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{FirstPag

The Covering Homotopy Extension Problem for Compact Transformation Groups

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Abstract—It is shown that the orbit space of universal (in the sense of Palais) G-spaces classifies G-spaces. Theorems on the extension of covering homotopy for G-spaces and on a homotopy representation of the isovariant category ISOV are proved .

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INTRODUCTION

This paper studies the isovariant category ISOV, whose objects are spaces with an action of a compact group G and morphisms are isovariant maps. Historically, the objects of this category are usually treated as generalized principal G-bundles and its injective objects ($\equiv$ Isov-AE-spaces), as universal generalized principal G-bundles. Objects with similar properties were first constructed by Palais for compact Lie groups acting on finite-dimensional spaces with finitely many orbit types [1, Sec. 2.6.]. Afterward, this result was extended to compact metrizable groups acting on spaces with finite-dimensional orbit space [2]. In [3], the last constraints were removed, and an existence theorem for Isov-AE-spaces was proved.

{th1:v803

Theorem 1. *If* $\mathbb{X}_i$ *is an* Isov*-generating* Equiv-AE*-space for any* $i \geq 1$*, then*

$$\prod\{\mathbb{X}_i \mid i \geq 1\} \in \text{Isov-AE}.$$

A space $\mathbb{X}$ is said to be Isov-*generating* if, for any metric G-space $\mathbb{Z}$, there exists an isovariant map $\eta\colon \mathbb{Z} \to \mathbb{X}$. It was proved in [4] that the countable power $\mathbb{J} \rightleftharpoons (\text{Con}\,\mathbb{T})^\omega$ of the metric cone $\text{Con}\,\mathbb{T}$ over the discrete union $\mathbb{T}$ of all homogeneous spaces $G/H \in G$-ANE is Isov-generating (see also [5]). Since $\mathbb{J}$ is also an Equiv-AE-space and $\mathbb{J} \cong \mathbb{J}^\omega$, it follows from Theorem 1 that $\mathbb{J} \in$ Isov-AE.

Each family $\mathcal{F}$ belonging to the set Conj_G of conjugate classes of closed subgroups of G generates a series of equivariant homotopy invariants, including $\mathcal{F}$-classifying G-spaces in the sense of [6], [7] and generalized cohomology groups [8], [9], which are closely related to them, as well as fundamental classes of G-spaces and other equivariant homotopy invariants [10]. Theorem 1 makes it possible to prove that, on $\mathcal{F}$-classifying G-spaces, the additional structure of isovariant absolute extensors can be defined, which opens new possibilities for calculating homotopy invariants of the orbit spaces of $\mathcal{F}$-classifying G-spaces. In turn, this gives important information about the generalized cohomology of compact groups.

Theorem 1 also implies the important conclusion that the equivariant homotopy type of Equiv-ANE-spaces coincides with the isovariant homotopy type of Isov-ANE-spaces. This clarifies the functorial nature of the operation of the passage to the bundle of orbits of a given type, which is not preserved by

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equivariant homotopy equivalences. The circle of questions outlined above will be considered elsewhere (see, e.g., [3], [11]).

In this paper, we apply Theorem 1 to prove that the orbit space E of any Isov-AE-space $\mathbb{E}$ classifies G-spaces in the sense of Palais. To this end, we carry over Palais' classical result on covering homotopy (see [1], [2]) to arbitrary compact groups and obtain a more general result on the extension of covering homotopy (Theorem 2). By virtue of this result, the category ISOV admits a homotopy representation, which largely reduces studying this category to considering homotopy properties of topological spaces (see Theorem 5).

Statement of the Covering Homotopy Extension Problem

Let G be a compact group. Consider the following commutative square G-diagram $\mathcal{D}$:

$$\begin{array}{ccc} \mathbb{S}\times[0,1]\cup\mathbb{T}\times\{0\} & \xrightarrow{\ \varphi\ } & \mathbb{X} \\ \bigcap & & \big\downarrow f \\ \mathbb{T}\times[0,1] & \xrightarrow{\ \psi\ } & \mathbb{Y}\,. \end{array}$$

In this diagram, $\mathbb{S}\subset\mathbb{T}$ is a closed G-subspace and f, φ, and ψ are G-maps. We say that the diagram $\mathcal{D}$ is *admissible* (*weakly admissible*) for the G-map f if

(a) φ is an isovariant map (respectively, φ is an isovariant map and ψ induces a homeomorphism $\psi\colon T\times[0,1]\to Y$ of the orbit spaces);

(b) $(f^{-1}\circ\psi)(G(u)\times I)$ has single orbit type (G_u) for each $u\in T$.

Any admissible commutative diagram $\mathcal{D}$ satisfies the following conditions:

1) the orbit types of the spaces $\mathbb{X}$ and $\mathbb{T}$ are related by $\mathrm{Orb}_{\mathbb{T}}\subset\mathrm{Orb}_{\mathbb{X}}$;

2) if the map f is isovariant, then so is the map ψ.

If $\mathcal{D}$ is an admissible diagram and f is a P-orbit projection, then

3) the induced map $\psi_P\colon\mathbb{T}/P\times[0,1]\to\mathbb{Y}$ given by $(\psi_P)(P\cdot u,t)=\psi(u,t)$ is well defined, and this map is an isovariant embedding.

We say that a G-map $\widehat{\varphi}\colon\mathbb{T}\times[0,1]\to\mathbb{X}$ *splits the diagram* $\mathcal{D}$ if $\widehat{\varphi}$ is a lift of ψ with respect to f (i.e., $f\circ\widehat{\varphi}=\psi$) and $\widehat{\varphi}=\operatorname{ext}\varphi$. By virtue of (b), the map $\widehat{\varphi}$ is necessarily isovariant (i.e., $\varphi=\widehat{\varphi}$ on $\mathbb{S}\times[0,1]\cup\mathbb{T}\times\{0\}$).

We say that *the isovariant covering homotopy extension problem is solvable* (*weakly solvable*) for a G-map $f\colon\mathbb{X}\to\mathbb{Y}$ if any admissible G-diagram (any weakly admissible G-diagram) $\mathcal{D}$ for f splits. In this case, we say that the G-map $f\colon\mathbb{X}\to\mathbb{Y}$ is a *Hurewicz* Isov-*bundle* (a *weak Hurewicz* Isov-*bundle*). Clearly, any Hurewicz Isov-bundle is a weak Hurewicz Isov-bundle. The following theorem plays the key role in this paper.

{th2:v803

Theorem 2. *Any orbit projection* $p\colon\mathbb{X}\to X$ *is a Hurewicz* Isov-*bundle.*

Now, applying the equimorphism criterion to an isovariant map $\widehat{\varphi}$ splitting a weakly admissible G-diagram $\mathcal{D}$ from which $\mathbb{S}$ is missing, we obtain a generalization to any compact groups of Palais' well-known theorem on the equivariant type of a space having the orbit type of the product $\mathbb{W}\times[0,1]$ (see [1]).

The proof of Theorem 2 is based on the following result, with is of independent interest.

{th3:v803

Theorem 3. *Let* $\pi\colon G\to H$ *be an epimorphism of compact groups whose kernel* $P=\operatorname{Ker}\pi$ *is a Lie group. Then any* P*-orbit projection* $f\colon\mathbb{X}\to\mathbb{Y}=\mathbb{X}/P$ *is a Hurewicz* Isov-*bundle.*

We reduce Theorem 2 to Theorem 3 as follows. First, we expand the G-space $\mathbb{X}$ in a Lie series $\{\mathbb{X}_\alpha,\pi_\alpha^\beta\}$. Since $P_\alpha/P_{\alpha+1}$ is a compact Lie group, it follows by Theorem 3 that the bonding $P_\alpha^{\alpha+1}$-orbit projection $\pi_\alpha^{\alpha+1}\colon\mathbb{X}_{\alpha+1}\to\mathbb{X}_\alpha$ is a Hurewicz Isov-bundle for each $\alpha<\tau$. Then, arguing by transfinite induction, we prove that the limit orbit projection $\pi\colon\mathbb{X}\to X$ is a Hurewicz Isov-bundle as well.

Homotopy Representation of the Category ISOV

On the orbit space X of a G-space $\mathbb{X}$, we consider the *stratification* $\{X_{(H)} \mid H < G\}$ generated by the orbit types (the space X itself is called an $\mathcal{S}$-*space*); a map $\alpha\colon X \to E$ of $\mathcal{S}$-spaces is said to be *stratification-preserving* (or, briefly, an $\mathcal{S}$-*map*) if

$$\alpha(X_{(H)}) \subset E_{(H)} \qquad \text{for any subgroup} \quad H < G.$$

An $\mathcal{S}$-*homotopy* $H\colon X \times I \to E$ between stratification-preserving maps is defined in a similar way (i.e., it is required that $H(X_{(H)} \times I) \subset E_{(H)}$ for any $H < G$).

The *fiber product* of spaces C and B with respect to maps $C \xrightarrow{g} A$ and $B \xrightarrow{f} A$ is defined as the set

$$\{(c,b) \mid g(c) = f(b)\} \subset C \times B,$$

which we denote by $C\,{}_g\!\times_f B$ or, briefly, by $g^*(B)$ or $f^*(C)$. The projections $D = C\,{}_g\!\times_f B$ onto the factors C and B determine maps $f^*\colon D \to C$ and $g^*\colon D \to B$. We refer to f^* as the *map parallel to* f and to the map g^* as the *map parallel to* g.

An equimorphism criterion (Proposition 2) relates isovariant maps to fiber products.

{pr1:v803

Proposition 1. *Let $h\colon \mathbb{Y} \to \mathbb{X}$ be an isovariant map, and let $\widetilde{h}\colon Y \to X$ be the map of orbit spaces generated by h. Then the G-map h and the G-map $(\widetilde{h})^*\colon (\widetilde{h})^*(\mathbb{X}) \to \mathbb{X}$ parallel to $\widetilde{h}$ are equimorphic, i.e., there exists a G-homeomorphism $\theta\colon (\widetilde{h})^*(\mathbb{X}) \to \mathbb{Y}$ for which $h \circ \theta = (\widetilde{h})^*$ (thereby, $\mathbb{Y}$ is identified with the fiber product $\widetilde{h}^*(\mathbb{X})$).*

Using Theorem 2 and Proposition 1, we prove that the G-spaces determined by homotopic $\mathcal{S}$-maps connected by an $\mathcal{S}$-homotopy are equimorphic; namely, the following theorem is valid.

{th4:v803

Theorem 4. *If $\mathcal{S}$-maps $\alpha\colon X \to E$ and $\beta\colon X \to E$ are connected by an $\mathcal{S}$-homotopy, then the fiber products*

$$\alpha^*(\mathbb{E}) = \mathbb{E}_\pi \times_\alpha X \qquad \textit{and} \qquad \beta^*(\mathbb{E}) = \mathbb{E}_\pi \times_\beta X$$

are equimorphic (here $\pi\colon \mathbb{E} \to E$ is the orbit projection).

If a G-space $\mathbb{E}$ is an Isov-AE*, then the converse is also true; that is, if $\alpha^*(\mathbb{E}) \cong_G \beta^*(\mathbb{E})$, then the $\mathcal{S}$-maps $\alpha\colon X \to E$ and $\beta\colon X \to E$ are connected by an $\mathcal{S}$-homotopy.*

Let $\mathcal{S}$-HOMOT$_E$ denote the category whose *objects* are $\mathcal{S}$-homotopy classes $[\alpha]\colon X \to E$ of $\mathcal{S}$-maps and the *morphism* between $\mathcal{S}$-maps $[\alpha]$ and $[\beta]\colon Y \to E$ is the $\mathcal{S}$-homotopy class of the $\mathcal{S}$-map $h\colon X \to Y$ for which $\alpha = \beta \circ h$.

To each object $[\alpha]\colon X \to E$ of the category $\mathcal{S}$-HOMOT$_E$ we assign the G-space coinciding with the fiber product $\mathbb{X} \rightleftharpoons \alpha^*(\mathbb{E})$, and to each morphism $[h]\colon X \to Y$ we assign the isovariant map

$$[h]_*\colon \mathbb{X} = \alpha^*(\mathbb{E}) \cong_G (\alpha')^*(\mathbb{E}) \xrightarrow{h_*} \beta^*(\mathbb{E}) \rightleftharpoons \mathbb{Y}, \qquad \text{where} \quad h_*(x,e) = (h(x), e).$$

It follows readily from Theorem 4 that, under certain conditions, the covariant functor Φ thus constructed is an isomorphism between the isovariant homotopy category ISOV-HOMOT and the category $\mathcal{S}$-HOMOT$_E$.

{th5:v803

Theorem 5. *If $\mathbb{E}$ is an* Isov-AE*-space, then*

$$\Phi\colon \mathcal{S}\text{-}\,\mathrm{HOMOT}_E \to \mathrm{ISOV\text{-}HOMOT}$$

is an equivalence of categories.

Theorem 5 can be regarded as an additional argument supporting the thesis that the isovariant category is a generalization of the category of principal G-bundles.

1. PRELIMINARY INFORMATION AND RESULTS {sec1:v80

From now on, all spaces (maps) if they do not arise as the result of certain constructions and if it is not otherwise specified, are assumed to be metric (continuous); we consider only actions of compact groups.

Below we recall the basic notions of the theory of G-spaces [12]. By an action of a compact group G on a space $\mathbb{X}$ we mean a continuous map μ from the product $G \times \mathbb{X}$ to $\mathbb{X}$ which satisfies the conditions

$$\mu(g, \mu(h, x)) = \mu(g \cdot h, x), \quad \mu(e, x) = x \qquad \text{for all} \quad x \in \mathbb{X}, \quad g, h \in G$$

(here e denotes the identity element of the group G). As a rule, instead of $\mu(g, x)$ we write $g \cdot x$ or simply gx. A space $\mathbb{X}$ with an action of a group G is called a *G-space*. A map $f : \mathbb{X} \to \mathbb{Y}$ of G-spaces is said to be a *G-map*, or an *equivariant map*, if $f(g \cdot x) = g \cdot f(x)$ for all $x \in \mathbb{X}$ and $g \in G$.

Note that all G-spaces and G-maps form a category, which we denote by G-TOP or by EQUIV, if it is clear what group G is considered. We freely use the symbols "G-" and "Equiv-", which mean "equivariant." If "$***$" is a certain notion from nonequivariant topology, then "G-$***$" and "Equiv-$***$" stand for the corresponding equivariant counterpart.

The *orbit* $G(x)$ of a point $x \in \mathbb{X}$ is defined as the subset $\{g \cdot x \mid g \in G\} = G \cdot x$; this subset is always closed. The natural map

$$\pi = \pi_{\mathbb{X}} \colon \mathbb{X} \to X, \qquad x \mapsto G(x),$$

of the space $\mathbb{X}$ to the space $X \rightleftharpoons \mathbb{X}/G$ of the quotient partition is called the *orbit projection*. The space X of the quotient partition endowed with the quotient topology generated by π is called the *orbit space*. A subset A is said to be *invariant*, or a *G-subset*, if $\pi^{-1}\pi(A) = G \cdot A$.

By Conj_G we denote the set of all conjugacy classes of the closed subgroups of G and by Orb_G, the family of all homogeneous spaces up to equimorphism. On these sets, we introduce the following partial orders:

$$(K) \le (H) \iff K \text{ is contained in some representative } H' \text{ of the class } (H),$$
$$G/K \ge G/H \iff \text{there exists an equivariant map } f \colon G/K \to G/H;$$

the bijection

$$(H) \in \mathrm{Conj}_G \mapsto G/H \in \mathrm{Orb}_G$$

reverses these orders. Taking into account this observation, we shall identify the sets specified above and use the same name, *the set of G-orbit types*, and the same notation Orb_G for these sets at all places where such an identification causes no ambiguity.

For each point $x \in \mathbb{X}$, the subset

$$G_x = \{g \in G \mid g \cdot x = x\}$$

is a closed subgroup in the group G; this subgroup is called the *stabilizer* of the point x. For any closed subgroup $H < G$, we define the following subsets of $\mathbb{X}$:

- $\mathbb{X}^H = \{x \in \mathbb{X} \mid H \cdot x = x\} = \{x \in \mathbb{X} \mid H \subset G_x\}$ (this is the *H-fixed point set*);
- $\mathbb{X}_H = \{x \in \mathbb{X} \mid H = G_x\}$;
- $\mathbb{X}_{(H)} = \{x \in \mathbb{X} \mid H \text{ is conjugate to } G_x\}$ (this is the *bundle of orbits of type* (H)).

By $\mathrm{Orb}_{\mathbb{X}}$ we denote the set $\{(G_x) \mid x \in \mathbb{X}\} \subset \mathrm{Orb}_G$, that is, the *family of orbit types of* $\mathbb{X}$.

An equivariant map $f \colon \mathbb{X} \to \mathbb{Y}$ is said to be *isovariant* if f preserves stabilizers, i.e., $G_x = G_{f(x)}$ for all $x \in \mathbb{X}$. The category formed by all G-spaces and isovariant maps is denoted by ISOV (it is always clear what group G is meant). The following *equimorphism criterion* is widely known (see [12, Chap. 1, Example 10]).

{pr2:v803

Proposition 2. *An isovariant continuous map is a homeomorphism if and only if the orbit map generated by this map is a homeomorphism.*

Consider a compact group G, a metric H-space $\mathbb{Y}$, where $H < G$, and the diagonal action

$$h \cdot (g, y) \rightleftharpoons (g \cdot h^{-1}, h \cdot y)$$

of the group H on the product $G \times \mathbb{Y}$. Let $[g, y]$ denote the element

$$H \cdot (g, y) = \{(g \cdot h^{-1}, h \cdot y) \mid h \in H\}$$

of the orbit space $(G \times \mathbb{Y})/H$. The relation

$$g_1 \cdot [g, y] = [g_1 \cdot g, y], \qquad \text{where} \quad g, g_1 \in G, \quad y \in \mathbb{Y},$$

specifies a well-defined continuous action of G on $(G \times \mathbb{Y})/H$, which is called the *twisted product* of G and $\mathbb{Y}$ and denoted by $G \times_H \mathbb{Y}$. Any G-space $\mathbb{A}$ admitting a G-map $\alpha\colon \mathbb{A} \to G/H$ to a homogeneous space is equimorphic to the twisted product $G \times_H \mathbb{S}$, where $\mathbb{S} \rightleftharpoons \alpha^{-1}([H])$ is an H-space, because the continuous G-map $\varphi\colon G \times_H \mathbb{S} \to \mathbb{A}$ defined by $\varphi([g, s]) = g \cdot s$ is an equimorphism. We leave the proof of the following fact to the reader.

{lem1:v80

Lemma 1. *Suppose that the G-orbit type of a twisted product $G \times_H \mathbb{S}$ is single. If the orbit space $\mathbb{S}/H$ is connected, then the H-orbit type of $\mathbb{S}$ is single as well.*

Below we introduce a number of notions related to the extension of G-maps in a category $\mathcal{C}$ coinciding with ISOV or EQUIV. We say that a space $\mathbb{X}$ with an action of a compact group G is an *absolute neighborhood $\mathcal{C}$-extensor* (and write $\mathbb{X} \in \mathcal{C}$-ANE) if each morphism $\varphi\colon \mathbb{A} \to \mathbb{X}$ from $\mathcal{C}$ defined on a closed G-subset $\mathbb{A} \subset \mathbb{Z}$ of a G-space $\mathbb{Z}$ (which is called a *partial $\mathcal{C}$-morphism*) can be extended over some G-neighborhood $\mathbb{U} \subset \mathbb{Z}$ of the set $\mathbb{A}$ to a morphism $\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X} \in \mathcal{C}$. If $\mathbb{U}$ can always be rendered equal to $\mathbb{Z}$, then we say that $\mathbb{X}$ is an *absolute $\mathcal{C}$-extensor* (and write $\mathbb{X} \in \mathcal{C}$-AE). In the case where the acting group G is trivial (i.e., spaces are considered without actions), this notion transforms into the notion of an *absolute* [*neighborhood*] *extensor* for metric spaces; the class of absolute [neighborhood] extensors is denoted by A[N]E (see [13]).

An absolute [neighborhood] $\mathcal{C}$-extensor is called

- an equivariant [neighborhood] extensor (or, briefly, an Equiv-A[N]E-space) if $\mathcal{C} = \text{EQUIV}$;
- an isovariant [neighborhood] extensor (or, briefly, an Isov-A[N]E-space) if $\mathcal{C}$ coincides with the category ISOV.

Note that the Isov-AE-space coincides with the universal G-space in the sense of Palais, and its orbit space classifies G-spaces. If there is a $\mathcal{C}$-homotopy $H\colon \mathbb{X} \times [0, 1] \to \mathbb{Y}$ between morphisms $f, g\colon \mathbb{X} \to \mathbb{Y}$ in the category $\mathcal{C}$, then we write

- $f \simeq_{\text{Equiv}} g$ in the case $\mathcal{C} = \text{EQUIV}$,
- $f \simeq_{\text{Isov}} g$ in the case $\mathcal{C} = \text{ISOV}$.

A closed subgroup $H < G$ of a compact group G is called an *extensor subgroup* if the homogeneous space G/H is a metrizable G-ANE-space. As is known, if $H < G$ is an extensor subgroup, then there exists a normal subgroup $P \lhd G$ for which (i) $P < H$ and G/P is a compact Lie group; (ii) G/H is a topological manifold; and (iii) G/H is finite-dimensional and locally connected. Each of these three properties characterizes extensor subgroups [14]. The existence of arbitrarily small normal extensor subgroups in any compact group implies the following assertion.

{pr3:v803

Proposition 3. *Given any neighborhood $\mathcal{O}(H) \subset G$ of a subgroup H in a compact group G, there exists an extensor subgroup $H' < G$ for which $H \subset H' \subset \mathcal{O}(H)$.*

This and other properties of extensor subgroups are presented in [15], [5]. The well-known notion of a *Euclidean neighborhood G-retract over a space* [7] can be generalized as follows.

{def1:v80

Definition 1. A G-map $f\colon \mathbb{X} \to \mathbb{Y}$ is said to be *locally equivariantly soft* if, for any G-diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{f} & \mathbb{Y} \\ \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \psi} \\ \mathbb{A} & \hookrightarrow & \mathbb{W} \end{array}$$

admissible with respect to f[1], there exists a neighborhood G-extension $\widehat{\varphi}\colon \mathcal{U}(\mathbb{A}) \to \mathbb{X}$ of φ which is a lift of ψ.

It is easy to strengthen T. tom Dieck's theorem [7, Sec. 7.6.4.] to the following assertion.

{th6:v803

Theorem 6. *Suppose that $K < H < G$ and $K < G$ is an extensor subgroup. Then the natural G-map*

$$p\colon G/K \to G/H, \qquad g \cdot K \mapsto g \cdot H,$$

is equivariantly locally soft.

An epimorphism $\pi\colon G \to H$ of compact groups with kernel P generates the equivalence relation $x \sim x' \Leftrightarrow x' \in P \cdot x$ on the G-space $\mathbb{X}$. Clearly, the quotient space $\mathbb{X}/P$ by this equivalence coincides with $\{P \cdot x \mid x \in \mathbb{X}\}$ and is an H-space:

$$h \cdot (P \cdot x) = P \cdot (g \cdot x), \qquad \text{where} \quad g \in \pi^{-1}(h).$$

If $y = P \cdot x$, then the stabilizer H_y coincides with $\pi(G_x)$.

We refer to the quotient map $f\colon \mathbb{X} \to \mathbb{X}/P$ as the *P-orbit projection*. If $P = G$, then f coincides with the orbit projection $p\colon \mathbb{X} \to \mathbb{X}/G$. Since the composition of the P-orbit projection f with the orbit projection of the H-space $(\mathbb{X}/P)/H$ is a perfect map, it follows that this composition itself is a perfect surjection and has the following properties:

$$f(gx) = \pi(g) \cdot f(x) \qquad \text{for all} \quad x \in \mathbb{X}, \quad g \in G, \tag{1}$$ {eq1:v803

$$\pi(G_x) = H_{f(x)} \qquad \text{for all} \quad x \in \mathbb{X}, \tag{2}$$ {eq2:v803

$$\text{if } f(x) = f(x'), \text{ then } x \text{ and } x' \text{ belong to the same } G\text{-orbit.} \tag{3}$$ {eq3:v803

It turns out that these properties completely characterize P-orbit projections [5].

{pr4:v803

Proposition 4. *A perfect surjection $f\colon \mathbb{X} \to \mathbb{Y}$ from a G-space $\mathbb{X}$ to an H-space $\mathbb{Y}$ is a P-orbit projection if and only if f has properties* (1)–(3).

Suppose that $K < G$, $L \rightleftharpoons \pi(K) < H$, and $\pi' \rightleftharpoons \pi\colon K \to L$ is an epimorphism of compact groups with kernel $Q = P \cap K$. Clearly,

$$\text{the map } p\colon G/K \to H/L,\ p(g \cdot K) = \pi(g) \cdot L, \text{ is a } P\text{-orbit projection.} \tag{4}$$ {eq4:v803

Suppose also that $f\colon \mathbb{X} \to \mathbb{Y}$ is a P-orbit projection and $\varphi\colon \mathbb{X} \to G/K$ is the slice map. Clearly, $\mathbb{X}' \rightleftharpoons \varphi^{-1}([K])$ is a K-space, and $\mathbb{Y}' \rightleftharpoons f(\mathbb{X})$ is an L-space. Proposition 4 readily implies the following heredity property of orbit projections.

{pr5:v803

Proposition 5. *The restriction map $f' \rightleftharpoons f\colon \mathbb{X}' \to \mathbb{Y}'$ is a Q-orbit projection.*

[1]That is, for a G-diagram in which $\mathbb{A} \subset \mathbb{W}$ is a closed G-subspace, f, φ, and ψ are G-maps, and $f \circ \varphi = \psi\restriction_{\mathbb{A}}$.

If $K < G$ and $P \setminus K \neq \varnothing$, then there exists an extensor subgroup $\overline{K} < G$ for which $K < \overline{K} < G$ and the map

$$\overline{p}\colon G/\overline{K} \to H/\overline{L}, \quad \overline{p}(g \cdot \overline{K}\,) = \pi(g) \cdot \overline{L}, \qquad \text{where} \quad \overline{L} \rightleftharpoons \pi(\overline{K}\,) < H,$$

is a nontrivial P-orbit projection (therefore, by Theorem 6, it is equivariantly locally soft).

Suppose given an extensor subgroup $K < G$, a G-map $f\colon \mathbb{X} \to \mathbb{Y}$, a closed G-subset $\mathbb{X}_0 \subset \mathbb{X}$, and a commutative G-diagram

$$\begin{array}{ccc} \mathbb{X}_0 & \xrightarrow{\ \varphi\ } & G/K \\ {\scriptstyle f\restriction}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{Y} & \xrightarrow{\ \psi\ } & G/H\,. \end{array}$$

By virtue of Theorem 6, the natural G-map $p\colon G/K \to G/H$, where $K < H < G$, is equivariantly locally soft, and the commutative G-diagram is easy to transform into an admissible G-diagram

$$G/K \xleftarrow{\varphi} \mathbb{X}_0 \subset \mathbb{X} \xrightarrow{\psi\circ f} G/H$$

with respect to p. This readily implies the following theorem, which can be regarded as a generalization of the slice extension theorem for G-spaces to the case of G-maps.

{th7:v803

Theorem 7. *There exists a G-extension $\widehat{\varphi}\colon \mathcal{O}(\mathbb{X}_0) \to G/K$ of the map φ to a G-neighborhood $\mathcal{O}(\mathbb{X}_0)$ for which $p \circ \widehat{\varphi} = \psi \circ f\restriction$.*

Lie Series for G-Spaces

In a compact group G, consider a *Lie series* $\{P_\alpha \lhd G\}$ of normal subgroups indexed by ordinals $\alpha < \tau$ (see [14]). This means that

$$\begin{gathered} P_1 = G, \qquad P_\beta < P_\alpha \quad \text{for all} \quad \alpha < \beta, \\ P_\alpha/P_{\alpha+1} \quad \text{is a compact Lie group for each} \quad \alpha < \tau, \qquad \bigcap\{P_\alpha \mid \alpha < \tau\} = \{e\}. \end{gathered} \tag{5}$$

{eq5:v803

In this case, G is the limit $\varprojlim\{G/P_\alpha, \varphi_\alpha^\beta\}$ of the inverse system of quotient groups G/P_α and natural epimorphisms $\chi_\alpha^\beta\colon G/P_\beta \to G/P_\alpha$, $\alpha < \beta$. The kernels $\chi_\alpha^{\alpha+1}$ of the bonding maps coincide with the compact Lie groups $P_\alpha/P_{\alpha+1}$.

Consider a more general construction. Let $\pi_\alpha^\beta\colon \mathbb{X}_\beta \to \mathbb{X}_\alpha$ be the natural projection of $\mathbb{X}_\beta = \mathbb{X}/P_\beta$ onto $\mathbb{X}_\alpha = \mathbb{X}/P_\alpha$. Then π_α^β is a P_α/P_β-orbit projection, and the map $\pi\colon \mathbb{X} \to \varprojlim\{\mathbb{X}_\alpha, \pi_\alpha^\beta\}$ defined by $\pi(x) = \{P_\alpha \cdot x\}$ is an equimorphism. This assertion is proved by directly applying the equimorphism criterion. Note that the bonding maps $\pi_\alpha^{\alpha+1}$ are P-orbit projections for the compact Lie group $P = P_\alpha/P_{\alpha+1}$. We say that the inverse system $\{\mathbb{X}_\alpha, \pi_\alpha^\beta\}$ is a *Lie series for the G-space* $\mathbb{X}$.

The converse situation, in which an inverse system is given but the G-space is not, often arises. The well-known Theorem 11 from [16, Chap. 2, Sec. 6] can be generalized as follows.

{lem2:v80

Lemma 2. *Let $\{P_\alpha \lhd G\}$ be a Lie series. Suppose that, for any $\alpha < \beta$, P_α^β is the kernel of the homomorphism φ_α^β and $g_\alpha^\beta\colon \mathbb{Z}_\beta \to \mathbb{Z}_\alpha$ is a P_α^β-orbit projection; moreover,*

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma \qquad \textit{for all} \quad \alpha < \beta < \gamma.$$

Then $\mathbb{Z} = \varprojlim\{\mathbb{Z}_\alpha, g_\alpha^\beta\}$ is a G-space, $\mathbb{Z}_\alpha = \mathbb{Z}/P_\alpha$, and the inverse system $\{\mathbb{Z}_\alpha, g_\alpha^\beta\}$ is a Lie series for $\mathbb{Z}$.

The following characterization of G-spaces of the form $G/H \times [0,1]$ is, possibly, be known.

{pr6:v803

Proposition 6. *If $\mathbb{Z} = \mathbb{Z}_{(H)}$ and $Z = [0,1]$, then $\mathbb{Z}$ is equimorphic to $G/H \times [0,1]$.*

Proof. The natural projection $\pi\colon \mathbb{Z}_H \to Z$ can be treated as the orbit projection under the action of the normalizer $N(H)$ on $\mathbb{Z}_H$; i.e., we can assume that $Z = \mathbb{Z}_H/N(H)$. According to Proposition 7, the map π has a section $\sigma\colon Z \to \mathbb{Z}_H$. Thus,

$$\varphi\colon G/H \times [0,1] \to \mathbb{Z}, \qquad \varphi(gH,t) = g\cdot\sigma(t),$$

is a well-defined equimorphism. □

{pr7:v803

Proposition 7. *If a compact group G acts on a space $\mathbb{X}$ of a single orbit type, then the orbit projection $p\colon \mathbb{X} \to X$ is a Hurewicz bundle.*

Proof. Consider the expansion of the G-space $\mathbb{X}$ in a Lie series $\{\mathbb{X}_\alpha, \pi_\alpha^\beta\}$. Since $P = P_\alpha/P_{\alpha+1}$ is a compact Lie group, it follows by Lemma 3 that the bonding P-orbit projections $\pi_\alpha^{\alpha+1}$ are locally trivial bundles and, therefore, Hurewicz bundles. □

{lem3:v80

Lemma 3. *Any P-orbit projection $f\colon \mathbb{X} \to \mathbb{Y}$ from a G-space $\mathbb{X}$ of a single orbit type to an H-space $\mathbb{Y}$, where the kernel P of the epimorphism $\pi\colon G \to H$ is a compact Lie group, is a locally trivial bundle.*

Proof of Lemma 3. The required assertion follows readily from the slice extension theorem. □

2. ISOVARIANT EXTENSORS

{sec2:v80

Obviously, Theorem 1 is implied by the following more precise theorem.

{th8:v803

Theorem 8. *Suppose that, for any $i \ge 1$, $\mathbb{X}_i$ is an* Isov-*generating* Equiv-AE-*space. Then,*

$$\begin{aligned}&\textit{for any partial } G\textit{-map } \mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{X} \rightleftharpoons \prod\{\mathbb{X}_i \mid i \ge 1\},\\ &\textit{there exists a } G\textit{-map } \psi\colon \mathbb{Z} \to \mathbb{X} \textit{ extending } \varphi\\ &\textit{for which } \psi\restriction_{\mathbb{Z}\setminus\mathbb{A}} \textit{ is an isovariant map.}\end{aligned} \tag{6}$$

{eq6:v803

Proof. Since $\mathbb{X} \in$ Equiv-AE, it is sufficient to find, given any equivariant map $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$, a G-map $\psi\colon \mathbb{Z} \to \mathbb{X}$ extending $\varphi = \widehat{\varphi}\restriction_{\mathbb{A}}$ for which $\psi\restriction_{\mathbb{Z}\setminus\mathbb{A}}$ is an isovariant map. Since $\mathbb{A} \subset \mathbb{X}$ is closed, we can choose a sequence of neighborhoods $\mathbb{Z} = \mathbb{U}_0 \Supset \mathbb{U}_1 \Supset \cdots$ and G-functions $\chi_i\colon \mathbb{Z} \to [0,1]$, $i \ge 1$, so that

$$\bigcap \mathbb{U}_i = \mathbb{A}, \qquad \chi_i^{-1}(0) \supset \mathbb{Z}\setminus\mathbb{U}_i, \quad \chi_i^{-1}(1) \supset \mathbb{U}_{i+1}\,{}^2.$$

Let us represent the map $\widehat{\varphi}$ in the form $\prod\widehat{\varphi}_i$, where $\widehat{\varphi}_i\colon \mathbb{Z} \to \mathbb{X}_i$ is an equivariant map. Take an isovariant map $e_i\colon \mathbb{Z} \to \mathbb{X}_i$ (which exists, because $\mathbb{X}_i$ is Isov-generating) and let $H_i\colon \mathbb{Z} \times I \to \mathbb{X}_i$ be an Equiv-homotopy between e_i and $\widehat{\varphi}_i$ (which exists as well, because $\mathbb{X}_i \in$ Equiv-AE). Then the required map ψ is defined by

$$\begin{gathered}\psi\restriction_{\mathbb{A}} = \widehat{\varphi}\restriction_{\mathbb{A}} = \varphi,\\ (\psi\restriction_{\mathbb{U}_i\setminus\mathbb{U}_{i+1}})(z) = \widehat{\varphi}_1 \times \cdots \times \widehat{\varphi}_{i-1} \times H_i(z,\chi_i(z)) \times e_{i+1} \times \cdots \qquad \text{for} \quad i \ge 0. \quad \square\end{gathered}$$

Take a closed topological embedding $j\colon X \hookrightarrow L$ of the orbit space of any G-space $\mathbb{X}$ into some normed linear space L [17]. Since the countable power $\mathbb{J}$ of the metric cone $\operatorname{Con}\mathbb{T}$ over

$$\mathbb{T} = \bigsqcup\{G/H \mid G/H \in G\text{-ANE}\}$$

is Isov-generating [5], it follows that there exists an isovariant map $f\colon \mathbb{X} \to \mathbb{J}$. Obviously, the product $(j\circ p)\times f$ is a closed topological G-embedding of $\mathbb{X}$ into the G-space $\mathbb{Y} \rightleftharpoons L \times \mathbb{J}$; obviously, this G-space is an Isov-AE. Thus, we have proved the following theorem.

{th9:v803

Theorem 9. *Any G-space admits a closed G-embedding into an* Isov-AE-*space of the form $L \times \mathbb{J}$.*

[2] We say that an embedding $A \subset B$ is *strict* and write $A \Subset B$ if $\operatorname{Cl} A \subset \operatorname{Int} B$.

An easy consequence of Theorems 9 and 8 is an important relation between the injective objects of the isovariant and the equivariant category.

{th10:v80

Theorem 10. *Any* Isov-A[N]E-*space* $\mathbb{X}$ *has property* (6); *therefore, any* Isov-A[N]E-*space is an* Equiv-A[N]E-*space.*

We mention without proof that the equivariant Hilbert cube $\mathbb{Q}$ (for a compact metric group G) is an Isov-AE. On the other hand, if $\mathbb{X}$ is a compact Isov-AE-space, then the product of $\mathbb{X}$ and the Hilbert cube Q is equimorphic to $\mathbb{Q}$. Other examples of Isov-AE-spaces are the equivariant space $\mathbb{L}_2$ and the space $\mathrm{C}(G, L)$ (with the metric of uniform convergence) of all continuous maps $f\colon G \to L$ to an equivariant Hilbert space L of weight $w(G)$ on with the group G continuously acts by the rule

$$(g \cdot f)(h) = f(g^{-1} \cdot h), \qquad \text{where} \quad f \in \mathrm{C}(G, L), \quad g, h \in G.$$

These and other results of the theory of isovariant extensors will be published elsewhere.

It is easy to give an example of an Equiv-AE-space $\mathbb{Z} \notin$ Isov-ANE. In what follows, we obtain several results which make it possible to assess the degree of dissimilarity between these two classes; namely, $\mathbb{X} \in$ Equiv-ANE implies the equi–local contractibility (equi-LC) property for the family $\{\mathbb{X}^H \mid H < G\}$, and $\mathbb{X} \in$ Isov-ANE implies $\{\mathbb{X}_H \mid H < G\} \in$ equi-LC; moreover, $\operatorname{Con}\mathbb{X} \in$ Isov-AE and $\mathbb{X}^G \in$ AE imply $\mathbb{X} \in$ Isov-AE.

Apparently, the following questions are of certain interest in relation to the Hilbert–Smith conjecture [18]. Suppose that the acting group G is nontrivial and $\mathbb{X} \in$ Isov-AE. Is it true that $\dim \mathbb{X} = \infty$? $\dim X = \infty$?

3. JOINING SPLITTINGS

{sec3:v80

The *restriction of an admissible diagram $\mathcal{D}$ to an invariant subset* $\mathbb{U} \subset \mathbb{T}$ is defined as the diagram

$$\begin{array}{ccc} \mathbb{U}_{\mathbb{S}} \times [0,1] \cup \mathbb{U} \times \{0\} & \xrightarrow{\varphi_{\mathbb{U}}} & \mathbb{X}_{\mathbb{U}} \rightleftharpoons f^{-1}(\mathbb{Y}_{\mathbb{U}}) \\ \bigcap & & \downarrow f \\ \mathbb{U} \times [0,1] & \xrightarrow{\psi_{\mathbb{U}}} & \mathbb{Y}_{\mathbb{U}} \rightleftharpoons \psi(\mathbb{U} \times [0,1]), \end{array}$$

denoted by $\mathcal{D}_{\mathbb{U}}$, in which $\varphi_{\mathbb{U}}$ is the restriction of φ to $\mathbb{U}_{\mathbb{S}} \times I \cup \mathbb{U} \times \{0\}$, $\psi_{\mathbb{U}} \rightleftharpoons \psi\restriction_{\mathbb{U}\times I}$, and $\mathbb{U}_{\mathbb{S}} \rightleftharpoons \mathbb{U} \cap \mathbb{S}$. Clearly, $\mathcal{D}_{\mathbb{U}}$ is an admissible diagram; if the diagram $\mathcal{D}$ splits, then so does $\mathcal{D}_{\mathbb{U}}$.

The following proposition on joining splittings is a modification of an argument used by Palais in [1] to a more general situation.

{pr8:v803

Proposition 8. *Let $\{\mathbb{T}_\lambda \mid \lambda \in \Lambda\}$ be an open locally finite G-cover of $\mathbb{T}$ such that the restriction $\mathcal{D}_\lambda$ of an admissible diagram $\mathcal{D}$ to $\mathbb{T}_\lambda$ splits for any $\lambda \in \Lambda$. Then the diagram $\mathcal{D}$ splits as well.*

First, consider the case of a two-element index set Λ.

{lem4:v80

Lemma 4 (on joining two splittings)**.** *Let $\{\mathbb{T}_1, \mathbb{T}_2\}$ be an open G-cover of $\mathbb{T}$, and let $\mathcal{D}_i$ be the restriction of an admissible diagram $\mathcal{D}$ to $\mathbb{T}_i$. If G-homeomorphisms $g_i\colon \mathbb{T}_i \times [0,1] \to \mathbb{X}_{\mathbb{T}_i}$, $i \le 2$, split the diagrams $(\mathcal{D})_i$, respectively, then there exists a G-homeomorphism $g\colon \mathbb{T} \times [0,1] \to \mathbb{X}$ splitting the diagram $\mathcal{D}$. Moreover,*

$$g = g_i \quad \textit{on} \quad \mathbb{T}_i \setminus \mathbb{T}_0, \qquad \textit{where} \quad \mathbb{T}_0 \rightleftharpoons \mathbb{T}_1 \cap \mathbb{T}_2. \tag{7}$$ {eq7:v803

Proof. The proof of Lemma 4 is based on the following homotopy splitting effect. □

{lem5:v80

Lemma 5 (on homotopy splitting). *If the G-homeomorphisms $\widehat{\varphi}_i\colon \mathbb{T}\times[0,1]\to\mathbb{X}$, $i=1,2$, split the admissible diagram $\mathcal{D}$, then there exist isovariant G-homotopies*

$$H_s\colon \mathbb{T}\times[0,1]\to\mathbb{X}, \qquad 0\le s\le 1,$$

between $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ such that the H_s split the diagram $\mathcal{D}$[3] *for all $0\le s\le 1$.*

(The proof of Lemma 5 is given after the proof of Proposition 8.)

According to Lemma 5 on homotopy splitting, there exist

isovariant G-homotopies $H_s\colon \mathbb{T}_0\times[0,1]\to\mathbb{X}_{\mathbb{T}_0}$, $0\le s\le 1$, between $g_1\restriction_{\mathbb{T}_0}$ and $g_2\restriction_{\mathbb{T}_0}$ such that the G-homeomorphisms H_s split the diagram $\mathcal{D}_{\mathbb{T}_0}$ for all $0\le s\le 1$. (8) {eq8:v803

Since $\mathbb{T}\setminus\mathbb{T}_2$ and $\mathbb{T}\setminus\mathbb{T}_1$ are disjoint closed subsets of the normal space $\mathbb{T}$, it follows that there exists an invariant function $\xi\colon \mathbb{T}\to[1,2]$ such that

$$\begin{aligned} \xi&\equiv 0 &&\text{in a neighborhood of}\quad \mathbb{T}\setminus\mathbb{T}_2,\\ \xi&\equiv 1 &&\text{in a neighborhood of}\quad \mathbb{T}\setminus\mathbb{T}_1. \end{aligned}\tag{9}$$

{eq9:v803

Finally, we set

$$g(u,t)=\begin{cases} g_1(u,t) & \text{if } (u,t)\in(T\setminus T_2)\times I,\\ H_{\xi(u)}(u,t) & \text{if } (u,t)\in T_0\times I,\\ g_2(u,t) & \text{if } (u,t)\in(T\setminus T_1)\times I. \end{cases}$$

The continuity of the resulting map $g\colon \mathbb{T}\times I\to\mathbb{Z}$ is checked directly by using (9). It follows from (8) that g splits the diagram $\mathcal{D}$. This completes the proof of Lemma 4.

The rest of the proof of Proposition 8 on joining splittings proceeds by standard transfinite induction on the well-ordered index set Λ by using Lemma 4 on joining two splittings. The stabilization of the map g (to be constructed) which splits the diagram $\mathcal{D}$ and its continuity follow from the local finiteness of the cover $\{\mathbb{T}_\lambda\}$ and property (7). We leave the details to the reader.

Proof of Lemma 5. It follows readily by assumption that $h\rightleftharpoons(\widehat{\varphi}_1)^{-1}\circ\widehat{\varphi}_2$ is a G-self-homeomorphism of $\mathbb{T}\times[0,1]$ which is the identity map on $\mathbb{S}\times[0,1]\cup\mathbb{T}\times\{0\}$. Since

$$\psi=p\circ\widehat{\varphi}_2=(\psi\circ\widehat{\varphi}_1^{-1})\circ\widehat{\varphi}_2=\psi\circ h,\tag{10}$$

{eq10:v80

it follows that h has the form

$$h(z,t)=(\varphi_t(z),t),$$

where $\varphi_t\colon \mathbb{T}\to\mathbb{T}$ is a continuous (in $t\in T$) family of G-homeomorphisms. Clearly,

$$\varphi_0=\mathrm{Id},\qquad \varphi_t\restriction_{\mathbb{S}}=\mathrm{Id}\quad \text{for all } t\in I.$$

By virtue of (10), we have

$$\psi(w)=\psi(P\cdot w)=\psi(h(P\cdot w))\qquad\text{for all}\quad w=(z,t)\in\mathbb{T}\times I;$$

hence

$$(\psi_P)(P\cdot z,t)=(\psi_P)(P\cdot\varphi_t(z),t).$$

Taking into account the fact that ψ_P is an H-homeomorphism, we obtain

$$P\cdot z=P\cdot\varphi_t(z)\qquad\text{for all}\quad t\in I.\tag{11}$$

{eq11:v80

Now, we set

$$h_s(u,t)=(\varphi_{s\cdot t}(u),t)\colon \mathbb{T}\times I\to\mathbb{T}\times I,\qquad 0\le s\le 1.$$

[3]Therefore, H_s is an G-homeomorphism.

It can be verified directly that h_s is a continuous family of G-homeomorphisms, and, moreover,

$$h_1 = h, \qquad h_0 = \mathrm{Id}, \qquad h_s = h \quad \text{on} \quad \mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\}. \tag{12}$$ {eq12:v80

By using (11), it is easy to derive that

$$\psi = \psi \circ h_s \qquad \text{for all} \quad s \in I. \tag{13}$$ {eq13:v80

Finally, we define the required isovariant homotopy by

$$H_s \rightleftharpoons \widehat{\varphi}_1 \circ h_s, \qquad 0 \le s \le 1.$$

All of the required properties of H_s follow from (12) and (13). □

4. PROOF OF THEOREM 3

{sec4:v80

It suffices to prove that the P-orbit projection $f\colon \mathbb{X} \to \mathbb{Y} = \mathbb{X}/P$ is a Hurewicz Isov-bundle. Indeed, if there is an admissible diagram $\mathcal{D}$ for the P-orbit projection f from Theorem 3, then we pass from $\mathbb{Y}$ to the image $\mathbb{Y}' \subset (T \times I) \times \mathbb{Y}$ of $\mathbb{T} \times I$ under the G-map

$$q \times \psi\colon \mathbb{T} \times I \to (T \times I) \times \mathbb{Y}$$

and from $\mathbb{X}$ to the preimage $\mathbb{X}' \subset (T \times I) \times \mathbb{X}$ of $\mathbb{Y}'$ under the P-orbit projection

$$f' = \mathrm{Id} \times f\colon (T \times I) \times \mathbb{X} \to (T \times I) \times \mathbb{Y}$$

(here $q\colon \mathbb{T} \times I \to T \times I$ is the orbit projection). Let φ' and ψ' denote the maps

$$q \times \varphi\colon \mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\} \to \mathbb{X}', \qquad q \times \psi\colon \mathbb{T} \times [0,1] \to \mathbb{Y}',$$

respectively. Clearly, the P-orbit projection $f'\colon \mathbb{X}' \to \mathbb{Y}'$ is naturally involved in an admissible diagram $\mathcal{D}'$, which is weakly admissible, because ψ' induces a homeomorphism of the orbit spaces. The validity of Theorem 3 for weakly admissible diagrams implies the existence of an isovariant G-map $\widehat{\varphi}'\colon \mathbb{T} \times [0,1] \to \mathbb{X}'$ which splits $\mathcal{D}'$. The required isovariant G-map $\widehat{\varphi}\colon \mathbb{T} \times [0,1] \to \mathbb{X}$ splitting $\mathcal{D}$ is the composition of $\widehat{\varphi}'$ with the projection $(T \times I) \times \mathbb{X}$ onto the factor $\mathbb{X}$.

In any completely admissible diagram $\mathcal{D}$, the isovariant map

$$\varphi\colon \mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\} \to \mathbb{X}$$

is a closed G-embedding. Thus, in the proof of Theorem 3, we can assume without loss of generality that

1) $\mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\}$ is a closed subset of $\mathbb{X}$, and φ is the identity embedding;

2) $\mathbb{Y} = (\mathbb{T}/P) \times I$, and $\psi_P\colon (\mathbb{T}/P) \times I \to \mathbb{Y}$ is the identity map;

3) the map $f\restriction_{\mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\}}$ coincides with the restriction of the canonical P-orbit projection $\mathbb{T} \times [0,1] \to T \times [0,1]$ to $\mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\}$.

The rest of the proof of Theorem 3 asserting that the weakly admissible G-diagram is split proceeds by induction on the compact Lie group P by using Palais' metatheorem [1], whose proof is based on the stabilization of a decreasing (by inclusion) sequence of compact Lie groups.

{pr9:v803

Proposition 9. *Let $\mathcal{P}(H)$ be a property depending on a compact Lie group H. Suppose that $\mathcal{P}(H)$ holds for the trivial group $H = \{e\}$ and $\mathcal{P}(H)$ holds if so does $\mathcal{P}(K)$ for any proper subgroup $K < H$. Then $\mathcal{P}(H)$ holds for all compact Lie groups H.*

If $|P| = 1$, then the epimorphism $\pi\colon G \to H$ is an isomorphism, which trivializes the situation under examination. Suppose that, for any proper subgroup $Q < P$, Theorem 3 is valid. Let us show that it is valid for any P-orbit projection $f\colon \mathbb{X} \to \mathbb{Y}$.

The following lemma reduces proving that the weakly admissible diagram $\mathcal{D}$ splits to the case in which there are no P-fixed points in $\mathbb{X}$.

{lem6:v80

Lemma 6. *If any weakly admissible diagram $\mathcal{D}$ with empty $\mathbb{X}^P$ splits, then so does any weakly admissible diagram.*

Proof. By assumption, there exists an isovariant map

$$\varphi' \colon (\mathbb{T} \setminus \mathbb{T}^P) \times I \to \mathbb{X} \setminus \mathbb{X}^P$$

splitting the weakly admissible diagram $\mathcal{D}_{\mathbb{T}\setminus\mathbb{T}^P}$. We extend the map φ' so that the extension coincides with ψ on $\mathbb{T}^P \times I$. It is easy to check that φ' is well defined and continuous. ☐

Let $\mathcal{D}$ be a weakly admissible diagram. By virtue of Lemma 6, in what follows, we can assume that $\mathbb{X}^P = \varnothing$ for the G-space $\mathbb{X}$ (or, equivalently, that $P \setminus G_x \neq \varnothing$ for all $x \in \mathbb{X}$). The product

$$\mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\}$$

is naturally embedded in $\mathbb{X}$; therefore, $\mathbb{T}^P = \varnothing$.

First, suppose that $\mathbb{T}$ admits a nontrivial slice map $\alpha \colon \mathbb{T} \to G/K$, where $K < G$ is an extensor subgroup with $P \setminus K \neq \varnothing$. Then the G-map

$$\beta \rightleftharpoons \alpha \circ \mathrm{pr}_1 \colon \mathbb{T} \times [0,1] \to G/K,$$

where $\mathrm{pr}_1 \colon \mathbb{T} \times I \to \mathbb{T}$ is the projection onto the first factor, generates the slice map

$$\beta_P \colon \mathbb{Y} = (\mathbb{T}/P) \times [0,1] \to G/(P \cdot K) = H/L, \qquad \text{where} \quad L \rightleftharpoons \pi(K) < H,$$

under the passage to the P-orbit projection. Let

$$p \colon G/K \to G/\pi^{-1}(\pi(K)) \cong H/L$$

denote the natural G-map, which is a P-orbit projection.

{lem7:v80

Lemma 7. *If the partial G-map*

$$\mathbb{X} \hookleftarrow \mathbb{S} \times [0,1] \cup \mathbb{T} \times \{0\} \xrightarrow{\beta\restriction} G/K$$

admits a G-extension $\widehat{\beta} \colon \mathbb{X} \to G/K$ for which $\beta_P \circ f = p \circ \widehat{\beta}$, then the diagram $\mathcal{D}$ splits.

Proof. Let $\pi' \rightleftharpoons \pi\restriction \colon K \to L$ be the epimorphism of compact groups with kernel $Q = P \cap K$. Since $P \setminus K \neq \varnothing$, it follows that Q is a proper subgroup in the Lie group P.

Now consider the K-space $\mathbb{T}' \rightleftharpoons \alpha^{-1}([K])$, its closed subspace $\mathbb{S}' \rightleftharpoons \mathbb{T}' \cap \mathbb{S}$, and the K-space $\mathbb{X}' \rightleftharpoons \widehat{\beta}^{-1}([K])$. Clearly, the L-space $\mathbb{Y}' \rightleftharpoons \beta_P^{-1}([L])$ coincides with $f(\mathbb{X}')$. It follows from Proposition 5 on the heredity of orbit projections that the restriction map $f' = f\restriction \colon \mathbb{X}' \to \mathbb{Y}'$ is a Q-orbit projection.

We have $\widehat{\beta} = \mathrm{ext}(\beta\restriction_{\mathbb{S}\times[0,1]\cup\mathbb{T}\times\{0\}})$; hence

$$\varphi(\mathbb{S}' \times I \cup \mathbb{T}' \times \{0\}) \subset \mathbb{X}', \qquad \psi(\mathbb{T}' \times I) = \mathbb{Y}'.$$

Let

$$\varphi' \colon \mathbb{S}' \times I \cup \mathbb{T}' \times \{0\} \hookrightarrow \mathbb{X}' \qquad \psi' \colon \mathbb{T}' \times I \to \mathbb{Y}'$$

be the restrictions of φ and ψ to the corresponding subsets. Consider the naturally arising commutative K-diagram

$$\begin{array}{ccc} \mathbb{S}' \times I \cup \mathbb{T}' \times \{0\} & \overset{\varphi'}{\hookrightarrow} & \mathbb{X}' \\ \bigcap\!\!\Big| & & \Big\downarrow f' \\ \mathbb{T}' \times I & \xrightarrow{\psi'} & \mathbb{Y}', \end{array}$$

in which ψ' induces a homeomorphism of the orbit spaces and φ' is a K-embedding; we denote this diagram by $\mathcal{D}'$. It follows readily from Lemma 1 that the G-space $\mathbb{Z} \rightleftharpoons (f')^{-1}(\psi'(t' \times I))$ is of a single orbit type; therefore, the commutative K-diagram $\mathcal{D}'$ is weakly admissible.

The subgroup $Q < P$ is proper. Therefore, by the induction hypothesis, there exists a K-map $\widehat{\varphi}'\colon \mathbb{T}' \times I \to \mathbb{X}'$ splitting $\mathcal{D}'$. A direct verification shows that the rule

$$\widehat{\varphi}([g, s]_K) = [g, \widehat{\varphi}'(s)]_K, \qquad s \in \mathbb{T}' \times I$$

well defines a G-map

$$\widehat{\varphi}\colon G \times_K (\mathbb{T}' \times I) \equiv \mathbb{T} \times I \to G \times_K \mathbb{X}' \equiv \mathbb{X},$$

which splits the diagram $\mathcal{D}$. □

Let us show that the assumptions of Lemma 7 (and, hence, its conclusions) hold for the elements of some invariant cover of $\mathbb{T}$. By virtue of the splitting joining theorem and the induction hypothesis, Theorem 3 will follow.

Take any point $t \in \mathbb{T}$. The relation $P \setminus G_t \neq \varnothing$ and Proposition 3 imply the existence of an extensor subgroup $K < G$ for which $G_t < K$ and $P \setminus K \neq \varnothing$. Since $G/K \in G$-ANE, it follows that there exists a slice map $\alpha\colon \mathbb{U} \to G/K$, where $\mathbb{U} = \mathbb{U}(t) \subset \mathbb{T}$ is a G-neighborhood of t.

Let $L \rightleftharpoons \pi(K) < H$, and let

$$p\colon G/K \to G/\pi^{-1}(\pi(K)) \cong H/L$$

be the natural G-map generated the embedding of groups $K < \pi^{-1}(\pi(K))$. Consider the slice map

$$\beta_P\colon \psi(\mathbb{U}) \times [0,1]) = (\mathbb{U}/P) \times [0,1] \to G/(P \cdot K) = H/L$$

obtained from the G-map $\beta \rightleftharpoons \alpha \circ \mathrm{pr}_1\colon \mathbb{U} \times [0,1] \to G/K$ under the passage to the P-orbit projection.

{pr10:v80

Proposition 10. *There exists a G-neighborhood $\mathbb{V} \subset \mathbb{U}$ of the point t for which the partial G-map*

$$(f^{-1} \circ \psi)(\mathbb{V} \times I) \hookleftarrow (\mathbb{S} \cap \mathbb{V}) \times I \cup \mathbb{V} \times \{0\} \xrightarrow{\beta} G/K$$

admits a G-extension

$$\widehat{\beta}\colon (f^{-1} \circ \psi)(\mathbb{V} \times I) \to G/K \qquad \textit{for which} \quad \beta_P \circ f = p \circ \widehat{\beta}.$$

Proof. Without loss of generality, we can assume that $\mathbb{U}$ coincides with $\mathbb{T}$, i.e., there exists a slice map $\alpha\colon \mathbb{T} \to G/K$.

Consider the G-space $\mathbb{A} \rightleftharpoons f^{-1}(\psi(\{t\} \times I)) \subset \mathbb{X}$. Clearly,

$$\mathbb{A} \subset \mathbb{S} \times I \quad \text{for } t \in \mathbb{S} \qquad \text{and} \qquad \mathbb{A} \cap (\mathbb{S} \times I \cup \mathbb{T} \times \{0\}) = G(t) \times \{0\} \quad \text{for } t \notin \mathbb{S}.$$

Note that $\mathbb{A}$ has single orbit type (G_t), and its orbit space is homeomorphic to I. Consequently, by virtue of Proposition 6, $\mathbb{A}$ is equimorphic to $G(t) \times I$. Hence there exists a slice map

$$\gamma\colon \mathbb{S} \times I \cup \mathbb{T} \times \{0\} \cup \mathbb{A} \to G/K$$

on $\mathbb{S} \times I \cup \mathbb{T} \times \{0\} \cup \mathbb{A}$ which coincides with β on $\mathbb{S} \times I \cup \mathbb{T} \times \{0\}$ and satisfies the condition $\beta_P \circ f = p \circ \gamma$.

According to Theorem 6, the natural G-map $p\colon G/K \to G/\pi^{-1}(\pi(K))$ is equivariantly locally soft, and according to Theorem 7, there exists a G-extension $\widehat{\beta}\colon \mathbb{W} \to G/K$ of γ to a G-neighborhood

$$\mathbb{W} \supset \mathbb{S} \times I \cup \mathbb{T} \times \{0\} \cup \mathbb{A}$$

for which $\beta_P \circ f = p \circ \widehat{\beta}$. Reducing, if necessary, the neighborhood $\mathbb{W}$ to a neighborhood of the form $(f^{-1} \circ \psi)(\mathbb{V} \times I)$, we obtain the required slice map $\widehat{\beta}$. □

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