Pure States, Mixed States and Hawking Problem in Generalized Quantum Mechanics

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Abstract

This paper is the continuation of a study into the information paradox problem started by the author in his earlier works. As previously, the key instrument is a deformed density matrix in Quantum Mechanics of the Early Universe. It is assumed that the latter represents Quantum Mechanics with Fundamental Length. It is demonstrated that the obtained results agree well with the canonical viewpoint that in the processes involving black holes pure states go to the mixed ones in the assumption that all measurements are performed by the observer in a well-known Quantum Mechanics. Also it is shown that high entropy for Planck's remnants of black holes appearing in the assumption of the Generalized Uncertainty Relations may be explained within the scope of the density matrix entropy introduced by the author previously. It is noted that the suggested paradigm is consistent with the Holographic Principle. Because of this, a conjecture is made about the possibility for obtaining the Generalized Uncertainty Relations from the covariant entropy bound at high energies in the same way as R.Bousso has derived Heisenberg's uncertainty principle for the flat space.

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1 Introduction

The present paper is a continuation of analysis of the Hawking information paradox started by the author in [1]-[3].In these works the above problem has been considered using a new object introduced by the author in [4] – a deformed density matrix in Quantum Mechanics of the Early Universe (at Planck scale). The principal idea is as follows: there is difference between Quantum Mechanics on the conventional scales and that at Planck's. In the first case we have a well-known Quantum Mechanics (QM), and in the second case we consider Quantum Mechanics with Fundamental Length (QMFL). And hence in the second case a density matrix differs from the standard quantum-mechanical one, being a deformation of the latter through the dimensionless quantity $\alpha = l_{min}^2/x^2$, where l_{min} is a minimal length, x is the measurement scale and the variation interval α is finite $0 < \alpha \leq 1/4$ [5]-[8],[4]. The deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition.

In the above-mentioned works an asymmetrical entropy density matrix depending on two values of the parameter α: α_1 and α_2 for the observer and for the observed has been introduced naturally. Introduction of such a matrix is important in two ways: first, it enables demonstration of a possible drastic increase in the entropy density close to the singularity, providing a new approach to solution of the information paradox problem [9]-[11]; and second, it indicates that entropy is a relative notion in a sense that it depends both on the energies at which an observed object is measured and on the fact to what scale, associated with α, belongs the observer.

And in [3] the author holds the viewpoint, similar to that of R.Penrose [12], concerning an increase in entropy close to the singularity: as the entropy density is the same for the origin (Big Bang) and final (singularity of Black Hole or Big Crunch) singularities, there is no information loss at all, i.e. Hawking's problem is solvable positively.

In the present paper it is demonstrated that with the developed formalism the information paradox problem may be solved in principle positively using S.Hawking [9]-[11] approach as well, when information is lost during the processes associated with the horizon of events of a black hole. And with the use of the proposed approaches it is shown that in processes involving black holes the initial pure states are always replaced by the mixed

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ones near the horizon of events of the black hole. As this takes place, the results [9] for thermal radiation of black holes are not used. (Note that these results have been open to question recently(for example [13])). In this case it is assumed that all measurements are made by the observer within QM, i.e. by the observer whose deformation parameter takes the only value $\alpha_1 = 0$ in conformity with the canonical viewpoint [9], as the primary result of [9] has been formulated in a semiclassical approximation corresponding to this value α_1.

In the second section some tentative results are given together with important refinements needed later on. The third section presents the primary result that within the generalized Quantum Mechanics Hawking's problem in a canonical formulation [9]-[11] may be solved positively. In the last section the author attempts at elucidation of the problem put forward by J.Bekenstein [15] in regard to entropy for Planck's remnants of black holes within the proposed paradigm that is adequately consistent with the holographic principle. Just in this paper [15] the problem is stated as follows: provided in the process of the black hole evaporation the masses of the formed remnants are on the order of Planck's mass M_p, these remnants are hardly characterized by high entropy. Whereas in the developed formalism this is possible due to a high entropy density at Planck scales, that is measured with the entropy density matrix $S_{\alpha_2}^{\alpha_1}$ for α_2 close to 1/4. As shown in [4], the relevant matrix element acts as a density of entropy per unit minimum area depending on the scales for the observer and observable. In this way for all the results obtained the principal object under study is the entropy density matrix introduced previously in [1],[3] and considered in Section 3 of the present work. In conclusion, as regards the holographic principle, a conjecture is made about the possibility for derivation of the Generalized Uncertainty Relations from the covariant entropy bound at high energies in the same way as R.Bousso [32] has obtained the Heisenberg uncertainty principle for the flat space.

2 Quantum Mechanics with Fundamental Length and Density Matrix

It should be recalled that in [4]-[8] a new object has been introduced – a deformation of the density matrix $\rho(\alpha)$ in QMFL. Here parameter $\alpha = l_{min}^2/x^2$, where l_{min} , is a minimal length, x is the scale of measurement.

The primary properties of $\rho(\alpha)$ are as follows [4].

Definition

Any system in QMFL is described by a density pro-matrix of the form

$$\rho(\alpha) = \sum_i \omega_\mathbf{i}(\alpha)|\mathbf{i} >< \mathbf{i}|,$$

where

1. $0 < \alpha \leq 1/4$.

2. The vectors $|i >$ form a full orthonormal system;

3. $\omega_i(\alpha) \geq 0$ and for all i the finite limit $\lim\limits_{\alpha \to 0} \omega_i(\alpha) = \omega_i$ exists;

4. $Sp[\rho(\alpha)] = \sum_i \omega_i(\alpha) < 1$, $\sum_i \omega_i = 1$.

5. For every operator B and any α there is a mean operator B depending on α:

$$< B >_\alpha = \sum_i \omega_i(\alpha) < i|B|i > .$$

6. the following condition must be fulfilled:

$$Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] \approx \alpha. \tag{1}$$

7. This suggests limitation for the parameter α:

$$Sp[\rho(\alpha)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}. \tag{2}$$

In the limit $\lim\limits_{\alpha \to 0} \rho(\alpha) = \rho$ a well-known QM density matrix appears.

It should be noted that in [4]-[8] the problems have been considered with reference to the Generalized Uncertainty Relations (GUR) [16]-[21], since based on the latter one arrives to the presence of the fundamental length in a quantum theory [21],[22]. However, there is no necessity to use GUR directly for studies of QMFL as all modern approaches to quantum gravity in some or other way lead to the availability of a minimal length [22]. Besides, at least heuristically, a theory of Big Bang should lead to the

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notion of a minimal length. The reasoning is as follows: since for its real-
ization the Big Bang should possess some maximum energy E_{max} (which
may be theoretically calculated for any specific inflation model), this en-
ergy places an upper limit for the particle momentum and hence some lower
limit for the length l_{min} even on the basis of the conventional Heisenberg's
Uncertainty Relations [23]. Introduction of GUR [16]-[21] just renders this
limit more precise, fixing it at Planck's level $l_{min} \sim l_p$.

3 Entropy Density Matrix, Pure and Mixed States

By the proposed approach, in QMFL there are no pure states in a sense
[4] that the states with a unit probability can appear only in the limit
$\alpha \to 0$, when all $\omega_i(\alpha)$ except one are equal to zero or when they tend to
zero in this limit. In our treatment pure states are the states which can be
represented in the form $|\psi><\psi|$, where $<\psi|\psi>=1$. That is pure states
may appear in the limiting transition from QMFL to QM only, when the
density matrix is $\lim_{\alpha \to 0} \rho(\alpha) = \rho$.

At the same time, for every solution of equation (1) one can describe the
states going in this limit to the pure ones in QM. These states may be
referred to as prototypes of pure states. Specifically, such states have been
used in [4] for the derivation of a well-known Bekenstein-Hawking formula
of the black hole entropy in a semiclassical approximation. In the process
for one of the solutions of equation (1) the author has used **exponential
ansatz**:

$$\rho^*(\alpha) = \sum_i \alpha_i exp(-\alpha)|i><i|, \tag{3}$$

where all $\alpha_i > 0$ are independent of α and their sum is equal to 1. In
this case the prototype of a pure state $\rho = |\psi><\psi|$ in QM is a density
pro-matrix $\rho(\alpha) = exp(-\alpha)|\psi><\psi|$ in QMFL. Similarly, the prototype
of any mixed state in QM with the use of an exponential ansatz is a density
pro-matrix, the general form of which is given by (3).

In [1],[3]the following entropy density matrix has been introduced:

$$S_{\alpha_2}^{\alpha_1} = -Sp[\rho(\alpha_1)\ln(\rho(\alpha_2))] = - <\ln(\rho(\alpha_2))>_{\alpha_1}, \tag{4}$$

where $0 < \alpha_1, \alpha_2 \le 1/4..$

As indicated in [1],[3], $S_{\alpha_2}^{\alpha_1}$ has a clear physical meaning: the entropy den-

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sity is computed on the scale associated with the deformation parameter α_2 by the observer who is at a scale corresponding to the deformation parameter α_1. Also note that the matrix $S^{\alpha_1}_{\alpha_2}$ is always meaningful irrespective of the fact, whether $\rho(\alpha_1)$ and $\rho(\alpha_2)$ form a prototype of pure or mixed state in QM.

Different approaches are taken to the information loss problem and unitarity violation in the black holes. Specifically, R.Penroze in [12] has demonstrated that information in the black hole may be lost and unitarity may be violated because of the singularity. In paper [3] it has been stated that in this case the information loss problem is solved positively. In the process the entropy density matrix $S^{\alpha_1}_{\alpha_2}$ has been primarily used for the situations when $\rho(\alpha_2 = 1/4$ is the case(being associated with the singularity). Also in this section we point to a possibility for solving the problem in principle positively with the use of the canonical Hawking approach [9]-[11], i.e. for **information losses near the horizon of events** of the black hole. And independently of the results presented in [9]-[11], from the developed formalism it follows that the state measured near the horizon of events is always mixed. In [9]-[11] this is established in view of thermal radiation exhibited by the black hole. However, in the last few years the fact of the existence of such a radiation is open to dispute (e.g., see [13]). Therefore, we exclude this fact from our consideration. Using the developed formalism, one is enabled to arrive to this result with the above-mentioned entropy density matrix. Actually, in recent works (e.g., [13],[14]), it has been shown that near the horizon of events the quantum-gravitational effects are considerable. Proceeding from the entropy density matrix, this means that for the matrix element $S^{\alpha_1}_{\alpha_2}$ we always have $\alpha_2 > 0$ as the quantum-gravitational effects are affecting small scales only.

Now we analyze a random matrix element $S^{\alpha_1}_{\alpha_2}$. Obviously, this element may be zero only for the case when $\rho(\alpha_2)$ is a pure state measured in QM, i.e. $\alpha_2 \approx 0$. A partial case of this situation has been considered in [9], where $\alpha_1 = \alpha_2 \approx 0$ and hence $\rho(\alpha_1) = \rho(\alpha_2) = \rho_{in}$ with zero entropy

$$S^0_0 = -Sp[\rho_{in}\ln(\rho_{in})] = 0. \tag{5}$$

Actually, this is the initial state entropy measured in the original statement of the information paradox problem [9]-[11]:

$$S^{in} = S^0_0 = -Sp[\rho_{in}\ln(\rho_{in})] = -Sp[\rho_{pure}\ln(\rho_{pure})] = 0. \tag{6}$$

The question is, what can be measured by an observer at the exit for ρ_{out}, when all measurements are performed in QM, i.e. when we have at hand

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only one deformation parameter $\alpha_1 \approx 0$? Simply this means that in QMFL due to the quantum-gravitational effects at a horizon of the above events in the black hole [13],[14] leads to:

$$S^{out} = S_{\alpha_2}^{\alpha_2} = -Sp[\rho_{out} \ln(\rho_{out})] = -Sp[\rho_{\alpha_2} \ln(\rho_{\alpha_2})] \neq 0 \qquad (7)$$

for some $\alpha_2 > 0$,unapproachable in QM, is conforming to a particular mixed state in QM with the same entropy

$$S^{out} = -Sp[\rho_{out} \ln(\rho_{out})] = -Sp[\rho_{mix} \ln(\rho_{mix})] \neq 0. \qquad (8)$$

It should be emphasized that a mixed state in (8) will be not uniquely defined. Of course, in this statement there is information loss, since

$$\Delta S = S^{out} - S^{in} > 0. \qquad (9)$$

However, as shown in [1],[3], it will be more correct to consider the state close to the origin singularity (or in the early Universe where the quantum-gravitational effects should be also included) as an initial state for which S^{in} is calculated, naturally with certain $\alpha > 0$ and with entropy

$$S^{in} = S_\alpha^\alpha = -Sp[\rho_{in} \ln(\rho_{in})] = -Sp[\rho_\alpha \ln(\rho_\alpha)] > 0. \qquad (10)$$

Again for the observer making measurements in QM and having no access to any $\alpha > 0$ (i.e. having access to $\alpha \approx 0$ only) this is associated with a certain mixed state

$$S^{in} = -Sp[\rho_{in} \ln(\rho_{in})] = -Sp[\rho_{mix} \ln(\rho_{mix})] > 0 \qquad (11)$$

that is also ambiguously determined. In this way the superscattering operator determined in [9]-[11]

$$\$: \rho^{in} \rightarrow \rho^{out} \quad or \quad \$: \rho_{pure} \rightarrow \rho_{mix}$$

in case under consideration will be of the form

$$\$: \rho_{mix} \rightarrow \rho_{mix}.$$

Since in this case $S^{in} > 0$ and $S^{out} > 0$, may be no information loss, then

$$\Delta S = S^{out} - S^{in} = 0. \qquad (12)$$

The following points of particular importance should be taken into consideration:

1) A study of the information paradox problem in the generalized Quantum Mechanics (QMFL) provides extended possibilities for interpretation of the notion of entropy. Indeed, in the classical problem statement [9]-[11] $S^{in} = -Sp[\rho_{in} \ln(\rho_{in})]$ is compared with $S^{out} = -Sp[\rho_{out} \ln(\rho_{out})]$,i.e. within the scope of the above-mentioned entropy density $S^{\alpha_1}_{\alpha_2}$, introduced in [1], [3], two different diagonal elements S^0_0 and S^α_α are compared. However, in the paradigm under consideration one is free to compare S^0_0 to S^0_α or $S^{in}_{in} = -Sp[\rho_{in} \ln(\rho_{in})]$ to $S^{in}_{out} = -Sp[\rho_{in} \ln(\rho_{out})]$.

2)Proceeding from this observation and from the results of [1],[3], we come to the conclusion that the notion of entropy is **relative** within the generalized Quantum Mechanics (QMFL) in a sense that it is dependent on two parameters α_1 and α_2 characterizing the positions of observer and observable, respectively. Certainly, as projected to QM, it becomes absolute since in this case only one parameter $\alpha \approx 0$; is measured.

3) As indicated above, within the scope of QMFL it is obvious that the states close to the origin singularity are always mixed, being associated with the parameter $\alpha > 0$ and hence with nonzero entropy. At the same time, within QM one can have an understanding (at least heuristically), in what way mixed states are generated by the origin singularity. Actually, in the vicinity of the origin singularity, i.e. at Planck's scale (where the quantum-gravitational effects are considerable) the space-time foam is formed [25],[26] that from the quantum-mechanical viewpoint is capable of generating only a mixed state, the components of which are associated with metrics from the space-time foam with certain probabilities arising from the partition function for quantum gravitation [25],[27].

4   Entropy Bounds and Entropy Density

In the last few years Quantum Mechanics of black holes has been studied under the assumption that GUR are valid [21],[24],[28]. As a result of this approach, it is indicated that the evaporation process of a black hole gives a stable remnant with a mass on the order of the Planck's M_p. However, J.Bekenstein in [15] has credited such an approach as problematic, since then the objects with dimensions on the order of the Planck length

$\sim 10^{-33}cm$ should have very great entropy thus making problems in regard to the entropy bounds of the black hole remnants [29].

In connection with this remark of J.Bekenstein [15] the following points should be emphasized:

I. An approach proposed in [1],[3] and in the present paper gives a deeper insight into the cause of high entropy for Planck's black hole remnants, namely: high entropy density that by this approach at Planck scales takes place for every fixed observer including that on a customary scale, i.e. on $\alpha \approx 0$. In [3] using the exponential ansatz (3) it has been demonstrated how this density can increase in the vicinity of the singularities with

$$S_{in} = S_0^0 \approx 0$$

up to

$$S_{out} = S_{\frac{1}{4}}^0 = - < ln[exp(-1/4)]\rho_{pure} >= - < \ln(\rho^*(1/4)) >= \frac{1}{4}.$$

when the initial state measured by the observer is pure.

As demonstrated in [1],[3], increase in the entropy density will be realized also for the observer moving together with the information flow: $S_{out} = S_{\frac{1}{4}}^{\frac{1}{4}} > S_0^0$, though to a lesser extent than in the first case. Obviously, provided the existing solutions for (1) are different from the exponential ansatz (3), the entropy density for them $S_{\alpha_2}^0$ will be increasing as compared to S_0^0 with a tendency of α_2 to 1/4.

II. In works of J.Bekenstein, [29] in particular, a "universal entropy bound" has been used [30]:

$$S \le 2\pi MR/\hbar, \tag{13}$$

where M is the total gravitational mass of the matter and R is the radius of the smallest sphere that barely fits around a system. This bound is, however, valid for a weakly gravitating matter system only. In case of black hole remnants under study it is impossible to assume that on Planck scales we are concerned with a weakly gravitating matter system, as in this case the transition to the Planck's energies is realized where quantum-gravitational effects are appreciable, and within the proposed paradigm parameter $\alpha \approx 0$ is changed by the parameter $\alpha > 0$ or equally QM is changed by QMFL.

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III.This necessitates mentioning of the recent findings of R.Bousso [31],[32], who has derived the Bekenstein's "universal entropy bound" for a weakly gravitating matter system, and among other things in flat space, from the covariant entropy bound [33] associated with the holographic principle of Hooft-Susskind [34],[35],[36].

Also it should be noted that the approach proposed in [3],[4] and in the present paper is consistent with the holographic principle [34]-[36]. Specifically, with the use of this approach one is enabled to obtain the entropy bounds for nonblack hole objects of L.Susskind [35]. Of course, in ([4], section 6) and ([3], section 4) it has been demonstrated, how a well-known semiclassical Bekenstein-Hawking formula for black hole entropy may be obtained using the proposed paradigm. Then we can resort to reasoning from [35]: "using gedanken experiment, take a neutral non-rotating spherical object containing entropy S which fits entirely inside a spherical surface of the area A, and it to collapse to black hole". Whence

$$S \le \frac{A}{4l_p^2}. \tag{14}$$

Note also that the entropy density matrix $S_{\alpha_2}^{\alpha_1}$ by its definition [1],[3] falls into 2D objects, being associated with $l_{min}^2 \sim l_p^2$ [4] and hence implicitly pointing to the holographic principle.

5 Conclusion

Qualitative analysis performed in this work reveals that the Information Loss Problem in black holes with the canonical problem statement [9]-[11] suggests in principle positive solution within the scope of the proposed method – high-energy density matrix deformation. Actually, this problem necessitates further (now quantitative) analysis. Besides, it is interesting to find direct relations between the described methods and the holographic principle. Of particular importance seems a conjecture following from [32]: is it possible to derive GUR for high energies (of strong gravitational field) with the use of the covariant entropy bound [33] in much the same manner as R.Bousso [32] has developed the Heisenberg uncertainty principle for the flat space?

References

[1] A.E.Shalyt-Margolin. Deformed density matrix, Density of entropy and Information problem,[gr-qc/0307096].

[2] A.E.Shalyt-Margolin, Non-Unitary and Unitary Transitions in Generalized Quantum Mechanics and Information Problem Solving, [hep-th/0309121]

[3] A.E.Shalyt-Margolin,Non-Unitary and Unitary Transitions in Generalized Quantum Mechanics, New Small Parameter and Information Problem Solving, *Mod. Phys. Lett. A*, Vol 19, No. 5 (2004) pp.391-403,[hep-th/0311239].

[4] A.E.Shalyt-Margolin and J.G.Suarez,Quantum Mechanics at Planck's scale and Density Matrix,*Intern.Journ.of Mod.Phys.* D.12(2003)1265,[gr-qc/0306081]

[5] A.E.Shalyt-Margolin, Fundamental Length,Deformed Density Matrix and New View on the Black Hole Information Paradox,[gr-qc/0207074]

[6] A.E.Shalyt-Margolin and A.Ya.Tregubovich, Generalized Uncertainty Relations,Fundamental Length and Density Matrix,[gr-qc/0207068]

[7] A.E.Shalyt-Margolin and J.G.Suarez. Density Matrix and Dynamical aspects of Quantum Mechanics with Fundamental Length, [gr-qc/0211083]

[8] A.E.Shalyt-Margolin and J.G.Suarez,Quantum Mechanics of the Early Universe and its Limiting Transition,[gr-qc/0302119]

[9] S.Hawking,Breakdown of Predictability in Gravitational Collapse, Phys.Rev.D14(1976)2460

[10] S.Giddings,The Black Hole Information Paradox,[hep-th/9508151]

[11] A.Strominger, Les Houches Lectures on Black Holes, [hep-th/9501071].

[12] R.Penrose,Quantum Theory and Space - Time, in S.Hawking and R.Penrose ,The Nature of Space and Time, Princeton University Press,Princeton,New Jersey,1994

[13] A.D.Helfer,Do black holes radiate?, *Rept.Prog.Phys.* 66(2003)943,[gr-qc/0304042]

[14] G.'t Hooft,Horizons,[gr-qc/0401027]

[15] J.D.Bekenstein,Black holes and information theory,[quant-ph/0311049].

[16] D. V. Ahluwalia,Quantum Measurement, Gravitation, and Locality, Phys.Lett. B339 (1994)301

[17] D.V.Ahluwalia,Wave-Particle duality at the Planck scale: Freezing of neutrino oscillations, Phys.Lett. A275 (2000)31;Interface of Gravitational and Quantum Realms, Mod.Phys.Lett. A17(2002)1135

[18] M.Maggiore, The algebraic structure of the generalized uncertainty principle,Phys.Lett.B319(1993)83

[19] M.Maggiore,Quantum Groups,Gravity and Generalized Uncertainty Principle, Phys.Rev.D49(1994)5182

[20] S.Capozziello,G.Lambiase and G.Scarpetta, The Generalized Uncertainty Principle from Quantum Geometry, Int.J.Theor.Phys. 39 (2000),15

[21] R.Adler,P.Chen and D.Santiago,The Generalised Uncertainty Principle and Black Hole Remnants,Gen.Rel.Grav.33(2001)2101; M. Cavaglia, S. Das,How classical are TeV-scale black holes? [hep-th/0404050]

[22] L.Garay,Quantum Gravity and Minimum Length Int.J.Mod.Phys.A.A10(1995)145[gr-qc/9403008]

[23] W.Heisenberg,Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Zeitsch.fur Phys,43(1927)172

[24] P.Chen and R.Adler, Black Hole Remnants and Dark Matter,Nucl.Phys.Proc.Suppl.124(2003)103

[25] J.A.Wheeler, Geometrodynamics and the Issue of the Final State, in:*Relativity, Groups and Topology*,p.317, eds.B.S. and C.M. DeWitt.Gordon and Breach,N.Y.,1963

[26] L.Garay,Quantum Evolution in Spacetime Foam Int.J.Mod.Phys.A14(1999)4079

[27] S.Hawking,Spacetime Foam,Nucl.Phys.B114(1978)349

[28] P.S.Custodio,J.E.Horvath,The Generalized Uncertainty Principle,entropy bounds and black hole (non-)evaporation in thermal bath,Class.Quant.Grav.20(2003)L197

[29] J.D.Bekenstein,Entropy bounds and black hole remnants,Phys.Rev. D49 (1994) 1912, [gr-qc/9307035].

[30] J.D.Bekenstein,A universal upper bound on the entropy to energy ratio for bounded systems,Phys.Rev. D23 (1981) 287.

[31] R.Bousso,Light-sheets and Bekenstein's bound,Phys.Rev.Lett. 90 (2003) 121302, [hep-th/0210295].

[32] R.Bousso,Flat space physics from holography,[hep-th/0402058].

[33] R.Bousso,A Covariant Entropy Conjecture,JHEP 9907 (1999) 004, [hep-th/9905177];

[34] G.'t Hooft,Dimension reduction in quantum gravity, [gr-qc/9310026]; The Holographic Principle (Opening Lecture),[hep-th/0003004]

[35] L.Susskind,The world as a hologram,J.of Math. Phys.36 (1995) 6377 [hep-th/9409089];

[36] R.Bousso,The holographic principle,Rev.Mod.Phys. 74 (2002) 825 [hep-th/9905177]