\mathbb{Z}$ by -1.

If $\mathbb{Z}_n < \mathbb{Z}_m$, then the induced homomorphism

$$(e_{nm})_2\colon \pi_2\bigl(\mathbb{V}_n/\mathrm{SO}(2)\bigr) = \mathbb{Z} \to \pi_2\bigl(\mathbb{V}_m/\mathrm{SO}(2)\bigr) = \mathbb{Z}$$

generated by the natural embedding $\widetilde{e}_{nm}\colon \mathbb{V}_n/\mathrm{SO}(2)) \hookrightarrow \mathbb{V}_m/\mathrm{SO}(2)$ of orbit spaces is multiplication by integer m/n (see § 12). Since the right diagram given above is also commutative, the homomorphism $(e_{nm})_2$ commutes with the induced homomorphisms $(\widetilde{\sigma}_n)_2\colon \pi_2(\mathbb{V}_n/\mathrm{SO}(2)) \to \pi_2(\mathbb{V}_n/\mathrm{SO}(2))$ and $(\widetilde{\sigma}_m)_2\colon \pi_2(\mathbb{V}_n/\mathrm{SO}(2)) \to \pi_2(\mathbb{V}_n/\mathrm{SO}(2))$. With the help of this fact it is established analogously to Theorem 12.1 that the homomorphism

$$(\widetilde{\sigma})_2\colon \pi_2\bigl(\mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2)\bigr) = \mathbb{Q} \to \pi_2\bigl(\mathbb{V}^*_{\mathscr{D}}/\mathrm{SO}(2)\bigr) = \mathbb{Q}$$

induced by the involution $\widetilde{\sigma}$ is multiplication of $\mathbb{Q}$ by -1.

By the Hurewicz theorem [22] it follows that $H_2(\mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2);\mathbb{Z}) = \mathbb{Q}$ and the induced homomorphism $(\widetilde{\sigma})_2\colon H_2(\mathbb{V}_{\mathscr{D}}/\mathrm{SO}(2);\mathbb{Z}) = \mathbb{Q} \to H_2(\mathbb{V}^*_{\mathscr{D}}/\mathrm{SO}(2);\mathbb{Z}) = \mathbb{Q}$ of homology groups is multiplication by -1. Further making use of the formula of universal coefficients, we obtain that

$$\check{H}^2\bigl(\mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Q}\bigr)=\operatorname{Hom}\bigl(H_2(\mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Z});\mathbb{Q}\bigr)=\mathbb{Q}$$

and the induced homomorphism

$$(\widetilde{\sigma})^2\colon \check{H}^2\bigl(\mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Q}\bigr)=\mathbb{Q}\to\check{H}^2\bigl(\mathbb{V}^*_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Q}\bigr)=\mathbb{Q}$$

also is multiplication by -1.

Recall that the cohomology ring $\check{H}^*(\mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Q})$ coincides with the polynomial ring $\mathbb{Q}[x]$ of the variable x of degree 2. In view of commutativity of multiplication in cohomologies and the induced homomorphism $\widetilde{\sigma}^*$, we conclude that $(\widetilde{\sigma})^{4k}$ is the identity map of $\mathbb{Q}$, and $(\widetilde{\sigma})^{4k+2}$ is multiplication of $\mathbb{Q}$ by -1. Therefore the homomorphism

$$\operatorname{Id}+\widetilde{\sigma}^*\colon \check{H}^*\bigl(\mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Q}\bigr)\to\check{H}^*\bigl(\mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2);\mathbb{Q}\bigr)$$

is an isomorphism only in dimensions $4k$.

In view of the Smith theory [12], the image $\operatorname{Im}(p^*)$ of the homomorphism induced by the orbit projection $p\colon \mathbb{V}_{\mathscr{D}}/\operatorname{SO}(2)\to\mathbb{V}_{\mathscr{D}}/\operatorname{O}(2)$ coincides with the fixed elements $(\check{H}^*(Y;\mathbb{Q}))^{\mathbb{Z}_2}$ of the homomorphism $\widetilde{\sigma}^*\colon \check{H}^*(Y;\mathbb{Q})\to\check{H}^*(Y;\mathbb{Q})$, and the arising homomorphism $p^*\colon \check{H}^*(\mathbb{V}_{\mathscr{D}}/\operatorname{O}(2);\mathbb{Q})\to(\check{H}^*(Y;\mathbb{Q}))^{\mathbb{Z}_2}$ is an isomorphism. In view of the calculations made above, the group $(\check{H}^i(Y;\mathbb{Q}))^{\mathbb{Z}_2}$ is isomorphic to $\mathbb{Q}$ for $i=4k$; in all the remaining cases $(\check{H}^i(Y;\mathbb{Q}))^{\mathbb{Z}_2}=0$. Therefore the cohomology ring $(\check{H}^i(Y;\mathbb{Q}))^{\mathbb{Z}_2}$ is the polynomial ring $\mathbb{Q}[y]$ of the variable y of degree 4, and therefore we establish

Theorem 13.2. *The homomorphism* $p^*\colon \check{H}^*(\mathbb{V}_{\mathscr{D}}/\operatorname{O}(2);\mathbb{Q})\to(\check{H}^*(Y;\mathbb{Q}))^{\mathbb{Z}_2}$ *is a ring isomorphism, and* $\check{H}^*(W_{\mathscr{D}};\mathbb{Q})$ *is the polynomial ring* $\mathbb{Q}[y]$ *of the variable* y *of degree* 4.

§ 14. Epilogue

We list a few problems and questions which are of interest in connection with the investigation of Isov-AE-spaces. For this purpose we fix a G-space $\mathbb{W}\in$ Isov-AE and a family $\mathscr{F}\subset\operatorname{Conj}_G$.

Investigation of homotopy type of Isov$_{\mathscr{F}}$-AE-spaces. In connection with the investigation of fundamental classes of G-spaces one must define (calculate) the homotopy type of the $\mathscr{F}$-orbit bundle $\mathbb{W}_{\mathscr{F}}$ and its orbit space $\mathbb{W}_{\mathscr{F}}/G$. In particular, of special interest is the so-called *ring of locking cohomology of the group G*, that is, the ring $\check{H}^*(\mathbb{W}_{<G}/G;\mathbb{Q})$ of the orbit spaces of nontrivial orbit types. The last two sections of the paper were devoted to the calculation of the ring of locking cohomology for the groups SO(2) and O(2). In connection with the investigation of the topology of the Banach-Mazur compactum the locking cohomology of the orthogonal group $\operatorname{O}(n)$, $n\geqslant 3$, is of interest (see [9]). In connection with Theorem 13.2 there arises the question of whether $W_{\mathscr{D}}$ is an Eilenberg-MacLane complex $\operatorname{K}(\mathbb{Q},4)$ for the $\operatorname{Isov}_{\operatorname{O}(2)}$-AE-space $\mathbb{W}$.

In the context of Theorem 6.3 it is of particular interest to find all families $\mathscr{F}\subset\mathscr{C}\subset\operatorname{Orb}_G$, for which $W_{\mathscr{F}}\hookrightarrow W_{\mathscr{C}}$ is a homotopy equivalence, $W_{\mathscr{F}}\in$ AE (which is equivalent to the solvability of the PEA for $\mathscr{F}$).

Investigation of the topological type of the Isov$_{\mathscr{F}}$-AE-spaces. When is $\mathbb{W}_{\mathscr{F}}$ topologically complete; locally compact; strongly universal; infinite-dimensional manifold modelled by (which) standard universal space? Is it true that an arbitrary Isov-AE-space closedly contains an arbitrary compact G-space? When will a compact Isov-AE-space $\mathbb{X}$ be equimorphic to the equivariant Hilbert cube?

Is it true that each free action of the integer p-adic numbers A_p on Hilbert space is a free Isov-AE-space? Is it true that for each Isov$_{\mathscr{F}}$-AE-space $\mathbb{X}$ there exists an Isov-AE-space $\mathbb{W}$ such that $\mathbb{W}_{\mathscr{F}} = \mathbb{X}$?

All G-isovariant extensors for nontrivial groups G known by now are infinite-dimensional. The following questions arising in connection with this observation are of definite interest in the Hilbert-Smith Conjecture [23]: if the acting group G is nontrivial and $\mathbb{X} \in$ Isov-ANE, then $\dim \mathbb{X} = \infty$? $\dim X = \infty$?

Many constructions of topology and analysis lead to Isov-AE-spaces. It seems likely that among such constructions are the space $\mathrm{C}(G, \mathbb{R}^1)$ of regular representation of the group G, the exponential space $\exp G$, the space of convex bodies of Euclidean space $\mathbb{R}^n$, the space of linear isomorphisms of a Hilbert G-space and many others. Of special interest is to find further constructions leading to Isov-AE-spaces.

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