Transitivity in the Theory of the Lorentz Group and the Stokes–Mueller Formalism in Polarization Optics

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Group-theoretical analysis of arbitrary polarization devices is performed, based on the theory of the Lorentz group. In effective "non-relativistic" Mueller case, described by 3-dimensional orthogonal matrices, results of the one polarization measurement $\mathbf{S} \stackrel{O}{\rightarrow} \mathbf{S}'$ determine group theoretical parameters within the accuracy of an arbitrary numerical variable. There are derived formulas, defining Muller parameter of the "non-relativistic" optical element uniquely and in explicit form by by the results of two independent polarization measurements.

Analysis is extended to Lorentzian optical devices, described by 4-dimensional Mueller matrices. In this case, any single polarization measurement $(S_0, \mathbf{S}) \stackrel{L}{\rightarrow} (S'_0, \mathbf{S}')$ fixes parameters of the corresponding Mueller matrix up to 3 arbitrary variables. Formulas, defining Muller parameter of any relativistic Mueller device uniquely can be found from results of four independent polarization measurements. Analytical expressions for Muller parameters of any optical element can be given the most simple form when using the results of 6 independent measurements, the corresponding formulas are written down in explicit form.

1. The transitivity problem in the theory of the Lorentz group

The polarization of light is one of the most remarkable phenomena in natur and has led to numerous discoveries and applications. Today it continues to play a vital role in optics. Before the nineteen fifties there was very little activity on the foundations of polarized light. For example, answers to questions such as the nature and mathematical formulation of unpolarized light and partially polarized light were not readily forthcoming. Fortunately, these questions along with the mathematical tools to deal with polarized light began to be addressed in earnest in the nineteen fifties. As a result, today there is a very

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good understanding of polarized light. In particular, the mathematical difficulties that had hindered complex polarization calculations were finally overcome with the introduction of the Mueller-Stokes matrix calculus and the Jones matrix calculus. Research in polarized light continues with much vigor as witnessed by the continued appearance of numerous publications and conferences

E. Collett. Field Guide to Polarization,
SPIE Press, Bellingham, WA (2005).

The Stokes parameters are a set of values that describe the polarization state of electromagnetic radiation. They were defined by G.G. Stokes in 1852 [1]. Mueller calculus is a matrix method for manipulating Stokes vectors, it was developed in 1943 by H. Mueller [2]; any optical element can be represented by a Mueller matrix. Any optical element can be represented by a Mueller matrix In optics, polarized light can be described using the Jones calculus, invented by R.C. Jones in 1941 [3–6]. Polarized light is represented by a 2-dimensional Jones complex vector, and linear optical elements are represented by Jones matrices. Note that Jones calculus is only applicable to light that is already fully polarized. Light which is unpolarized, partially polarized, or incoherent must be treated using Mueller calculus.

It is well known (the bibliography on the subject is enormous, many references are given in the present list [7–124]) that in describing (fully or partly) polarized light noticeable role may be given to the group of 3 + 1-pseudo orthogonal transformations consisting of a group $SO(3,1)$ isomorphic to the Lorentz group. Therefore, techniques developed in the frames of the Lorentz group (for instance, see [127–131], the big list on the theory of the Lorentz group is given in [132]), in particular within relativistic kinematics, may play heuristic role in exploring optical problems. In recent papers [125, 126], generalization of Jones formalism to describe a partly polarized light was given. The goal of the present paper is to construct mathematical tools for determining Mueller matrices of any optical elements uniquely from results of several independent polarization measurements.

Let us recall the known transitivity problem in relativistic kinematics: in Stokes – Mueller approach it reads

$$L_b{}^a(k,\bar{k}^*)\; S_a = +S'_b\;. \tag{1}$$

From the very beginning, one peculiarity shout be noted: due to existence of the concept of little Lorentz group initial and final Stokes 4-vectors S and S', one can write down the transitivity condition in the form $L\;(L_{little} S) = L'_{little} S'$,so that

$$[\;(L'_{little})^{-1} L\; L_{little}\;]\; S = S'\;. \tag{2}$$

This means that the transitive matrix L cannot be defined uniquely in terms of vectors S and S'. Let us use the factorized representation for Lorentzian matrices (we adhere notation given in [132]), eq. (1) gives

$$\hat{A}^{*} S = A^{-1} S' \,, \qquad \text{and} \qquad \hat{A}\, S = (\hat{A}^{*})^{-1}\, S' \,, \tag{3}$$

or in more detailed form (conjugate equation is written down too)

$$\left| \begin{array}{cccc} k_0^* & -k_1^* & -k_2^* & -k_3^* \\ -k_1^* & k_0^* & ik_3^* & -ik_2^* \\ -k_2^* & -ik_3^* & k_0^* & ik_1^* \\ -k_3^* & ik_2^* & -ik_1^* & k_0^* \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{cccc} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & ik_3 & -ik_2 \\ k_2 & -ik_3 & k_0 & ik_1 \\ k_3 & ik_2 & -ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| ,$$

$$\left| \begin{array}{cccc} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{cccc} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -ik_3^* & ik_2^* \\ k_2^* & ik_3^* & k_0^* & -ik_1^* \\ k_3^* & -ik_2^* & ik_1^* & k_0^* \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| . \tag{4}$$

Below, the notation will be used

$$k_0 = n_0 + im_0 \,, \qquad k_j = -in_j + m_j \,, \qquad k_0 - \mathbf{k}^2 = 1 \,.$$

Summing and subtracting eqs we get

$$\left| \begin{array}{cccc} n_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & n_0 & -n_3 & n_2 \\ -m_2 & n_3 & n_0 & -n_1 \\ -m_3 & -n_2 & n_1 & n_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{cccc} n_0 & m_1 & m_2 & m_3 \\ m_1 & n_0 & n_3 & -n_2 \\ m_2 & -n_3 & n_0 & n_1 \\ m_3 & n_2 & -n_1 & n_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| ,$$

$$\left| \begin{array}{cccc} -m_0 & -n_1 & -n_2 & -n_3 \\ -n_1 & -m_0 & m_3 & -m_2 \\ -n_2 & -m_3 & -m_0 & m_1 \\ -n_3 & m_2 & -m_1 & -m_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| = \left| \begin{array}{cccc} m_0 & -n_1 & -n_2 & -n_3 \\ -n_1 & m_0 & m_3 & -m_2 \\ -n_2 & -m_3 & m_0 & m_1 \\ -n_3 & m_2 & -m_1 & m_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| .$$

So, we arrive at two homogeneous linear systems under 8 varianles

$$\begin{array}{r} n_0\,(S_0 - S'_0) - m_1\,(S_1 + S'_1) - m_2\,(S_2 + S'_2) - m_3\,(S_3 + S'_3) = 0 \,, \\ -m_1\,(S_0 + S'_0) + n_0\,(S_1 - S'_1) + n_2\,(S_3 + S'_3) - n_3\,(S_2 + S'_2) = 0 \,, \\ -m_2\,(S_0 + S'_0) + n_0\,(S_2 - S'_2) + n_3\,(S_1 + S'_1) - n_1\,(S_3 + S'_3) = 0 \,, \\ -m_3\,(S_0 + S'_0) + n_0\,(S_3 - S'_3) + n_1\,(S_2 + S'_2) - n_2\,(S_1 + S'_1) = 0 \,, \end{array}$$

$$\begin{array}{r} -m_0\,(S_0 + S'_0) - n_1\,(S_1 - S'_1) - n_2\,(S_2 - S'_2) - n_3\,(S_3 - S'_3) = 0 \,, \\ -n_1\,(S_0 - S'_0) - m_0\,(S_1 + S'_1) - m_2\,(S_3 - S'_3) + m_3\,(S_2 - S'_2) = 0 \,, \\ -n_2\,(S_0 - S'_0) - m_0\,(S_2 + S'_2) - m_3\,(S_1 - S'_1) + m_1\,(S_3 - S'_3) = 0 \,, \\ -n_3\,(S_0 - S'_0) - m_0\,(S_3 + S'_3) - m_1\,(S_2 - S'_2) + m_2\,(S_1 - S'_1) = 0 \,. \end{array} \tag{5}$$

2. "Non-relativistic" 3-dimensional Mueller matrices

First, let us consider more simple (non-relativistic) case when $S'_0 = S_0 = I = \text{inv}$. Eqs. (5) takes the form (because we search solutions in 3-dimensional rotations, we require $m_0 = 0, \; m_j = 0$):

$$
\begin{aligned}
n_0 \; (S_1 - S'_1) + n_2 \; (S_3 + S'_3) - n_3 \; (S_2 + S'_2) = 0 \; , \\
n_0 \; (S_2 - S'_2) + n_3 \; (S_1 + S'_1) - n_1 \; (S_3 + S'_3) = 0 \; , \\
n_0 \; (S_3 - S'_3) + n_1 \; (S_2 + S'_2) - n_2 \; (S_1 + S'_1) = 0 \; , \\
-n_1 \; (S_1 - S'_1) - n_2 \; (S_2 - S'_2) - n_3 \; (S_3 - S'_3) = 0 \; .
\end{aligned}
\tag{6}
$$

The fourth equation in not independent of three remaining – it follows from them. Therefore we have the system of 3 independent ones

$$
\begin{aligned}
n_2 \; (S_3 + S'_3) - n_3 \; (S_2 + S'_2) = -n_0 \; (S_1 - S'_1) \; , \\
n_3 \; (S_1 + S'_1) - n_1 \; (S_3 + S'_3) = -n_0 \; (S_2 - S'_2) \; , \\
n_1 \; (S_2 + S'_2) - n_2 \; (S_1 + S'_1) = -n_0 \; (S_3 - S'_3) \; .
\end{aligned}
\tag{7}
$$

They may be written in 3-vector form

$$
\mathbf{n} \times (\mathbf{S} + \mathbf{S}') = -n_0 \; (\mathbf{S} - \mathbf{S}') \; . \tag{8}
$$

General solutions for $\mathbf{n}$ can be searched with the aid of substitution

$$
\mathbf{n} = \alpha \; \mathbf{S} + \rho \; \mathbf{S}' + \beta \; \mathbf{S} \times \mathbf{S}' \; ,
$$

then eq. (8) leads to (below note $S^2 = \mathbf{S}\mathbf{S}$)

$$
(\alpha - \rho) \; \mathbf{S} \times \mathbf{S}' + \beta \; [\; \mathbf{S}' \; S^2 + \mathbf{S}' \; (\mathbf{S}\mathbf{S}') - \mathbf{S} \; S^2 - \mathbf{S} \; (\mathbf{S}\mathbf{S}') \;] = -n_0 \mathbf{S} + n_0 \mathbf{S}' \; ,
$$

from whence it follow $\rho = \alpha$, α ia arbitrary, and

$$
n_0 = \beta \; (S^2 + \mathbf{S} \; \mathbf{S}') \; , \qquad \mathbf{n} = \alpha \; (\mathbf{S} + \mathbf{S}') + \beta \; \mathbf{S} \times \mathbf{S}' \; . \tag{9}
$$

One must to take into account additional restriction for parameters of rotation matrices

$$
n_0^2 + \mathbf{n}^2 = 1 \; , \tag{10}
$$

which results in

$$
\beta^2 \; (S^2 + \mathbf{S} \; \mathbf{S}')^2 + [\alpha \; (\mathbf{S} + \mathbf{S}') + \beta \; \mathbf{S} \times \mathbf{S}']^2 = 1 \; ,
$$

or

$$
\beta^2 \; [\; S^4 + 2S^2 \; (\mathbf{S} \; \mathbf{S}') + (\mathbf{S} \; \mathbf{S}')^2] + \beta^2 \; [S^4 - (\mathbf{S}\mathbf{S}')^2] + 2\alpha^2 \; (S^2 + \mathbf{S} \; \mathbf{S}') = 1 \; ;
$$

and ultimately eq. (10) gives

$$\beta^2 \; S^2 \; + \alpha^2 = \frac{1}{2(S^2 + \mathbf{S} \; \mathbf{S}')} \; . \tag{11}$$

General solution of eq. (11) can be presented in terms of sin- and cos-functions of an angular variable

$$\alpha = \frac{\sin \Gamma}{\sqrt{2(S^2 + \mathbf{S} \; \mathbf{S}')}} \; , \qquad \beta = \frac{\cos \Gamma}{S \; \sqrt{2(S^2 + \mathbf{S} \; \mathbf{S}')}} \; , \qquad \Gamma \in [0, 2\pi] \; . \tag{12}$$

Thus, relations (9) read (here $\Gamma \in [0, 2\pi]$ stands for arbitrary parameter)

$$n_0^2 + \mathbf{n}^2 = 1 \; , \qquad n_0 = \frac{\cos \Gamma}{S \; \sqrt{2(S^2 + \mathbf{S} \; \mathbf{S}')}} \; (S^2 + \mathbf{S} \; \mathbf{S}') \; ,$$
$$\mathbf{n} = \frac{\sin \Gamma}{\sqrt{2(S^2 + \mathbf{S} \; \mathbf{S}')}} \; (\mathbf{S} + \mathbf{S}') + \frac{\cos \Gamma}{S \; \sqrt{2(S^2 + \mathbf{S} \; \mathbf{S}')}} \; \mathbf{S} \times \mathbf{S}' \; . \tag{13}$$

Note that when $\mathbf{S}' = \mathbf{S}$, relations (13) describe the case of little rotation group

$$n_0^2 + \mathbf{n}^2 = 1 \; , \qquad n_0 = \cos \Gamma \; , \qquad \mathbf{n} = \sin \Gamma \; \frac{\mathbf{S}}{S} \; . \tag{14}$$

When $\Gamma = 0$, solution (13) becomes of the most simple form

$$n_0 = \frac{S^2 + \mathbf{S} \; \mathbf{S}'}{S \; \sqrt{2 \; (S^2 + \mathbf{S} \; \mathbf{S}')}} \; , \qquad \mathbf{n} = \frac{\mathbf{S} \times \mathbf{S}'}{S \; \sqrt{2 \; (S^2 + \mathbf{S} \; \mathbf{S}')}} \; . \tag{15}$$

Note, that we may transform all the relations to a Gibbs 3-vector parameter in the rotation group (the full treatment of the theory in this parametrization see in [130])

$$\mathbf{c} = \frac{\mathbf{n}}{n_0} \; , \tag{16}$$

then eqs. (13) give

$$\mathbf{c} = \; \mathrm{tg} \, \Gamma \; \frac{S}{S^2 + \mathbf{S} \; \mathbf{S}'} \; (\mathbf{S} + \mathbf{S}') + \frac{\mathbf{S} \times \mathbf{S}'}{S^2 + \mathbf{S} \; \mathbf{S}'} \; . \tag{17}$$

Note that in the non-relativistic case, for Stokes vectors one can use the following parametrization (I is intensity of the light beam, p is a polarization degree)

$$S_0 = I, \qquad \mathbf{S} = Ip \; \mathbf{N}, \qquad I - \mathrm{inv} \; , \qquad \mathbf{N}^2 = 1 \; ; \tag{18}$$

at this (13) and (15) change to

$$n_0^2 + \mathbf{n}^2 = 1 \; , \qquad n_0 = \cos \Gamma \frac{1 + \mathbf{N} \; \mathbf{N}'}{\sqrt{2(1 + \mathbf{N} \; \mathbf{N}')}} \; ,$$
$$\mathbf{n} = \sin \Gamma \; \frac{\mathbf{N} + \mathbf{N}'}{\sqrt{2(1 + \mathbf{N} \; \mathbf{N}')}} \; + \; \cos \Gamma \; \frac{\mathbf{N} \times \mathbf{N}'}{\sqrt{2(1 + \mathbf{N}\mathbf{N}')}} \; , \tag{19}$$

and

$$\mathbf{c} = \; \mathrm{tg} \, \Gamma \; \frac{\mathbf{N} + \mathbf{N}'}{1 + \mathbf{N} \; \mathbf{N}'} \; + \frac{\mathbf{N} \times \mathbf{N}'}{1 + \mathbf{N} \; \mathbf{N}'} \; . \tag{20}$$

3. On defining Mueller 3-matrices from the results of polarization measurements

Because a single polarization measurement relating $\mathbf{S} \xrightarrow{L} \mathbf{S}'_1$ cannot fix Mueller 3-matrix uniquely, to obtain result values for parameters of the Mueller 3-matrix, one need to perform two independent measurements $\mathbf{S}_1 \xrightarrow{L} \mathbf{S}'_1$, $\mathbf{S}_2 \xrightarrow{L} \mathbf{S}'_2$. Mathematically, the problem of finding a definite Mueller 3-matrix can be formulated as a system to solve, describing two polarization measurement with one the same Mueller matrix.

First, let us consider this task with the aid of Gibbs 3-paramere

$$\mathbf{c} = \text{ tg } \Gamma \, \frac{\mathbf{N}_1 + \mathbf{N}'_1}{1 + \mathbf{N}_1\mathbf{N}'_1} + \frac{\mathbf{N}_1 \times \mathbf{N}'_1}{1 + \mathbf{N}_1\mathbf{N}'_1} \, ,$$
$$\mathbf{c} = \text{ tg } \Gamma \, \frac{\mathbf{N}_2 + \mathbf{N}'_2}{1 + \mathbf{N}_2\mathbf{N}'_2} + \frac{\mathbf{N}_2 \times \mathbf{N}'_2}{1 + \mathbf{N}_2\mathbf{N}'_2} \, ; \qquad (21)$$

so we have a vector equation

$$\text{tg } \Gamma \, \left[\frac{\mathbf{N}_1 + \mathbf{N}'_1}{1 + \mathbf{N}_1\mathbf{N}'_1} - \frac{\mathbf{N}_2 + \mathbf{N}'_2}{1 + \mathbf{N}_2\mathbf{N}'_2} \right] + \frac{\mathbf{N}_1 \times \mathbf{N}'_1}{1 + \mathbf{N}_1\mathbf{N}'_1} - \frac{\mathbf{N}_2 \times \mathbf{N}'_2}{1 + \mathbf{N}_2\mathbf{N}'_2} = 0 \, . \qquad (22)$$

Multiplying it by $\mathbf{N}_1, \mathbf{N}'_1, \mathbf{N}_2, \mathbf{N}'_2$, we obtain four scalar equations

$$\text{tg } \Gamma \left[1 - \frac{\mathbf{N}_1(\mathbf{N}_2 + \mathbf{N}'_2)}{1 + \mathbf{N}_2\mathbf{N}'_2} \right] - \frac{\mathbf{N}_1(\mathbf{N}_2 \times \mathbf{N}'_2)}{1 + \mathbf{N}_2\mathbf{N}'_2} = 0 \, ,$$
$$\text{tg } \Gamma \left[1 - \frac{\mathbf{N}'_1(\mathbf{N}_2 + \mathbf{N}'_2)}{1 + \mathbf{N}_2\mathbf{N}'_2} \right] - \frac{\mathbf{N}'_1(\mathbf{N}_2 \times \mathbf{N}'_2)}{1 + \mathbf{N}_2\mathbf{N}'_2} = 0 \, ,$$
$$\text{tg}\Gamma \left[\frac{\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}'_1)}{1 + \mathbf{N}_1\mathbf{N}'_1} - 1 \right] + \frac{\mathbf{N}_2(\mathbf{N}_1 \times \mathbf{N}'_1)}{1 + \mathbf{N}_1\mathbf{N}'_1} = 0 \, ,$$
$$\text{tg}\Gamma \left[\frac{\mathbf{N}'_2(\mathbf{N}_1 + \mathbf{N}'_1)}{1 + \mathbf{N}_1\mathbf{N}'_1} - 1 \right] + \frac{\mathbf{N}'_2\mathbf{N}_1 \times \mathbf{N}'_1}{1 + \mathbf{N}_1\mathbf{N}'_1} = 0 \, . \qquad (23)$$

From whence it follow

$$\text{tg } \Gamma = \frac{\mathbf{N}_1 \, (\mathbf{N}_2 \times \mathbf{N}'_2)}{(\mathbf{N}_2 - \mathbf{N}_1)(\mathbf{N}_2 + \mathbf{N}'_2)} \, , \qquad \text{tg } \Gamma = -\frac{\mathbf{N}'_1 \, (\mathbf{N}'_2 \times \mathbf{N}_2)}{(\mathbf{N}'_2 - \mathbf{N}'_1)(\mathbf{N}'_2 + \mathbf{N}_2)} \, ,$$
$$\text{tg } \Gamma = \frac{\mathbf{N}_2 \, (\mathbf{N}_1 \times \mathbf{N}'_1)}{(\mathbf{N}_1 - \mathbf{N}_2)(\mathbf{N}_1 + \mathbf{N}'_1)} \, , \qquad \text{tg } \Gamma = -\frac{\mathbf{N}'_2 \, (\mathbf{N}'_1 \times \mathbf{N}_1)}{(\mathbf{N}'_1 - \mathbf{N}'_2)(\mathbf{N}'_1 + \mathbf{N}_1)} \, . \qquad (24)$$

Thus, we have a simple expression for tg Γ , together with four additional constraints, which determine the whole aggregate of all possible couples of Stokes 3-vectors related by one the same Mueller matrices.

Now let us detail considering of the task in the frames of unitary group $SU(2)$ – evidently, two solutions cannot contradict each other. Here we have

$$n_0 = \beta_1 \, \mathbf{S}_1(\mathbf{S}_1 + \mathbf{S}'_1) \, , \qquad \mathbf{n} = \alpha_1 \, (\mathbf{S}_1 + \mathbf{S}'_1) + \beta_1 \, \mathbf{S}_1 \times \mathbf{S}'_1 \, ,$$
$$n_0 = \beta_2 \, \mathbf{S}_2(\mathbf{S}_2 + \mathbf{S}'_2) \, , \qquad \mathbf{n} = \alpha_2 \, (\mathbf{S}_2 + \mathbf{S}'_2) + \beta_2 \, \mathbf{S}_2 \times \mathbf{S}'_2 \, , \qquad (25)$$

what is equivalent to

$$n_0 = \cos\Gamma \, \frac{1 + \mathbf{N}_1\mathbf{N}'_1}{\sqrt{2(1 + \mathbf{N}_1\mathbf{N}'_1)}} \, ,$$
$$\mathbf{n} = \sin\Gamma \, \frac{\mathbf{N}_1 + \mathbf{N}'_1}{\sqrt{2(1 + \mathbf{N}_1\mathbf{N}'_1)}} + \cos\Gamma \, \frac{\mathbf{N}_1 \times \mathbf{N}'_1}{\sqrt{2(1 + \mathbf{N}_1\mathbf{N}'_1)}} \, ,$$

$$n_0 = \cos\Gamma \, \frac{1 + \mathbf{N}_2\mathbf{N}'_2}{\sqrt{2(1 + \mathbf{N}_2\mathbf{N}'_2)}} \, ,$$
$$\mathbf{n} = \sin\Gamma \, \frac{\mathbf{N}_2 + \mathbf{N}'_2}{\sqrt{2(1 + \mathbf{N}_2\mathbf{N}'_2)}} + \cos\Gamma \, \frac{\mathbf{N}_2 \times \mathbf{N}'_2}{\sqrt{2(1 + \mathbf{N}_2\mathbf{N}'_2)}} \, . \tag{26}$$

From two different expressions for n_0, it follows

$$\mathbf{N}_1\mathbf{N}'_1 = \mathbf{N}_2\mathbf{N}'_2 \, . \tag{27}$$

Taking this into account, from two different expressions for $\mathbf{n}$ we derive

$$\sin\Gamma \, [\, (\mathbf{N}_1 + \mathbf{N}'_1) - (\mathbf{N}_2 + \mathbf{N}'_2) \,] + \cos\Gamma \, [\, (\mathbf{N}_1 \times \mathbf{N}'_1) - (\mathbf{N}_2 \times \mathbf{N}'_2) \,] = 0 \, . \tag{28}$$

It should be noted that due to (27), relation (22) becomes much more simpler

$$\mathrm{tg}\,\Gamma \, [\, (\mathbf{N}_1 + \mathbf{N}'_1) - (\mathbf{N}_2 + \mathbf{N}'_2) \,] + \mathbf{N}_1 \times \mathbf{N}'_1 - \mathbf{N}_2 \times \mathbf{N}'_2 = 0 \, . \tag{29}$$

In fact, (28) and (29) coincide, difference consist in the following: (28) cannot distinguish between two solutions: $(+\cos\Gamma, \, +\sin\Gamma)$ and $(-\cos\Gamma, \, -\sin\Gamma)$.

4. Relativistic Mueller matrices relating two Stokes 4-vectors

Let us turn back to general (relativistic) case of Mueller matrices (5):

$$m_1 \, (S_1 + S'_1) + m_2 \, (S_2 + S'_2) + m_3 \, (S_3 + S'_3) = n_0 \, (S_0 - S'_0) \, ,$$
$$m_1 \, (S_0 + S'_0) - n_2 \, (S_3 + S'_3) + n_3 \, (S_2 + S'_2) = n_0 \, (S_1 - S'_1) \, ,$$
$$m_2 \, (S_0 + S'_0) - n_3 \, (S_1 + S'_1) + n_1 \, (S_3 + S'_3) = n_0 \, (S_2 - S'_2) \, ,$$
$$m_3 \, (S_0 + S'_0) - n_1 \, (S_2 + S'_2) + n_2 \, (S_1 + S'_1) = n_0 \, (S_3 - S'_3) \, ,$$
$$-n_1 \, (S_1 - S'_1) - n_2 \, (S_2 - S'_2) - n_3 \, (S_3 - S'_3) = m_0 \, (S_0 + S'_0) \, ,$$
$$-n_1 \, (S_0 - S'_0) - m_2 \, (S_3 - S'_3) + m_3 \, (S_2 - S'_2) = m_0 \, (S_1 + S'_1) \, ,$$
$$-n_2 \, (S_0 - S'_0) - m_3 \, (S_1 - S'_1) + m_1 \, (S_3 - S'_3) = m_0 \, (S_2 + S'_2) \, ,$$
$$-n_3 \, (S_0 - S'_0) - m_1 \, (S_2 - S'_2) + m_2 \, (S_1 - S'_1) = m_0 \, (S_3 + S'_3) \, . \tag{30}$$

Because we search solutions among proper orthochronous Lorentzian transformations, unknown parameters must obey additional relations

$$n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1 , \qquad n_0 m_0 + \mathbf{n}\mathbf{m} = 0 ; \tag{31}$$

by this reason, the trivial solution $n_a = 0, m_a = 0$ for (30) is of no interest. Eqs. (30) can be rewritten in 3-vector form

$$\begin{gathered}
\mathbf{m}\,(\mathbf{S} + \mathbf{S}') = n_0\,(S_0 - S_0') , \\
\mathbf{n}\,(\mathbf{S} - \mathbf{S}') = -m_0\,(S_0 + S_0') , \\
\mathbf{m}\,(S_0 + S_0') + (\mathbf{S} + \mathbf{S}') \times \mathbf{n} = n_0\,(\mathbf{S} - \mathbf{S}') , \\
\mathbf{n}\,(S_0 - S_0') - (\mathbf{S} - \mathbf{S}') \times \mathbf{m} = -m_0\,(\mathbf{S} + \mathbf{S}') .
\end{gathered} \tag{32}$$

Note that the (non-relativity) requirement $S_0 - S_0' = 0$ immediately leads us to additional relations $\mathbf{m} = 0$ and $m_0 = 0$, and we get eqs. (7)–(8).

Let us introduce notation

$$S_0 + S_0' = A , \qquad S_0 - S_0' = B , \qquad \mathbf{S} + \mathbf{S}' = \mathbf{A} , \qquad \mathbf{S} - \mathbf{S}' = \mathbf{B} . \tag{33}$$

The complete system od equations to solve is

$$n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1 , \qquad n_0 m_0 + \mathbf{n}\mathbf{m} = 0 ; \tag{34}$$

$$\mathbf{m}\,\mathbf{A} = n_0\,B , \qquad \mathbf{n}\,\mathbf{B} = -m_0\,A ; \tag{35}$$

$$\mathbf{m}\,A + \mathbf{A} \times \mathbf{n} = n_0\,\mathbf{B} , \qquad \mathbf{n}\,B - \mathbf{B} \times \mathbf{m} = -m_0\,\mathbf{A} . \tag{36}$$

It is convenient to use linear expansions for both 3-vectors

$$\mathbf{n} = N_+\mathbf{A} + N_-\mathbf{B} + N\mathbf{A} \times \mathbf{B} , \qquad \mathbf{m} = M_+\mathbf{A} + M_-\mathbf{B} + M\mathbf{A} \times \mathbf{B} . \tag{37}$$

From the first equation in (36) it follows

$$A(M_+\mathbf{A} + M_-\mathbf{B} + M\mathbf{A} \times \mathbf{B}) + \mathbf{A} \times (N_-\mathbf{B} + N\mathbf{A} \times \mathbf{B}) = n_0\,\mathbf{B} ,$$

which gives three equations

$$AM_+ + \mathbf{A}\mathbf{B}\,N = 0 , \qquad AM_- - \mathbf{A}^2 N = n_0 , \qquad AM + N_- = 0 . \tag{38}$$

In the same manner, from the second equation in (36) we get

$$B\,(N_+\mathbf{A} + N_-\mathbf{B} + N\mathbf{A} \times \mathbf{B}) - \mathbf{B} \times (M_+\mathbf{A} + M\mathbf{A} \times \mathbf{B}) = -m_0\,\mathbf{A} ,$$

and further

$$BN_- + \mathbf{A}\mathbf{B}M = 0 , \qquad BN_+ - \mathbf{B}^2\,M = -m_0 , \qquad BN + M_+ = 0 . \tag{39}$$

Thus, two vector equations (36) provide us with the system for six parameters

$$AM_+ + \mathbf{AB}\, N = 0 \; , \qquad AM_- - \mathbf{A}^2 N = n_0 \; , \qquad AM + N_- = 0 \; ;$$
$$BN_- + \mathbf{AB} M = 0 \; , \qquad BN_+ - \mathbf{B}^2\, M = -m_0 \; , \qquad BN + M_+ = 0 \; . \tag{40}$$

After excluding the variables N_-, M_+:

$$N_- = -AM \; , \qquad M_+ = -BN \; , \tag{41}$$

eqs. (40) read

$$-ABN + \mathbf{AB}\, N = 0 \; , \qquad AM_- - \mathbf{A}^2 N = n_0 \; ,$$
$$-ABM + \mathbf{AB} M = 0 \; , \qquad BN_+ - \mathbf{B}^2\, M = -m_0 \; . \tag{42}$$

Note that equations 1 and 3 are identities. In fact, eqs. (42) are equivalent to two equations only

$$AM_- - \mathbf{A}^2 N = n_0 \; , \qquad BN_+ - \mathbf{B}^2\, M = -m_0 \; . \tag{43}$$

Substituting expressions

$$\mathbf{n} = N_+ \mathbf{A} - M\, A\mathbf{B} + N \mathbf{A} \times \mathbf{B} \; , \qquad \mathbf{m} = M_- \mathbf{B} - N\, B\mathbf{A} + M \mathbf{A} \times \mathbf{B} \; ; \tag{44}$$

into (35), we arrive at

$$M_- \mathbf{BA} - N\, B\mathbf{A}^2 = n_0\, B \quad \Longrightarrow \quad M_- A - N\mathbf{A}^2 = n_0 \; ,$$
$$N_+ \mathbf{AB} - M\, A\mathbf{B}^2 = -m_0\, A \quad \Longrightarrow \quad N_+ B - M\, \mathbf{B}^2 = -m_0 \; ;$$

which coincide with (43). This means that eqs. (35) can be removed. The above substitutions for two vectors (44) are to be allowed in the conditions

$$n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2 \; , \qquad n_0 m_0 = -\mathbf{nm} = 0 \; .$$

Let us simplify notation $M_- = x \; , \; N = y \; , \; N_+ = z \; , \; M = w$. In these variables, the main equations to solve read

$$n_0 = A\, x - \mathbf{A}^2 y \; , \qquad \mathbf{n} = z\mathbf{A} - wA\mathbf{B} + y\mathbf{A} \times \mathbf{B} \; ,$$
$$m_0 = -B\, z + \mathbf{B}^2\, w \; , \qquad \mathbf{m} = x\mathbf{B} - y\, B\, \mathbf{A} + w\mathbf{A} \times \mathbf{B} \; ;$$
$$n_0 m_0 = -\mathbf{nm} \; , \qquad n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2 \; . \tag{45}$$

First, let us detail $n_0 m_0 = -\mathbf{nm}$. Taking into account

$$n_0 m_0 = -xz\, AB + wx\, A\mathbf{B}^2 \; + yz\, B\mathbf{A}^2 - wy\, \mathbf{A}^2\mathbf{B}^2 \; ,$$
$$-\mathbf{nm} = -(z\mathbf{A} - wA\mathbf{B} + y\mathbf{A} \times \mathbf{B})\,(x\mathbf{B} - y\, B\, \mathbf{A} + w\mathbf{A} \times \mathbf{B}) =$$
$$= -xz\, \mathbf{AB} + yz\, B\mathbf{A}^2 + wx\, A\mathbf{B}^2 - yw\, AB\mathbf{AB} - yw\mathbf{A}^2\mathbf{B}^2 + yw(\mathbf{AB})^2 \; ,$$

we arrive at

$$0 = xz\,(AB - \mathbf{A}\mathbf{B}) - yw\,AB\mathbf{A}\mathbf{B} + yw(\mathbf{A}\mathbf{B})^2\,. \tag{46}$$

Because

$$AB - \mathbf{A}\mathbf{B} = (S_0^2 - \mathbf{S}^2) - (S_0^{\prime 2} - \mathbf{S}^{\prime 2}) = 0\,, \tag{47}$$

eq. (46) takes the form of an identity $0 = 0$, subsequently, this equation can be excluded from (45). Remaining and independent relations are

$$\begin{aligned} & n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2\,, \\ n_0 = A\,x - \mathbf{A}^2 y\,, \qquad & \mathbf{n} = z\mathbf{A} - wA\mathbf{B} + y\mathbf{A}\times\mathbf{B}\,, \\ m_0 = -B\,z + \mathbf{B}^2\,w\,, \qquad & \mathbf{m} = x\mathbf{B} - y\,B\,\mathbf{A} + w\mathbf{A}\times\mathbf{B}\,. \end{aligned} \tag{48}$$

Each of vector equation in (48) can be changed into three scalar ones; those are obtained through multiplying them by $\mathbf{A}, \mathbf{B}, \mathbf{A}\times\mathbf{B}$:

$$\begin{aligned} \mathbf{A}\mathbf{n} &= z\,\mathbf{A}^2 - w\,A^2 B\,, \\ \mathbf{B}\mathbf{n} &= z\,AB - w\,A\mathbf{B}^2\,, \\ (\mathbf{A}\times\mathbf{B})\mathbf{n} &= +y\,\mathbf{A}^2\mathbf{B}^2 - y\,A^2B^2\,, \\[6pt] \mathbf{A}\mathbf{m} &= x\,AB - y\,B\,\mathbf{A}^2 \\ \mathbf{B}\mathbf{m} &= x\,\mathbf{B}^2 - y\,B^2 A\,, \\ (\mathbf{A}\times\mathbf{B})\mathbf{m} &= +w\,\mathbf{A}^2\mathbf{B}^2 - w\,A^2B^2\,. \end{aligned} \tag{49}$$

These equations are easy to solve

$$\begin{aligned} y &= \frac{(\mathbf{A}\times\mathbf{B})\mathbf{n}}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}\,, \qquad y = \frac{1}{B}\frac{(\mathbf{B}\mathbf{m})AB - (\mathbf{A}\mathbf{m})\mathbf{B}^2}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}\,, \\ w &= -\frac{1}{A}\frac{(\mathbf{B}\mathbf{n})\mathbf{A}^2 - (\mathbf{A}\mathbf{n})AB}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}\,, \qquad w = \frac{(\mathbf{A}\times\mathbf{B})\mathbf{m}}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}\,, \\ z &= -\frac{(\mathbf{B}\mathbf{n})AB - (\mathbf{A}\mathbf{n})\mathbf{B}^2}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}\,, \qquad x = \frac{-(\mathbf{A}\mathbf{m})AB + (\mathbf{B}\mathbf{m})\mathbf{A}^2}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}\,. \end{aligned} \tag{50}$$

Taking (48), we may turn back to a starting complex parameter k_a:

$$\begin{aligned} k_0 = n_0 + im_0 = (xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)\,, \\ \mathbf{k} = \mathbf{m} - i\mathbf{n} = -(y\,B + i\,z)\,\mathbf{A} + (x + iwA)\,\mathbf{B} + (w - iy)\mathbf{A}\times\mathbf{B}\,. \end{aligned} \tag{51}$$

Note that one can derive a more simple 3-vector, parameter for Lorentz group [130]

$$\mathbf{q} = \frac{\mathbf{k}}{k_0} = \frac{-(y\,B + i\,z)\,\mathbf{A} + (x + iwA)\,\mathbf{B} + (w - iy)\mathbf{A}\times\mathbf{B}}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}\,. \tag{52}$$

It may be formally simplified

$$
\begin{aligned}
\mathbf{q} = \alpha\ \mathbf{A} + \beta\ \mathbf{B} + \gamma\ \mathbf{A}\times\mathbf{B}\ ,\\
\alpha = \frac{-(y\ B + i\ z)}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}\ ,\\
\beta = [\frac{x + iwA}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}\ ,\\
\gamma = \frac{w - iy}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}\ .
\end{aligned} \tag{53}
$$

The formulas allow transition to a more simple non-relativistic case (when $x \equiv 0\ ,\ w \equiv 0\ ,\ B = 0$)

$$
\begin{aligned}
\mathbf{c} = i\ \mathbf{q} = i\alpha\ \mathbf{A} + i\beta\ \mathbf{B} + i\gamma\ \mathbf{A}\times\mathbf{B}\ ,\\
i\ \alpha = -\frac{1}{\mathbf{A}^2}\ \frac{z}{y}\ , \qquad i\ \beta = 0\ , \qquad i\gamma = -\frac{1}{\mathbf{A}^2}\ ;
\end{aligned} \tag{54}
$$

these relations describe 1-parametric set of 3-rotations. In relations (48), the non-relativistic case is reached as follow

$$
n_0^2 + \mathbf{n}^2 = 1\ , \qquad n_0 = y\mathbf{A}^2\ ,\ \ \mathbf{n} = z\mathbf{A} + y\mathbf{A}\times\mathbf{B}\ . \tag{55}
$$

let us obtain an explicit form of the relationship $n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2$ from (48). We have

$$
\begin{aligned}
n_0^2 - m_0^2 = (A\ x - \mathbf{A}^2 y)^2 - (-B\ z + \mathbf{B}^2\ w)^2 =\\
= A^2x^2 - B^2z^2 - 2A\mathbf{A}^2\ xy + 2B\mathbf{B}^2\ zw + (\mathbf{A}^2)^2\ y^2 - (\mathbf{B}^2)^2\ w^2\ ,
\end{aligned}
$$

and further

$$
\begin{aligned}
\mathbf{m}^2 = (x\mathbf{B} - y\ B\ \mathbf{A} + w\mathbf{A}\times\mathbf{B})\ (x\mathbf{B} - y\ B\ \mathbf{A} + w\mathbf{A}\times\mathbf{B}) =\\
x^2\ \mathbf{B}^2 - xy\ B(\mathbf{B}\mathbf{A}) - xy\ B(\mathbf{B}\mathbf{A}) + y^2\ B^2\mathbf{A}^2 + w^2\mathbf{A}^2\mathbf{B}^2 - w^2(\mathbf{A}\mathbf{B})^2\ ,
\end{aligned}
$$

that is

$$
\mathbf{m}^2 = x^2\ \mathbf{B}^2 - 2xy\ AB^2 + y^2\ B^2\mathbf{A}^2 + w^2\mathbf{A}^2\mathbf{B}^2 - w^2A^2B^2\ .
$$

In the same manner, we derive

$$
\begin{aligned}
\mathbf{n}^2 = (z\mathbf{A} - wA\mathbf{B} + y\mathbf{A}\times\mathbf{B})\ (z\mathbf{A} - wA\mathbf{B} + y\mathbf{A}\times\mathbf{B}) =\\
= z^2\ \mathbf{A}^2 - 2zw\ BA^2 + w^2\ A^2\mathbf{B}^2 + y^2\mathbf{A}^2\mathbf{B}^2 - y^2A^2B^2\ ,
\end{aligned}
$$

and further

$$
\begin{aligned}
1 + \mathbf{m}^2 - \mathbf{n}^2 = 1 + x^2\ \mathbf{B}^2 - 2xy\ AB^2 + y^2\ B^2\mathbf{A}^2 + w^2\mathbf{A}^2\mathbf{B}^2 - w^2A^2B^2 -\\
-z^2\ \mathbf{A}^2 + 2zw\ BA^2 - w^2\ A^2\mathbf{B}^2 - y^2\mathbf{A}^2\mathbf{B}^2 + y^2A^2B^2\ ,
\end{aligned}
$$

that is

$$1 + \mathbf{m}^2 - \mathbf{n}^2 = 1 + x^2\ \mathbf{B}^2 - z^2\ \mathbf{A}^2 - 2xy\ AB^2 + 2zw\ BA^2 +$$
$$+y^2[(\ B^2 - \mathbf{B}^2)\mathbf{A}^2 + A^2B^2] - w^2[(A^2 - \mathbf{A}^2)\mathbf{B}^2 + A^2B^2]\ .$$

The quadratic equation for parameters of the Mueller matrix takes the form

$$x^2(A^2 - \mathbf{B}^2) + 2xy\ A(B^2 - \mathbf{A}^2) + y^2[\ (\mathbf{A}^2 + \mathbf{B}^2 - B^2)\mathbf{A}^2 - A^2B^2\ ] =$$
$$= z^2(B^2 - \mathbf{A}^2) + 2zw\ B(A^2 - \mathbf{B}^2) + w^2[\ (\mathbf{A}^2 + \mathbf{B}^2 - A^2)\mathbf{B}^2 - A^2B^2\ ] + 1\ . \tag{56}$$

5. On defining 4-dimensional Mueller matrix from polarization measurements

As shown above, each polarization measurement

$$S_a \stackrel{L}{\Longrightarrow} S'_a \qquad \text{or} \qquad (A_a, B_a) \stackrel{L}{\Longrightarrow} (A'_a, B'_a)$$

allows to obtain the quadratic constraint on Mueller's characteristics of a polarization device

$$x^2(A^2 - \mathbf{B}^2) + 2xyA(B^2 - \mathbf{A}^2) + y^2[(\mathbf{A}^2 + \mathbf{B}^2 - B^2)\mathbf{A}^2 - A^2B^2] =$$
$$= z^2(B^2 - \mathbf{A}^2) + 2zwB(A^2 - \mathbf{B}^2) + w^2[(\mathbf{A}^2 + \mathbf{B}^2 - A^2)\mathbf{B}^2 - A^2B^2] + 1\ ; \tag{57}$$

it has a 3-parametric set of solutions which describe all the possible Mueler matrices of the given optical device

$$n_0 = x\ A - y\ \mathbf{A}^2\ , \qquad \mathbf{n} = z\ \mathbf{A} - w\ A\mathbf{B} + y\ \mathbf{A} \times \mathbf{B}\ ,$$
$$m_0 = -z\ B + w\ \mathbf{B}^2\ , \qquad \mathbf{m} = x\ \mathbf{B} - y\ B\ \mathbf{A} + w\ \mathbf{A} \times \mathbf{B}\ . \tag{58}$$

It is evident, that to fix Mueller matrix uniquely, one should perform several polarization tests. Let start with four ones – the problem to solve is formulate as a system of 4 equations

$$x^2\ (A_1^2 - \mathbf{B}_1^2) + 2xy\ A_1(B_1^2 - \mathbf{A}_1^2) + y^2\ [\ (\mathbf{A}_1^2 + \mathbf{B}_1^2 - B_1^2)\mathbf{A}_1^2 - A_1^2B_1^2\ ] =$$
$$= z^2\ (B_1^2 - \mathbf{A}_1^2) + 2zw\ B_1(A_1^2 - \mathbf{B}_1^2) + w^2\ [\ (\mathbf{A}_1^2 + \mathbf{B}_1^2 - A_1^2)\mathbf{B}_1^2 - A_1^2B_1^2\ ] + 1\ .$$

$$x^2\ (A_2^2 - \mathbf{B}_2^2) + 2xy\ A_2(B_2^2 - \mathbf{A}_2^2) + y^2\ [\ (\mathbf{A}_2^2 + \mathbf{B}_2^2 - B_2^2)\mathbf{A}_2^2 - A_2^2B_2^2\ ] =$$
$$= z^2\ (B_2^2 - \mathbf{A}_2^2) + 2zw\ B_2(A_2^2 - \mathbf{B}_2^2) + w^2\ [\ (\mathbf{A}_2^2 + \mathbf{B}_2^2 - A_2^2)\mathbf{B}_2^2 - A_2^2B_2^2\ ] + 1\ .$$

$$x^2\ (A_3^2 - \mathbf{B}_3^2) + 2xy\ A_3(B_3^2 - \mathbf{A}_3^2) + y^2\ [\ (\mathbf{A}_3^2 + \mathbf{B}_3^2 - B_3^2)\mathbf{A}_3^2 - A_3^2B_3^2\ ] =$$
$$= z^2\ (B_3^2 - \mathbf{A}_3^2) + 2zw\ B_3(A_3^2 - \mathbf{B}_3^2) + w^2\ [\ (\mathbf{A}_3^2 + \mathbf{B}_3^2 - A_3^2)\mathbf{B}_3^2 - A^2B_3^2\ ] + 1\ .$$

$$x^2\ (A_4^2 - \mathbf{B}_4^2) + 2xy\ A_4(B_4^2 - \mathbf{A}_4^2) + y^2\ [\ (\mathbf{A}_4^2 + \mathbf{B}_4^2 - B_4^2)\mathbf{A}_4^2 - A_4^2B_4^2\ ] =$$
$$= z^2\ (B_4^2 - \mathbf{A}_4^2) + 2zw\ B_4(A_4^2 - \mathbf{B}_4^2) + w^2\ [\ (\mathbf{A}_4^2 + \mathbf{B}_4^2 - A_4^2)\mathbf{B}_4^2 - A_4^2B_4^2\ ] + 1\ .$$

It may be presented in a symbolical form as

$$\begin{aligned}
a_1 x^2 + 2b_1 xy + c_1 y^2 &= \alpha_1 z^2 + 2\beta_1 zw + \sigma_1 w^2 + 1 \ , \\
a_2 x^2 + 2b_2 xy + c_2 y^2 &= \alpha_2 z^2 + 2\beta_2 zw + \sigma_2 w^2 + 1 \ , \\
a_1 x^2 + 2b_3 xy + c_3 y^2 &= \alpha_3 z^2 + 2\beta_3 zw + \sigma_3 w^2 + 1 \ , \\
a_4 x^2 + 2b_4 xy + c_4 y^2 &= \alpha_4 z^2 + 2\beta_4 zw + \sigma_4 w^2 + 1 \ .
\end{aligned} \tag{59}$$

In general, this mathematical task should have a definite solution, though rather cumbersome one. Indeed, we could successively exclude the variables as follows

$$\begin{array}{r}
(1) \quad \Longrightarrow \quad x = x(y,z,w) \ , \\
(2) \quad \Longrightarrow \quad y = y(z,w) \ , \quad x = x(y(z,w),z,w) = \bar{x}(z,w) \ , \\
(3) \quad \Longrightarrow \quad z = z(w) \ , \\
(4) \quad \Longrightarrow \quad w = w(...) \ , \quad z = z(w(...)) \ .
\end{array}$$

However, there exist another and more beautiful way to solve the problem. Indeed, let us consider 6 independent polarization measurements – they provide us with 6 linear equations under 6 variables

$$x^2 \ , \ \ y^2 \ , \ \ 2xy \ , \qquad z^2 \ , \ \ w^2 \ , \ \ 2zw \ ;$$

$$\begin{aligned}
a_1 x^2 + 2b_1 xy + c_1 y^2 - \alpha_1 z^2 - 2\beta_1 zw - \sigma_1 w^2 &= +1 \ , \\
a_2 x^2 + 2b_2 xy + c_2 y^2 - \alpha_2 z^2 - 2\beta_2 zw - \sigma_2 w^2 &= +1 \ , \\
a_1 x^2 + 2b_3 xy + c_3 y^2 - \alpha_3 z^2 - 2\beta_3 zw - \sigma_3 w^2 &= +1 \ , \\
a_4 x^2 + 2b_4 xy + c_4 y^2 - \alpha_4 z^2 - 2\beta_4 zw - \sigma_4 w^2 &= +1 \ , \\
a_5 x^2 + 2b_5 xy + c_5 y^2 - \alpha_5 z^2 - 2\beta_5 zw - \sigma_5 w^2 &= +1 \ , \\
a_6 x^2 + 2b_6 xy + c_6 y^2 - \alpha_6 z^2 - 2\beta_6 zw - \sigma_6 w^2 &= +1 \ .
\end{aligned} \tag{60}$$

By physical reasons, we cam presuppose existence of a unique solution of the task. This is given by Kramer's rule

$$\begin{aligned}
x^2 &= \frac{\Delta_{x^2}}{\Delta} \ , \qquad y^2 = \frac{\Delta_{y^2}}{\Delta} \ , \qquad 2xy = \frac{\Delta_{2xy}}{\Delta} \ , \\
z^2 &= \frac{\Delta_{z^2}}{\Delta} \ , \qquad w^2 = \frac{\Delta_{w^2}}{\Delta} \ , \qquad 2zw = \frac{\Delta_{2zw}}{\Delta} \ ,
\end{aligned} \tag{61}$$

from whence it follows (evidently, arising subtleties with $\pm$ should be examined additionally)

$$\begin{aligned}
x + y &= \sqrt{\frac{\Delta_{x^2} + \Delta_{y^2} + \Delta_{2xy}}{\Delta}} \ , \qquad x - y = \sqrt{\frac{\Delta_{x^2} + \Delta_{y^2} - \Delta_{2xy}}{\Delta}} \ , \\
z + w &= \sqrt{\frac{\Delta_{z^2} + \Delta_{w^2} + \Delta_{2zw}}{\Delta}} \ , \qquad z - w = \sqrt{\frac{\Delta_{z^2} + \Delta_{w^2} - \Delta_{2zw}}{\Delta}} \ .
\end{aligned} \tag{62}$$

Recall (see (51) that Muller's matrices are defined by k-parameter

$$k_0 = (xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2) \,,$$
$$\mathbf{k} = -(y\, B + i\, z)\, \mathbf{A} + (x + iwA)\, \mathbf{B} + (w - iy)\mathbf{A} \times \mathbf{B} \,;$$

evidently, any orthogonal Lorentz matrix cannot distinguish between $(+k_0, +\mathbf{k})$ and $(-k_0, -\mathbf{k})$.

We may employ the same method in non-relativistic case as well. See (55); with the notation $z = \nu, y = N$ we have

$$n_0^2 + \mathbf{n}^2 = 1 \,, \qquad n_0 = y\mathbf{A}^2 \,, \;\; \mathbf{n} = z\mathbf{A} + y\mathbf{A} \times \mathbf{B} \,. \tag{63}$$

Note that because

$$\mathbf{A}^2 = (\mathbf{S} + \mathbf{S}')^2 = \mathbf{S}^2 + \mathbf{S}^{'2} + 2\mathbf{S}\mathbf{S} = 2(S^2 + \mathbf{S}\mathbf{S}) \,,$$
$$\mathbf{A} \times \mathbf{B} = 2\mathbf{S} \times \mathbf{S}' \,,$$

eqs. (63) are equivalent to

$$n_0 = 2y\, (S^2 + \mathbf{S}\mathbf{S}) \,, \;\; \mathbf{n} = z\mathbf{A} + 2y\, \mathbf{S} \times \mathbf{S}' \,, \tag{64}$$

and thereby coincide with (9)

$$n_0 = \beta\, (S^2 + \mathbf{S}\, \mathbf{S}') \,, \qquad \mathbf{n} = \alpha\, (\mathbf{S} + \mathbf{S}') + \beta\, \mathbf{S} \times \mathbf{S}' \,. \tag{65}$$

In this notation two independent polarization test provide us with a linear system

$$\begin{gathered} y^2[\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2] + z^2\mathbf{A}_1^2 = 1 \,, \\ y^2[\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2] + z^2\mathbf{A}_2^2 = 1 \,, \end{gathered} \tag{66}$$

its solution is

$$\begin{gathered} y^2 = \frac{(\mathbf{A}_1\mathbf{B}_1)^2 - (\mathbf{A}_2\mathbf{B}_2)^2}{[\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2]\mathbf{A}_2^2 - [\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2]\mathbf{A}_1^2} \,, \\ z^2 = \frac{[\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2] - [\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2]}{[\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2]\mathbf{A}_2^2 - [\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2]\mathbf{A}_1^2} \,. \end{gathered} \tag{67}$$

6. On diagonalizing the transitivity equation

The transitivity equation $LS = S'$ led us to a 3-surface in 4-parametric space

$$x^2\, (A^2 - \mathbf{B}^2) + 2xy\, A(B^2 - \mathbf{A}^2) + y^2\, [\, (\mathbf{A}^2 + \mathbf{B}^2 - B^2)\mathbf{A}^2 - A^2B^2\,] -$$
$$-z^2\, (B^2 - \mathbf{A}^2) - 2zw\, B(A^2 - \mathbf{B}^2) - w^2\, [\, (\mathbf{A}^2 + \mathbf{B}^2 - A^2)\mathbf{B}^2 - A^2B^2\,] = 1 \,,$$

or in symbolical form

$$ax^2 + 2bxy + cy^2 - \alpha z^2 - 2\beta zw - \sigma w^2 = +1 \ . \tag{68}$$

Let us examine the possibility to transform an elementary quadratic form to a diagonal form by mens of 3-rotation in 2-plane

$$\begin{gathered} ax^2 + 2bxy + cy^2 = FX^2 + GY^2 \ , \\ x = \cos\phi \ X + \sin\phi \ Y \ , \qquad y = -\sin\phi \ X + \cos\phi \ Y \ . \end{gathered} \tag{69}$$

Eqs. (69) yield

$$\begin{aligned} a(\cos\phi \ X + \sin\phi \ Y)^2 + 2b(\cos\phi \ X + \sin\phi \ Y)(-\sin\phi \ X + \cos\phi \ Y) + \\ +c(-\sin\phi \ X + \cos\phi \ Y)^2 = FX^2 + GY^2 \qquad \Longrightarrow \end{aligned}$$

$$\begin{aligned} a(2XY \ \sin\phi\cos\phi + X^2\cos^2\phi + Y^2\sin^2\phi) + \\ +2b[(Y^2 - X^2)\sin\phi \ \cos\phi + XY(\cos^2\phi - \sin^2\phi)] + \\ +c(-2XY \ \sin\phi\cos\phi + X^2\sin^2\phi + Y^2\cos^2\phi) = FX^2 + GY^2 \ . \end{aligned} \tag{70}$$

So we have three equations

$$\begin{aligned} X^2 : \qquad & a\cos^2\phi - 2b\sin\phi\cos\phi + c\sin^2\phi = F \, , \\ Y^2 : \qquad & a\sin^2\phi + 2b\sin\phi\cos\phi + c\cos^2\phi = G \, , \\ 2XY : \qquad & a\sin\phi\cos\phi + b(\cos^2\phi - \sin^2\phi) - c\sin\phi\cos\phi = 0 \, . \end{aligned}$$

With the help of the variable 2ϕ, these are written as

$$\begin{gathered} a\,\frac{\cos 2\phi + 1}{2} - b\sin 2\phi + c\,\frac{1-\cos 2\phi}{2} = F \, , \\ a\,\frac{1-\cos 2\phi}{2} + b\sin 2\phi + c\,\frac{\cos 2\phi + 1}{2} = G \, , \\ \frac{a-c}{2}\sin 2\phi + b\cos 2\phi = 0 \ . \end{gathered} \tag{71}$$

This results in

$$\sin 2\phi = \frac{2b}{\sqrt{(c-a)^2 + 4b^2}} \, , \qquad \cos 2\phi = \frac{c-a}{\sqrt{(c-a)^2 + 4b^2}} \ ; \tag{72}$$

and

$$\begin{aligned} F = \frac{a+c}{2} + \frac{a-c}{2}\cos 2\phi - b\sin 2\phi = \frac{a+c}{2} - \frac{\sqrt{(a-c)^2 + 4b^2}}{2} \ , \\ G = \frac{a+c}{2} - \frac{a-c}{2}\cos 2\phi + b\sin 2\phi = \frac{a+c}{2} + \frac{\sqrt{(a-c)^2 + 4b^2}}{2} \ . \end{aligned} \tag{73}$$

In the same manner, the second quadratic form is considered

$$-\alpha z^2 - 2\beta zw - \sigma w^2 = \Delta\ Z^2 + \Gamma W^2$$
$$z = \cos\rho\ Z + \sin\rho\ W\ , \qquad w = -\sin\rho\ Z + \cos\rho\ W\ , \tag{74}$$

$$\sin 2\rho = \frac{2\beta}{\sqrt{(\sigma-\alpha)^2 + 4\beta^2}}\ , \qquad \cos 2\rho = \frac{\sigma - \alpha}{\sqrt{(\sigma-\alpha)^2 + 4\beta^2}}\ ; \tag{75}$$

$$\Delta = \frac{\alpha+\sigma}{2} - \frac{\sqrt{(\alpha-\sigma)^2 + 4\beta^2}}{2}\ ;$$
$$\Gamma = \frac{\alpha+\sigma}{2} + \frac{\sqrt{(\alpha-\sigma)^2 + 4\beta^2}}{2}\ . \tag{76}$$

For instance, conditions at which F and G are positive, and $\Delta, \Gamma$are negative, are formulated in the form

$\underline{(F, G, \Delta, \Gamma) \sim (+, +, -, -),}$

$$a > 0\ ,\ c > 0\ , \qquad a + c > +\sqrt{(a-c)^2 + 4b^2} > 0 \qquad \Longrightarrow \qquad ac > b^2\ .$$
$$\alpha < 0\ ,\ \sigma < 0\ , \qquad \alpha + \sigma < -\sqrt{(\alpha-\sigma)^2 + 4\beta^2} \qquad \Longrightarrow \qquad \alpha\sigma > \beta^2\ . \tag{77}$$

When specifying expressions for $a, b, c, \alpha, \beta, \sigma$ we should distinguish between a partly and completely polarized light. In the case of a partly polarized and completely polarized light we have respectively

$$S_0^2 - \mathbf{S}^2 = S_0'^2 - \mathbf{S}'^2 = 0\ , \qquad S_0 = +\mid \mathbf{S} \mid\ ,$$
$$S_0^2 - \mathbf{S}^2 = S_0'^2 - \mathbf{S}^2 > 0, \qquad S_0 >\mid \mathbf{S} \mid\ \ .$$

For the main invariant let us use the notation $S_0^2 - \mathbf{S}^2 = S_0'^2 - \mathbf{S}'^2 = \Sigma^2$.

Expression for a, b, α, β are given by

$$a = (S_0 + S_0')^2 - (\mathbf{S} - \mathbf{S}')^2 = 2\Sigma^2 + 2(S_0 S_0' + \mathbf{S}\mathbf{S}')\ ,$$
$$\frac{b}{A} = (S_0 - S_0')^2 - (\mathbf{S} + \mathbf{S}')^2 = 2\Sigma^2 - 2(S_0 S_0' + \mathbf{S}\mathbf{S}')\ ,$$
$$\alpha = (S_0 - S_0')^2 - (\mathbf{S} + \mathbf{S}')^2 = 2\Sigma^2 - 2(S_0 S_0' + \mathbf{S}\mathbf{S}')\ ,$$
$$\frac{\beta}{B} = (S_0 + S_0')^2 - (\mathbf{S} - \mathbf{S}')^2 = 2\Sigma^2 + 2(S_0 S_0' + \mathbf{S}\mathbf{S}')\ . \tag{78}$$

they become simpler for a completely polarized light

$$a_{polar} = +\ 2(S_0 S_0' + \mathbf{S}\mathbf{S}') > 0\ , \qquad \frac{b_{polar}}{A} = -\ 2(S_0 S_0' + \mathbf{S}\mathbf{S}') < 0\ ,$$
$$\alpha_{polar} = -\ 2(S_0 S_0' + \mathbf{S}\mathbf{S}') < 0\ , \qquad \frac{\beta_{polar}}{B} = +\ 2(S_0 S_0' + \mathbf{S}\mathbf{S}') > 0\ . \tag{79}$$

Let us specify $c = (\mathbf{A}^2 + \mathbf{B}^2 - B^2)\mathbf{A}^2 - A^2 B^2$; accounting for

$$\mathbf{A}^2 + \mathbf{B}^2 - B^2 = (\mathbf{S} + \mathbf{S}')^2 + (\mathbf{S} - \mathbf{S}')^2 - (S_0 - S_0')^2 = -4\Sigma^2 + (S_0 + S_0')^2 \; ,$$
$$\mathbf{A}^2 = (\mathbf{S} + \mathbf{S}')^2 \; , \qquad A^2 B^2 = (S_0 + S_0')^2 (S_0 - S_0')^2$$

we get

$$c = [-4\Sigma^2 + (S_0 + S_0')^2](\mathbf{S} + \mathbf{S}')^2 - (S_0 + S_0')^2 (S_0 - S_0')^2 \; ,$$
$$c_{polar} = 2(S_0 + S_0')^2 \; (S_0 S_0 + \mathbf{S}\mathbf{S}') \; . \qquad (80)$$

In the same mater, for $\sigma = \sigma = (\mathbf{B}^2 + \mathbf{A}^2 - A^2)\mathbf{B}^2 - B^2 A^2$ with relations

$$\mathbf{B}^2 + \mathbf{A}^2 - A^2 = (\mathbf{S} - \mathbf{S}')^2 + (\mathbf{S} + \mathbf{S}')^2 - (S_0 + S_0')^2 = -4\Sigma^2 + (S_0 - S_0')^2 \; ,$$
$$\mathbf{B}^2 = (\mathbf{S} - \mathbf{S}')^2 \; , \qquad B^2 A^2 = (S_0 - S_0')^2 (S_0 + S_0')^2$$

we obtain

$$\sigma = [-4\Sigma^2 + (S_0 - S_0')^2](\mathbf{S} - \mathbf{S}')^2 - (S_0 - S_0')^2 (S_0 + S_0')^2 \; ,$$
$$\sigma_{polar} = -2(S_0 - S_0')^2 \; (S_0 S_0 + \mathbf{S}\mathbf{S}') \; . \qquad (81)$$

7. On the Lorentz little group for a partly polarized light

In the context of polarization optics, some interest may have the known problem of the little Lorentz group. What is the majority of Mueller matrices leaving invariant a given Stokes 4-vector.

The problem is reduced to

$$L_b{}^a(k, \bar{k}^*) \; S_a = +S_b \; , \qquad S^a S_a = \text{inv} > 0 \; ; \qquad (82)$$

with the use of a factorized form of the Lorentz matrix $L = \hat{A}\hat{A}^* = \hat{A}^*\hat{A}$, the previous equations is written as

$$\hat{A} \; S = (\hat{A}^*)^{-1} \; S \qquad \Longrightarrow \qquad [\hat{A} - (\hat{A}^*)^{-1} \;] \; S = 0 \; , \qquad (83)$$

where

$$\hat{A} = \left| \begin{array}{rrrr} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array} \right| \; , \; (\hat{A}^*)^{-1} = \left| \begin{array}{rrrr} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -ik_3^* & ik_2^* \\ k_2^* & ik_3^* & k_0^* & -ik_1^* \\ k_3^* & -ik_2^* & ik_1^* & k_0^* \end{array} \right| .$$

So we arrive at

$$\left|\begin{array}{cccc} (k_0-k_0^*) & -(k_1+k_1^*) & -(k_2+k_2^*) & -(k_3+k_3^*) \\ -(k_1+k_1^*) & (k_0-k_0^*) & -i(k_3-k_3^*) & i(k_2-k_2^*) \\ -(k_2+k_2^*) & i(k_3-k_3^*) & (k_0-k_0^*) & -i(k_1-k_1^*) \\ -(k_3+k_3^*) & -i(k_2-k_2^*) & i(k_1-k_1^*) & (k_0-k_0^*) \end{array}\right| \left|\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\right| = 0 \;, \tag{84}$$

which with notation $k_0 = n_0 + im_0$, $k_j = -in_j + m_j$ reads

$$\left|\begin{array}{cccc} im_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & im_0 & -n_3 & n_2 \\ -m_2 & n_3 & im_0 & -n_1 \\ -m_3 & -n_2 & n_1 & im_0 \end{array}\right| \left|\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\right| = 0 \;. \tag{85}$$

Note that imposing restrictions $m_0 = 0, m_j = 0$, we oftain a more simple equation

$$\left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_3 & n_2 \\ 0 & n_3 & 0 & -n_1 \\ 0 & -n_2 & n_1 & 0 \end{array}\right| \left|\begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\right| = 0 \qquad \Longrightarrow \qquad \mathbf{n} = \frac{\mathbf{S}}{S} \tag{86}$$

which describes a 1-parametric group of 3-rotations $O(\phi, \mathbf{n})$ about the axis $\mathbf{S} = S\mathbf{n}$. In general case, eq. (85) can be presented in the vector form

$$im_0 S_0 - \mathbf{m}\mathbf{S} = 0 \;, \qquad -\mathbf{m}S_0 + im_0\mathbf{S} + \mathbf{n}\times\mathbf{S} = 0 \;. \tag{87}$$

To jave solutions in real variables, we must require $m_0 = 0$. Therefore, an explicit expression for $\mathbf{m}$ is

$$\mathbf{m} = \frac{\mathbf{n}\times\mathbf{S}}{S_0} = \mathbf{n}\times\mathbf{p} \;. \tag{88}$$

Thus, solution for the problem of little Lorentz group is

$$L_b{}^a(k,\bar{k}^*)\; S_a = +S_b \;, \qquad S^a S_a = \text{inv} > 0 \;;$$
$$k_0 = n_0 + i0 \;, \qquad \mathbf{k} = -i\;\mathbf{n} + \mathbf{n}\times\mathbf{p} \;. \tag{89}$$

Explicitly, additional condition for parameters looks

$$k_0^2 - \mathbf{k}^2 = 1 \qquad \Longrightarrow \qquad n_0^2 + \mathbf{n}^2(1-\mathbf{p}^2) + (\mathbf{n}\mathbf{p})^2 = 1 \;. \tag{90}$$

This relationship determines a 3-parametric majority of ¡ueller matrices leaving invariant the polarization vector $S_a = (S_0, S_0 p_i)$ of the partly polarized light. As known, this set of transformations consists of a group isomorphic to $SU(2)$.

8. On the Lorentz little group for a completely polarized light

Analogous problem for a completely polarized light looks much the same

$$L_b{}^a(k,\bar{k}^*)\; S_a = +S_b \;, \qquad S^a S_a = 0 \;;$$

we again have equations

$$im_0 S_0 - \mathbf{m}\mathbf{S} = 0 \;, \qquad -\mathbf{m}S_0 + im_0\mathbf{S} + \mathbf{n}\times\mathbf{S} = 0 \;,$$

in which restriction $m_0 = 0$ must hold. Solution looks as follows

$$\begin{gathered} L_b{}^a(k,\bar{k}^*)\; S_a = +S_b \;, \qquad S^a S_a = 0 \;; \\ k_0 = n_0 + i0 \;, \qquad \mathbf{k} = -i\;\mathbf{n} + \mathbf{n}\times\mathbf{p} \;, \qquad \mathbf{p}^2 = 1 \;. \end{gathered} \tag{91}$$

The difference arises due to the relation $\mathbf{p}^2 = 1$,

$$k_0^2 - \mathbf{k}^2 = 1 \qquad \Longrightarrow \qquad n_0^2 + (\mathbf{np})^2 = 1 \;. \tag{92}$$

This relationship determines a 3-parametric majority of Mueller matrices leaving invariant a given isotropic Stokes 4-vector $S_a = (S_0, S_0 p_i)$, $\mathbf{p}^2 = 1$.

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