THE EXACT VALUES FOR THE COEFFICIENT OF ASYMPTOTIC EXPANSIONS OF THE FUNCTION'S DEVIATION FROM THEIR POISSON INTEGRAL ON THE HOLDER'S CLASS

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When approximating the functions from class H^1 there appear the asymptotic expansions, whose coefficients are not expressed in explicit form. This problem has been solved by the instrumentality of Riemann zeta-function, which gives us the possibility to find the exact values of Kolmogorov–Nikolskiy constants. The program that allows us to find the constants of given infinitesimal order has been developed.

Harmonic and Biharmonic Poisson Integral as method of Fourier series summation. Let $f(x)$ be 2π-periodic Lebesgue summable function and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

its Fourier Series. Let also $\Lambda = \{\lambda_\rho(k)\}$ – the set of functions of positive integer argument $\rho \in E_\Lambda \subset R$, such that $\lambda_\rho(0) = 1$.

Each function $f(x)$,whose assumption expansion in Fourier series represents with (1), we will associate with the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \lambda_\rho(k)(a_k \cos kx + b_k \sin kx). \tag{2}$$

In case that (2) for every $\rho \in E_\Lambda$ is the Fourier series of the certain continuous function, we will denote it as $U_\rho(f; x; \Lambda)$, and say that the set $\Lambda = \{\lambda_\rho(k)\}$ determines the sequence of operators $U_\rho(f; x; \Lambda)$.

When substituting in (2) the value of coefficients a_k и b_k, we will find that

$$\begin{aligned} U_\rho(f; x; \Lambda) =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} + \sum_{k=1}^{\infty} \lambda_\rho(k) \cos kt \right) dt = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K(\rho; t) dt, \end{aligned} \tag{3}$$

where $K(\rho; t) = \frac{1}{2} + \sum_{k=1}^{\infty} \lambda_\rho(k) \cos kt$ – the kernel of operator $U_\rho(f; x; \Lambda)$.

Assuming that in (3) $\lambda_\rho(k) = \rho^k,\ 0 \leqslant \rho < 1$, we will get

$$U_\rho(f; x; \Lambda) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_1(\rho; t) dt =: P_\rho(f; x), \tag{4}$$

$$K_1(\rho;t)=\frac{1}{2}+\sum_{k=1}^{\infty}\rho^k\cos kt=\frac{1-\rho^2}{2\left(\rho^2-2\rho\cos t+1\right)},$$

and when $\lambda_\rho(k)=\left(1+\frac{k}{2}\left(1-\rho^2\right)\right)\rho^k,\ 0\leqslant\rho<1$, then

$$U_\rho(f;x;\Lambda)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)K_2(\rho;t)dt=:B_\rho(f;x), \tag{5}$$

$$K_2(\rho;t)=\frac{1}{2}+\sum_{k=1}^{\infty}\left(1+\frac{k}{2}\left(1-\rho^2\right)\right)\rho^k\cos kt=\frac{\left(1-\rho^2\right)^2\left(1-\rho\cos t\right)}{2\left(\rho^2-2\rho\cos t+1\right)^2}.$$

The expressions (4) и (5) are referred as Harmonic [1–4] and Biharmonic [5–8] Poisson Integral f respectively.

The approximation of functions of Holder's class. Let C be the space of 2π-periodic functions, in which we define norm as following

$$\|f\|_C=\max_t|f(t)|.$$

The class of functions $f\in C$, for which holds

$$|f(x+h)-f(x)|\leqslant|h|.$$

we will call the Holder's class of order 1. We will denote this class as H^1.

The problem for finding the asymptotic equalities for the value

$$\mathcal{E}(\mathfrak{M};U_\rho(\Lambda))_C=\sup_{f\in\mathfrak{M}}\|f(x)-U_\rho(f;x;\Lambda)\|_C,$$

where $\mathfrak{M}\subseteq C$ – class of functions that is given, $U_\rho(f;x;\Lambda)$ – the operators (3), we will call, according to A.I. Stepanetz [9], Kolmogorov–Nikolskiy problem. In our case we will consider two values: $\mathcal{E}(H^1;P_\rho)_C$ and $\mathcal{E}(H^1;B_\rho)_C$, which correspond to the values of divergence harmonic and biharmonic Poisson Integral from the function at class H^1. In case that the function $g(\rho)=g(\mathfrak{M},\rho)$ such that $\rho\to1-0$

$$\mathcal{E}(\mathfrak{M};U_\rho(\Lambda))_C=g(\rho)+o(g(\rho)),$$

has been found, then we will say, that the Kolmogorov–Nikolskiy problem has been solved for the operator $U_\rho(f;x;\Lambda)$ at class $\mathfrak{M}$ in the metric space C.

The formal expansion $\sum_{n=0}^{\infty}g_n(\rho)$ is called the asymptotic expansion for the function $f(\rho)$ when $\rho\to1-0$, if for every positive integer N

$$f(\rho)=\sum_{n=0}^{N}g_n(\rho)+o(g_N(\rho)),\quad \rho\to1-0,$$

and for all $n \in \mathbb{N}$

$$|g_{n+1}(\rho)| = o\left(|g_n(\rho)|\right).$$

Briefly, we will denote this as following

$$f(\rho) \cong \sum_{n=0}^{\infty} g_n(\rho).$$

The question about the investigation of the approximative properties of Poisson's Integral remain to be urgent for many mathematicians.

For instance, E.L.Stark in the work [2] established, that

$$\mathcal{E}(H^1;P_\rho)_C = \frac{2}{\pi}\sum_{k=1}^{\infty}\left\{\frac{1}{k}(1-\rho)^k \ln\frac{1}{1-\rho} + \beta_k (1-\rho)^k\right\},\ \rho \to 1-0,$$

$$\beta_k = \frac{1}{k}\left\{\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\right\}, k = 1, 2, \ldots.$$

In the works [6] and [7] independent of each other the following result was established:

$$\mathcal{E}(H^1;B_\rho)_C = \frac{2}{\pi}(1-\rho) + \frac{2}{\pi}(1-\rho)^2 \ln\frac{1}{1-\rho} + \frac{1+2\ln 2}{\pi}(1-\rho)^2 +$$

$$+\frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\frac{1}{k}(1-\rho)^k \ln\frac{1}{1-\rho} + \gamma_k (1-\rho)^k\right\},\ \rho \to 1-0,$$

$$\gamma_k = \frac{1}{k}\left(\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\right) - \frac{1}{(k-2)(k-1)2^{k-1}}.$$

The asymptotic expansion for the value $\mathcal{E}(H^1;P_\rho)_C$ in powers of $\frac{1}{\delta}$ $\left(\delta = -\frac{1}{\ln\rho}\right)$ was found by V.A.Baskakov [3]

$$\begin{aligned}\mathcal{E}(H^1;P_\delta)_C \cong \frac{2}{\pi}\left\{\frac{1}{\delta}\ln\delta + \frac{1}{\delta}\left[\ln\pi + \textstyle\int_\pi^\infty \frac{(t)_{2\pi}}{t^2}dt\right]\right\} + \\ +\frac{2}{\pi}\textstyle\sum_{k=1}^{\infty}(-1)^k\left[\int_\pi^\infty \frac{(t)_{2\pi}}{t^{2(k+1)}}dt - \frac{1}{2k\pi^{2k}}\right]\frac{1}{\delta^{2k+1}},\end{aligned} \tag{6}$$

where under the symbol $(t)_{2\pi}$ we mean even 2π-periodic extension of the function $\varphi(t) = t$, $0 \leqslant t \leqslant \pi$.

The asymptotic expansion that is formulated either in the terms of $\frac{1}{\delta}$, and $(1-\rho)$, was established in the works [8] by Amanov.T.I and Falaleev

L.P., namely that $\delta \to \infty$ $(\rho \to 1-)$

$$\begin{gathered}\mathcal{E}(H^1; B_\rho)_C = \\ = \tfrac{1-\rho^2}{\pi}\left\{1 + \sum_{k=1}^{\infty}(-1)^{k+1}\left[2k\int_{\pi}^{\infty}\tfrac{(t)_{2\pi}}{t^{2k+2}}dt - \tfrac{1}{\pi^{2k}}\right]\tfrac{1}{\delta^{2k}}\right\} + \\ + \left(\tfrac{2}{\pi}\tfrac{1}{\delta} - \tfrac{1-\rho^2}{\pi}\right)\left\{\int_{\pi}^{\infty}\tfrac{(t)_{2\pi}}{t^2}dt + \ln\delta + \ln\pi + \right. \\ \left. + \sum_{k=1}^{\infty}(-1)^{k+1}\left(\tfrac{1}{2k\pi^{2k}} - \int_{\pi}^{\infty}\tfrac{(t)_{2\pi}}{t^{2k+2}}dt\right)\tfrac{1}{\delta^{2k}}\right\}.\end{gathered} \tag{7}$$

In the obtained expansions (6) and (7) the coefficients in which appears 2π-periodic extension $\varphi(t) = t$ were unknown yet.

In the current work the expressions for $\int_{\pi}^{\infty}\frac{(t)_{2\pi}}{t^{2(k+1)}}\,dt$ were found, namely we got that:

$$\int_{\pi}^{\infty}\frac{(t)_{2\pi}}{t^{2(k+1)}}\,dt = \sum_{i=1}^{\infty}\int_{(2i-1)\pi}^{2i\pi}\frac{2i\pi - t}{t^{2(k+1)}}\,dt + \sum_{i=1}^{\infty}\int_{2i\pi}^{(2i+1)\pi}\frac{t-2i\pi}{t^{2(k+1)}}\,dt =$$

$$= \sum_{i=1}^{\infty}\left(-\frac{2i\pi}{2k+1}\frac{1}{t^{2k+1}} + \frac{1}{2k}\frac{1}{t^{2k}}\right)\Bigg|_{(2i-1)\pi}^{2i\pi} +$$

$+$

$$\sum_{i=1}^{\infty}\left(-\frac{1}{2k}\frac{1}{t^{2k}} + \frac{2i\pi}{2k+1}\frac{1}{t^{2k+1}}\right)\Bigg|_{2i\pi}^{(2i+1)\pi} =$$

$$\begin{gathered}= \sum_{i=1}^{\infty}\frac{2i}{(2k+1)\,\pi^{2k}}\left[\frac{1}{(2i-1)^{2k+1}} - \frac{2}{(2i)^{2k+1}} + \frac{1}{(2i+1)^{2k+1}}\right] + \\ + \sum_{i=1}^{\infty}\frac{1}{2k\pi^{2k}}\left[\frac{2}{(2i)^{2k}} - \frac{1}{(2i-1)^{2k}} - \frac{1}{(2i+1)^{2k}}\right].\end{gathered} \tag{8}$$

For the further transformation we will use Riemann–Zeta function [10]

$$\zeta(z) = \sum_{s=1}^{\infty}\frac{1}{s^z} \tag{9}$$

and the equality (9.535) [10]

$$\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}} = \zeta(2k) - 2^{-2k}\zeta(2k). \tag{10}$$

Then we get

$$\sum_{i=1}^{\infty}\frac{2i}{2k+1}\left[\frac{1}{(2i-1)^{2k+1}} + \frac{1}{(2i+1)^{2k+1}}\right] =$$

$$= \frac{1}{2k+1}\sum_{i=1}^{\infty}(2i-1+1)\left[\frac{1}{(2i-1)^{2k+1}}+\frac{1}{(2i+1)^{2k+1}}\right]=$$

$$= \frac{1}{2k+1}\sum_{i=1}^{\infty}\left[\frac{1}{(2i-1)^{2k}}+\frac{2i-1}{(2i+1)^{2k+1}}+\frac{1}{(2i-1)^{2k+1}}+\frac{1}{(2i+1)^{2k+1}}\right]=$$

$$= \frac{1}{2k+1}\sum_{i=1}^{\infty}\left[\frac{1}{(2i-1)^{2k}}+\frac{1}{(2i+1)^{2k}}+\frac{1}{(2i-1)^{2k+1}}-\frac{1}{(2i+1)^{2k+1}}\right]=$$

$$= \frac{1}{2k+1}\left(\sum_{i=1}^{\infty}\left[\frac{1}{(2i-1)^{2k}}+\frac{1}{(2i+1)^{2k}}\right]+1\right)=$$

$$= \frac{1}{2k+1}\left(2\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\right)=\frac{2}{2k+1}\left(\zeta(2k)-2^{-2k}\zeta(2k)\right), \qquad (11)$$

$$\frac{1}{2k}\sum_{i=1}^{\infty}\left(\frac{1}{(2i-1)^{2k}}-1\right)=\frac{1}{2k}\left(\zeta(2k)-2^{-2k}\zeta(2k)-1\right). \qquad (12)$$

Using (9), (10) and substituting (11), (12) in (8), we get

$$\int_{\pi}^{\infty}\frac{(t)_{2\pi}}{t^{2(k+1)}}dt=\frac{2}{(2k+1)\,\pi^{2k}}\left(\zeta\,(2k)-2^{-2k}\zeta\,(2k)-\frac{1}{2^{2k}}\zeta\,(2k)\right)+$$

$$+\frac{1}{2k\pi^{2k}}\left(2\cdot\frac{1}{2^{2k}}\zeta\,(2k)-\zeta\,(2k)+2^{-2k}\zeta\,(2k)-\zeta\,(2k)+2^{-2k}\zeta\,(2k)+1\right)=$$

$$=\frac{2}{(2k+1)\,\pi^{2k}}\zeta\,(2k)\left[1-2^{-2k+1}\right]+\frac{2}{2k\pi^{2k}}\zeta\,(2k)\left[2^{-2k+1}-1\right]+\frac{1}{2k\pi^{2k}}=$$

$$=\frac{2}{\pi^{2k}}\zeta\,(2k)\left[1-2^{-2k+1}\right]\left(\frac{1}{2k+1}-\frac{1}{2k}\right)+\frac{1}{2k\pi^{2k}}=$$

$$=\frac{1}{2k\pi^{2k}}-\frac{2\zeta\,(2k)}{\pi^{2k}}\frac{1-2^{-2k+1}}{2k\,(2k+1)}. \qquad (13)$$

On case that $k=0$ we find

$$\int_{\pi}^{\infty}\frac{(t)_{2\pi}}{t^{2}}dt=\sum_{i=1}^{\infty}\int_{(2i-1)\pi}^{2i\pi}\frac{2i\pi-t}{t^{2}}dt+\sum_{i=1}^{\infty}\int_{2i\pi}^{(2i+1)\pi}\frac{t-2i\pi}{t^{2}}dt=$$

$$=\sum_{i=1}^{\infty}\left(-\frac{2i\pi}{t}-\ln t\right)\Bigg|_{(2i-1)\pi}^{2i\pi}+\sum_{i=1}^{\infty}\left(\ln t+\frac{2i\pi}{t}\right)\Bigg|_{2i\pi}^{(2i+1)\pi}=$$

$$= -2 + \sum_{i=1}^{\infty}\left(\frac{2i}{2i-1} + \frac{2i}{2i+1}\right) + \sum_{i=1}^{\infty}\ln\frac{(2i-1)\,(2i-1)}{2i\cdot 2i} =$$

$$= 1 + \sum_{i=1}^{\infty}\ln\left(1 - \frac{1}{(2i)^2}\right) = 1 + \ln\prod_{i=1}^{\infty}\left(1 - \frac{1}{(2i)^2}\right).$$

Considering that the (0.262.2) [10] takes place

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi},$$

we will have

$$\int_{\pi}^{\infty}\frac{(t)_{2\pi}}{t^2}dt = 1 + \ln\frac{2}{\pi}. \tag{14}$$

Using (13) and (14) with (6) and (7) we will have correspondingly

$$\mathcal{E}(H^1;P_\rho)_C \cong \frac{2}{\pi}\left\{\frac{1}{\delta}\ln\delta +]\frac{1}{\delta}\left[1+\ln 2\right]\right\} +$$

$$+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\pi^{2k}}\zeta(2k)\frac{1-2^{-2k+1}}{2k(2k+1)}\frac{1}{\delta^{2k+1}}, \tag{15}$$

$$\mathcal{E}(H^1;B_\rho)_C \cong \frac{1-\rho^2}{\pi}\left\{1 + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi^{2k}}\zeta(2k)\frac{1-2^{-2k+1}}{2k+1}\frac{1}{\delta^{2k}}\right\} +$$

$$+\left(\frac{2}{\pi}d\tfrac{1}{\delta} - \tfrac{1-\rho^2}{\pi}\right)\left\{1 + \ln 2 + \ln\delta + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi^{2k}}\zeta(2k)\frac{1-2^{-2k+1}}{2k+1}\frac{1}{\delta^{2k}}\right\}, \tag{16}$$

where $\zeta(2k)$ – Riemann–Zeta function, that is defined by (9).

In order to have a possibility of finding the values of the coefficients of the extensions for the values $\mathcal{E}(H^1;P_\rho)_C$ и $\mathcal{E}(H^1;B_\rho)_C$ by the powers of $\frac{1}{\delta}$ the application was developed.

The application was made in Visual C# 2008. We give the fragment of the source code of the program's main part below.

```
using System;
using System.Collections.Generic;
using System.Text;
namespace VAApp
{
public static class CMath
{
// alpha
```

```
// infiniteSmall – порядок малости
public static double CalcHarm(double alpha, int
infiniteSmall)
{
double result = 0;
switch (infiniteSmall)
{
case 1:
if (alpha == 1)
result = (2 / Math.PI);
else
result = (1 / Math.Cos(alpha * Math.PI / 2));
break;
case 2:
if (alpha == 1)
result = Math.Log(??)+1;
else
result = 2 / Math.PI * (Math.Log(2 * Math.E /
Math.Pow(Math.PI, 2)) - 1 / ((1 - alpha) *
Math.Pow(Math.PI, 1 - alpha)));
break;
default:
if (alpha ==1)
if (infiniteSmall % 2 == 0)
{
result = 2 / Math.PI * common_calc1(infiniteSmall - 2);
}
else
{
result = -2 / Math.PI * common_calc1(infiniteSmall - 2);
}
else
if (infiniteSmall % 2 == 0)
{
result = 2 / Math.PI * common_calc(alpha, infiniteSmall);
}
else
{
result = -2 / Math.PI * common_calc(alpha,infiniteSmall);
```

```
}
break;
}
return result;
}
public static double CalcBiHarm(double alpha, int
infiniteSmall)
{
double result = 0;
switch (infiniteSmall)
{
case 1:
if (alpha == 1)
result = (4 / Math.PI);
else
result = (1 - 2 * alpha / Math.PI) /
Math.Cos(alpha * Math.PI / 2);
break;
case 2:
if (alpha == 1)
result = (-2 / Math.PI) * (1 + Math.Log(2 * Math.E /
Math.Pow(Math.PI, 2))) + 2 / Math.PI *
Math.Log((Math.Pow(Math.PI, 2) / (2 * Math.E)));
else
result = 4 / Math.PI * common_calc(alpha, 0);
break;
default:
if (infiniteSmall % 2 == 0)
{
if ((infiniteSmall / 2) % 2 == 0)
{
// четные
if (alpha == 1)
{ double lDiv2 = infiniteSmall / 2 ;
// result = (1/Math.PI)*Math.Pow(4*Math.E/infiniteSmall,
infiniteSmall/2)* // (1/Math.Sqrt(Math.PI*infiniteSmall)) +
(1/Math.PI)*summa2p(infiniteSmall) +
(2/Math.PI)*addition2(infiniteSmall);
```

```
result = 2 * Math.Pow(-1, lDiv2) *
Math.Pow((2 * Math.E / lDiv2), lDiv2)
* (1 / Math.Sqrt(2 * Math.PI * lDiv2)) + Math.Log(Math.PI) +
1 + 2 / Math.PI + (1 / Math.PI) * summa2p(infiniteSmall) +
(1 / Math.PI) * summa2p_sec(infiniteSmall);
}
else
{
result = (1 / Math.PI) * summa20p(infiniteSmall, alpha) +
(1 / Math.PI) * summa21p(infiniteSmall, alpha);
}
}
else // нечетные
{
if (alpha == 1)
{
double lDiv2 = infiniteSmall / 2;
// result = (1 / Math.PI) * Math.Pow(4 * Math.E /
infiniteSmall, infiniteSmall / 2) * (1 / Math.Sqrt(Math.PI *
infiniteSmall)) + (1 / Math.PI) * summa2p(infiniteSmall);
result = 2 * Math.Pow(-1, lDiv2) * Math.Pow((2 * Math.E /
lDiv2), lDiv2) * (1 / Math.Sqrt(2 * Math.PI * lDiv2)) +
Math.Log(Math.PI) + 1 + 2 / Math.PI + (1 / Math.PI) *
summa2p(infiniteSmall) + (1 / Math.PI) *
summa2p_sec(infiniteSmall);
}
else
{
result = (1 / Math.PI) * summa20p(infiniteSmall, alpha) +
(1 / Math.PI) * summa21p(infiniteSmall, alpha);
}
}
}
else
{
double lDiv2plus1 = infiniteSmall / 2 + 1;
double comm_expr = Math.Pow(-1, lDiv2plus1) *
Math.Pow((2 * Math.E / lDiv2plus1), lDiv2plus1) *
(1 / Math.Sqrt(2 * Math.PI * lDiv2plus1));
```

```
if (alpha == 1)
{
result = 1 / Math.PI * comm_expr;
}
else
{
result = 1 / Math.Cos(alpha * Math.PI / 2) *
Math.Pow(-1, infiniteSmall / 2 + 1) *
Math.Pow((2 * Math.E / infiniteSmall / 2 + 1),
infiniteSmall / 2 + 1) * (1 / Math.Sqrt(2 * Math.PI *
infiniteSmall / 2 + 1));
}
}
break;
}
return result;
}
private static double zeta(double i)
{
return math_fnc.Math.riemann_zeta(i);
}
private static double summa20np(int n, double alpha)
{
double summ = 0;
for (int k = 0; k <= (n / 4); k++)
{
summ += summa_common20(n, k, alpha);
}
return summ;
}
private static double summa21np(int n, double alpha)
{
double summ = 0;
for (int k = 0; k <= (n / 4); k++)
{
summ += 2 * k * summa_common20(n, k, alpha);
}
return summ;
}
```

}
}
So we obtain a method to calculate according to the given order of infinitesimality and to the values (15) and (16) the Kolmogorov-Nikolskiy constants for asymptotic expansions of the expressions $\mathcal{E}(H^1; P_\rho)_C$ and $\mathcal{E}(H^1; B_\rho)_C$.

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