Quantum mechanical particle in magnetic field in the Lobachevsky space

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Various possibilities to define analogs of the uniform magnetic field in the Lobachevsky space are considered, using different coordinate systems in this space. Quantum mechanical problem of motion in the defined fields is also discussed.

1. Introduction

Since Schrödinger [1] first solved quantum mechanical Kepler - Coulomb problem on a three-dimensional sphere, a great number of authors have studied different aspects of this problem in spaces of constant, positive and negative, curvature (see a concise historical review in [2]). Solutions of this problem have been used to describe bound states in nuclear physics and nanophysics [3, 4].

Much less is known about motion of particles in other configurations of electromagnetic fields in such spaces. We mention here paper [5], where static solutions of Maxwell equations on a three-sphere were found and applied to the study of Stark and Zeeman effects. Static fields introduced in [5] do not allow to solve exactly the problem of motion of particles in these fields. Recently, a simple generalization of the uniform magnetic field to the three-dimensional Lobachevsky space has been proposed, and corresponding quantum mechanical and classical problems have been solved [6, 7]. The generalization is based on a certain choice of solutions of Maxwell equations which give the uniform magnetic field in the limit of zero curvature.

In this paper, we consider some alternative possibilities to define analogs of the uniform magnetic field in the Lobachevsky space, using different coordinate systems in this space. The quantum mechanical problem of motion in the defined fields is also discussed, and for some cases exact solutions of Schrödinger equation are obtained.

2. Analogs of the uniform magnetic field in the Lobachevsky space

The first generalization of the uniform magnetic field to the Lobachevsky space was done in the paper [6]. The magnetic field in this paper was found as a solution of Maxwell equations in the Robertson – Walker metrics with Lobachevsky space as spatial part

$$dS^2 = c^2dt^2 - dl^2, \quad dl^2 = \rho^2[\cosh^2 z_1(dr_1^2 + \sinh^2 r_1 d\phi_1^2) + dz_1^2]. \tag{1}$$

The coordinates z_1, r_1, ϕ_1 of the Lobachevsky space are defined by relations

$$\begin{gathered} u_0 = \rho\cosh z_1 \cosh r_1, u_1 = \rho\cosh z_1 \sinh r_1 \cos\phi_1, \\ u_2 = \rho\cosh z_1 \sinh r_1 \sin\phi_1, u_3 = \rho\sinh z_1, \\ -\infty < z_1 < \infty,\ 0 < r_2 < \infty,\ \leq \phi < 2\pi,\ u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2, \end{gathered} \tag{2}$$

where u_a, $a = 0, 1, 2, 3$ are coordinates in the ambient four-dimensional pseudo-Euclidean space.

In the limit when $\rho \to \infty$, $\rho r_1 \to r$, $\rho z_1 \to z$, the spatial part of the line element (1) reduces to that of Euclidean space in which r, z are cylindrical coordinates:

$$dS^2 = c^2dt^2 - (dr^2 + r^2d\phi^2 + dz^2). \tag{3}$$

The vector potential of the uniform magnetic field in the flat space with metrics (3) can be written in the form

$$A_t = A_r = A_z = 0,\ A_\phi = -\frac{Br^2}{2}. \tag{4}$$

The generalization of this potential to the Lobachevsky space reads [6]

$$A_{r_1} = A_{z_1} = 0,\ A_{\phi_1} = -B\rho^2(\cosh r_1 - 1). \tag{5}$$

The corresponding magnetic field is

$$F_{\phi_1 r_1} = B\rho^2 \sinh r_1. \tag{6}$$

Besides the coordinates r_1, z_1, ϕ_1 (2), usually named hyperbolic, there are some other coordinate systems in Lobachevsky space, which have the cylindrical coordinate system of Euclidean space as their limit. We consider here two such systems. First one is the cylindrical coordinate system:

$$\begin{gathered} u_0 = \rho\cosh z_2 \cosh r_2,\ u_1 = \rho\sinh r_2 \cos\phi_2, \\ u_2 = \rho\sinh r_2 \sin\phi_2,\ u_3 = \rho\cosh r_2 \sinh z_2, \\ -\infty < z_2 < \infty,\ 0 < r_2 < \infty,\ \leq \phi < 2\pi. \end{gathered} \tag{7}$$

The line element in Lobachevsky space is here

$$dl^2 = \rho^2(dr_2^2 + \sinh^2 r_2 d\phi_2^2 + \cosh^2 r_2 dz_2^2),$$

and in the limit $\rho \to \infty,\ \rho r_2 \to r,\ \rho z_2 \to z$, we get the metrics of Euclidean space in cylindrical coordinates.

Expressions for nonzero components of potential and electromagnetic field are

$$A_{\phi_2} = -B\rho^2 \ln \cosh r_2,\ \ F_{\phi_2 r_2} = B\rho^2 \tanh r_2. \tag{8}$$

We also consider the horospheric coordinate system in Lobachevsky space defined by relations

$$\begin{aligned} u_0 &= \frac{\rho}{2}[e^{z_3} + (r_3^2+1)e^{-z_3}],\ u_1 = \rho r_3 e^{-z_3}\cos\phi_3, \\ u_2 &= \rho r_3 e^{-z_3}\sin\phi_3,\ \ u_3 = \frac{\rho}{2}[e^{z_3} + (r_3^2-1)e^{-z_3}], \\ &-\infty < z_3 < \infty,\ 0 < r_3 < \infty,\ 0 \le \phi < 2\pi. \end{aligned} \tag{9}$$

Metrics of Lobachevsky space in coordinates $r_3,\ z_3,\ \phi_3$ takes the form

$$dl^2 = \rho^2[e^{-2z_3}(dr_3^2 + r_3^2 d\phi_3^2) + dz_3^2],$$

and in the limit $\rho \to \infty,\ \rho r_3 \to r,\ \rho z_3 \to z$, it also goes over into the metrics of Euclidean space in cylindrical coordinates.

Nonzero components of potential and electromagnetic field are found to be

$$A_\phi = -\frac{\rho^2 B r^2}{2}, \quad F_{\phi r} = \rho^2 B r. \tag{10}$$

Electromagnetic fields and potentials (8) and(10) were found by close analogy with approach of paper [6], but Maxwell equations were solved in different coordinate systems. As a result, we obtained different magnetic fields in the Lobachevsky space which reduce to the same uniform magnetic field in Euclidean space when $\rho \to \infty$.

Besides the coordinates in the Lobachevsky space which reduce in the limit $\rho \to \infty$ to the cylindrical coordinates of Euclidean space, we also consider a coordinate system which goes over in the same limit into the Cartesian one. It is defined as follows:

$$\begin{aligned} u_0 &= \frac{1}{2}(e^z + (1 + x_3^2 + y_3^2)e^{-z_3}),\ u_1 = x_3 e^{-z_3}, \\ u_2 &= ye^{-z_3},\ u_3 = \frac{1}{2}(e^{z_3} - (1 - x_3^2 - y_3^2)e^{-z_3}), \\ &-\infty < z_3 < \infty,\ 0 < r_3 < \infty,\ 0 \le \phi < 2\pi, \end{aligned} \tag{11}$$

and the line element is

$$dl^2 = \rho^2[e^{-2z_3}(dx_3^2 + dy_3^2) + dz_3^2].$$

Coordinates x_3, y_3 are related to the horospheric coordinates r_3, ϕ_3 as

$$x_3 = r_3 \cos\phi_3,\ y_3 = r_3 \sin\phi_3.$$

Transforming potential (10) to the coordinates x_3, y_3, z_3 we find

$$A_{x_3} = -\frac{B\rho^2 y_3}{2},\ A_{y_3} = \frac{B\rho^2 x_3}{2},\ A_{z_3} = 0,\ F_{x_3 y_3} = -B\rho^2. \tag{12}$$

3. The quantum mechanical problem for a particle in a magnetic field in the Lobachevsky space

Here we consider the quantum mechanical problem for a particle in a magnetic field.

The Schrödinger equation in the Robertson – Walker metric

$$dS^2 = (dx^0)^2 + g_{kl}dx^k dx^l, \quad g = \det g_{kl}.$$

in the presence of magnetic field is (we use units such that $\hbar = M = 1$)

$$i\partial_t \Psi = H\Psi, \quad H = \frac{1}{2}\left[\left(\frac{i}{\sqrt{-g}}\partial_k \sqrt{-g} + \frac{e}{c}A_k\right)(-g^{kl})\left(i\partial_l + \frac{e}{c}A_l\right)\right]\Psi. \tag{13}$$

In the hyperbolic coordinate system (2) Hamiltonian (13) with potential (5) is

$$H = -\frac{1}{2\rho^2\cosh^2 z_1}\left[\frac{1}{\sinh r_1}\frac{\partial}{\partial r_1}\sinh r_1 \frac{\partial}{\partial r_1} - \frac{1}{\sinh^2 r_1}\left(i\frac{\partial}{\partial\phi_1} + \frac{2eB\rho^2}{c}\sinh\frac{r_1^2}{2}\right)^2 - \frac{\partial}{\partial z_1}\cosh^2 z_1 \frac{\partial}{\partial z_1}\right]. \tag{14}$$

Solutions of the Schrödinger equation (14) were found in ([6]). Here we note that with Hamiltonian (14) commute operators

$$J_1 = \sin\phi_1 \frac{\partial}{\partial r_1} + \coth r_1 \cos\phi_1 \frac{\partial}{\partial\phi_1} - \frac{iB\rho^2 \sinh r_1 \cos\phi_1}{\cosh r_1 + 1},$$

$$J_2 = \cos\phi_1 \frac{\partial}{\partial r_1} - \coth r_1 \sin\phi_1 \frac{\partial}{\partial\phi_1} + \frac{iB\rho^2 \sinh r_1 \sin\phi_1}{\cosh r_1 + 1}, \quad J_3 = \frac{\partial}{\partial\phi_1} - iB\rho^2, \tag{15}$$

with commutation relations of generators of group O(2,1)

$$[J_1, J_2] = -J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2. \tag{16}$$

In the cylindrical coordinate system (7) Hamiltonian (13) with potential (8) is

$$-\frac{1}{2\rho^2}\left[\frac{1}{\sinh r_2 \cosh r_2}\frac{\partial}{\partial r_2}\sinh r_2 \cosh r_2 \frac{\partial}{\partial r_2} - \frac{1}{\sinh^2 r_2}\left(i\frac{\partial}{\partial\phi_2} - \frac{2eB\rho^2}{c}\ln\cosh r_2\right)^2 + \frac{1}{\cosh^2 r_2}\frac{\partial^2}{\partial z_2^2}\right]. \tag{17}$$

After substitution

$$\Psi = e^{-iEt} e^{im\phi_2} e^{ikz_2} R_2(r_2)$$

we obtain equation

$$-\frac{1}{\sinh r_2 \cosh r_2} \frac{d}{dr_2} \sinh r_2 \cosh r_2 \frac{dR_2}{dr_2}$$
$$+\frac{1}{\sinh^2 r_2}\left(m + \frac{eB}{c}\rho^2 \ln\cosh r_2\right)^2 R_2 - \left(\frac{k^2}{\cosh^2 r_2} + 2E\rho^2\right) R_2 = 0.$$

This equation could not be solved in terms of known functions.

In the horospherical coordinate system (9) Hamiltonian (13) with potential (10) is

$$H = \frac{e^{2z_3}}{\rho^2}\left[-\frac{1}{r_3}\frac{\partial}{\partial r_3} r_3 \frac{\partial}{\partial r_3} + \frac{1}{r_3^2}\left(i\frac{\partial}{\partial \phi_3} - \frac{eB\rho^2 r_3^2}{2c}\right)^2 - \frac{\partial}{\partial z_3} e^{-2z_3}\frac{\partial}{\partial z_3}\right]. \tag{18}$$

With Hamiltonian (18) commute operators

$$P_1 = \cos\phi_3 \frac{\partial}{\partial r_3} - \frac{1}{r_3}\sin\phi_3 \frac{\partial}{\partial \phi_3} + \frac{iB\rho^2 r_3 \sin\phi_3}{2},$$
$$P_2 = \sin\phi_3 \frac{\partial}{\partial r_3} + \frac{1}{r_3}\cos\phi_3 \frac{\partial}{\partial \phi_3} - \frac{iB\rho^2 r_3 \cos\phi_3}{2}, \quad L_3 = \frac{\partial}{\partial \phi_3}.$$

The substitution

$$\Psi = e^{-iEt} e^{im\phi_3} Z_3(z_3) R_3(r_3)$$

leads to separation of variables, and we obtain differential equations

$$\frac{d}{dz_3} e^{-2z_3} \frac{dZ_3}{dz_3} + 2E\rho^2 e^{-2z_3} Z_3 = \lambda Z_3, \tag{19}$$
$$-\frac{1}{r_3}\frac{d}{dr_3} r_3 \frac{dR_3}{dr_3} + \frac{1}{r_3^2}\left(m - \frac{eB\rho^2 r_3^2}{2c}\right)^2 R_3 = \lambda R_3. \tag{20}$$

By substitution $x = i\sqrt{\lambda}e^z$ the equation (19) is reduced to the Bessel equation

$$\frac{d^2 Z_3(x)}{dx^2} - \frac{1}{x}\frac{dZ_3(x)}{dx} + \left(1 + \frac{2E\rho^2}{x^2}\right) Z_3(x) = 0$$

with solutions

$$Z_3(x) = C_1 x J_{\sqrt{1-2E\rho^2}}(x) + C_2 x Y_{\sqrt{1-2E\rho^2}}(x).$$

In the equation (20) we denote $eB\rho^2/c = B_1$ and introduce a new variable $y = B_1 r_3^2/2$. Then equation takes the form

$$y\frac{d^2 R_3(y)}{dy^2} + \frac{dR_3(y)}{dy} - \frac{1}{4}\left(y + \frac{m^2}{y} - \frac{2(B_1 m + \lambda)}{B_1}\right) R_3(y) = 0. \tag{21}$$

We set in (21) $R_3(y) = y^{|m/2|}e^{-y/2}F(y)$ and obtain a confluent hypergeometric equation [8]

$$y\frac{d^2F}{dy^2} + (|m| + 1 - y)\frac{dF}{dy} - \left(\frac{|m| - m}{2} - \frac{\lambda}{2B_1}\right)F(y) = 0,$$

which has solutions $\Phi(\alpha, \gamma; y)$ with

$$\alpha = \frac{|m| - m}{2} - \frac{\lambda}{2B_1}, \ \gamma = |m| + 1.$$

Requirement of finiteness of solutions of radial equation (20) leads to the quantization rule for separation constant λ

$$\lambda = B_1(|m| - m + 2n + 1), \quad n = 0,\, 1,\, 2, \ldots .$$

In order to find solutions of Schrödinger equation in the coordinates (11) we take the vector potential in the form

$$A_{y_3} = B\rho^2 x_3, \ A_{x_3} = A_{z_3} = 0, \tag{22}$$

which can be obtained from (12) by a gauge transformation. Then the Hamiltonian takes the form

$$H = \frac{e^{2z_3}}{\rho^2}\left[-\frac{\partial^2}{\partial x^2} + \left(i\frac{\partial}{\partial y} + \frac{eB\rho^2}{c}\right)^2 - \frac{\partial}{\partial z_3}e^{-2z_3}\frac{\partial}{\partial z_3}\right].$$

Separation of variables is carried out by using the substitution $\Psi = e^{-iEt}e^{ipy_3}X(x_3)Z(z_3)$, and separated equations are

$$\frac{d}{dz_3}e^{-2z_3}\frac{dZ_3}{dz_3} + 2E\rho^2 e^{-2z_3}Z_3 = \lambda' Z_3, \tag{23}$$

$$-\frac{d^2X}{dx^2} + \left(p - \frac{eB\rho^2 x}{c}\right)^2 X = \lambda' X. \tag{24}$$

Equation (23) is identical to (19). In the equation (24) we introduce a new variable $u = \sqrt{B_1}(x - p/B_1)$. Then we obtain the harmonic oscillator equation

$$\frac{d^2X(u)}{du^2} - u^2X(u) + \frac{\lambda'}{B_1} = 0. \tag{25}$$

As is well known, from finiteness of solutions of equation (25) follows the quantization rule

$$\lambda' = B_1(2n + 1), \quad n = 0,\, 1,\, 2, \ldots .$$

Conclusion

By solving Maxwell equations in different coordinate systems in the Lobachevsky space one may obtain alternative generalizations of the notion of the uniform magnetic field. All the fields found in this way have the same flat space limit. In some cases, problem of the motion of charged particles in these fields can be solved exactly. The found solutions may be of interest for modelling physical interactions in various fields.

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