THE LOWER WEYL SPECTRUM OF A POSITIVE OPERATOR

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Abstract. For the lower Weyl spectrum $\sigma_{\mathrm{w}}^{-}(T) = \bigcap_{0 \le K \in \mathcal{K}(E) \le T} \sigma(T-K)$, where T is a positive operator on a Banach lattice E, the conditions for which the equality $\sigma_{\mathrm{w}}^{-}(T) = \sigma_{\mathrm{w}}^{-}(T^*)$ holds, are established. In particular, it is true if E has order continuous norm. An example of a weakly compact positive operator T on ℓ_∞ such that the spectral radius $r(T) \in \sigma_{\mathrm{w}}^{-}(T) \setminus (\sigma_{\mathrm{f}}(T) \cup \sigma_{\mathrm{w}}^{-}(T^*))$, where $\sigma_{\mathrm{f}}(T)$ is the Fredholm spectrum, is given. The conditions which guarantee the order continuity of the residue T_{-1} of the resolvent $R(.,T)$ of an order continuous operator $T \ge 0$ at $r(T) \notin \sigma_{\mathrm{f}}(T)$, are discussed. For example, it is true if T is o-weakly compact. It follows from the proven results that a Banach lattice E admitting an order continuous operator $T \ge 0$, $r(T) \notin \sigma_{\mathrm{f}}(T)$, can not have the trivial band of order continuous functionals $E_n^{\sim}$ in general. It is obtained that a non-zero order continuous operator $T : E \to F$ can not be approximated in the r-norm by the operators from $E_\sigma^{\sim} \otimes F$, where F is a Banach lattice, $E_\sigma^{\sim}$ is a disjoint complement of the band $E_n^{\sim}$ of E^*.

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1 Introduction and preliminaries.

This paper is a continuation of research which was begun by the author in notes [6, 7] and devoted to special subsets of the spectrum of a positive operator T on a Banach lattice E.

For terminology, notions, and properties on the theory of Banach lattices and operators on them not explained or proved in this note, we refer to [1, 8]; see also [15, 16]. Throughout the note, unless otherwise stated, Banach lattices E and F will be assumed to be infinite dimensional and an operator T from E into F (or into E) will be assumed linear and (norm) bounded. In Sections 3-6 where the spectral properties are considered, spaces will be assumed complex, and in Section 7 spaces will be assumed real. By the term operator, we mean a linear operator.

Let Z be a Banach space, T be a bounded operator on Z. We denote by $N(T)$ and $R(T)$ *the null space* and *the range* of T, respectively. That is,

$$N(T) = \{z \in Z : Tz = 0\}, \;\; R(T) = \{Tz : z \in Z\}.$$

An operator T is said to be *Fredholm* ([1], p. 156; [3], p. 33) if the dimension of the null space $N(T)$ and the dimension of the quotient space $Z/R(T)$ are both finite.

As usual, the spectrum of an operator T on Z will be denoted by $\sigma(T)$. *The Fredholm spectrum* ([1], p. 299; [3], p. 41) of an operator T is the set

$$\sigma_{\mathrm{f}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not a Fredholm operator on } Z\},$$

and *the Weyl spectrum* ([1], p. 312; [3], p. 133, 135) of an operator T is the set

$$\sigma_{\mathrm{w}}(T) \; = \bigcap_{K \in \mathcal{K}(Z)} \sigma(T + K),$$

where $\mathcal{K}(Z)$ is the set of all compact operators on Z.

In the case, when T is a positive operator on a Banach lattice E, *the lower Weyl spectrum* [6] of an operator T is the set

$$\sigma_{\mathrm{w}}^{-}(T) \; = \bigcap_{0 \le K \in \mathcal{K}(E) \le T} \sigma(T - K).$$

Clearly, the inclusions

$$\sigma_{\mathrm{f}}(T) \subseteq \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{w}}^{-}(T) \subseteq \sigma(T) \tag{1}$$

hold. In particular, if E is an infinite dimensional Banach lattice, then $\sigma_{\mathrm{w}}^{-}(T) \neq \emptyset$. In [7] the example of an operator $T \geq 0$ for which all inclusions of (1) are proper, was given.

This paper is devoted to investigate some properties of the lower Weyl spectrum $\sigma_{\mathrm{w}}^{-}(T)$ of a positive operator T on a Banach lattice E and a problems which are related to this.

Before the statement of the main results, recall some definitions and notations in Riesz spaces and Banach lattices which will be used further on. Let E be a (Archimedean) Riesz space. The cone of all positive elements of E is denoted by E^{+}, i.e., $E^{+} = \{x \in E : x \geq 0\}$. The band B of E is called *a projection band* ([8], p. 32) whenever $B \oplus B^{\mathrm{d}} = E$. For a Banach lattice E *the Lorenz seminorm* on E is defined by the formula

$$\|x\|_L = \inf \{\sup_{\alpha} \|z_\alpha\|_E : 0 \leq z_\alpha \uparrow |x|\}. \tag{2}$$

In a real Riesz space E a net x_α is said to be *order convergent* to $x \in E$, $x_\alpha \xrightarrow{o} x$, ([8], p. 30) whenever there exists a net y_α satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$. An operator $T : E \to F$, where E and F are real Riesz spaces, is said to be *a regular operator* ([8], p. 10) whenever it can be written as a difference of two positive operators, and is said to be *order continuous* ([8], p. 42) whenever $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F. The collection of all regular operators and all order continuous operators from E into F will be denoted by $\mathcal{L}_r(E, F)$ and $\mathcal{L}_n(E, F)$, respectively. For an operator $T : E \to F$ we say that its *modulus* $|T|$ exists ([8], p. 9) whenever the supremum $|T| := T \vee (-T)$ exists in the canonical order of the space of all linear maps from E into F. Obviously, if an operator T possesses a modulus, then T is regular. In the case, when E and F are complex Riesz spaces, an operator T from E into F is called regular if its real and imaginary parts are both regular; similarly for an order continuous operator. Every operator $T \in \mathcal{L}_n(E, F)$ is [2] order bounded, that is, mapping order bounded sets of E onto order bounded of F.

If E and F are (real or complex) Banach lattices, then ([1], p. 22) every order bounded operator T from E into F is bounded. Therefore, the inclusions $\mathcal{L}_n(E, F) \subseteq \mathcal{L}(E, F)$ and

$\mathcal{L}_r(E,F) \subseteq \mathcal{L}(E,F)$ hold, where, of course, $\mathcal{L}(E,F)$ is the space of all bounded operators between E and F; in particular, *every positive operator T from E into F is bounded*.

The operator $T : E \to F$ acting from a Banach lattice E to a Banach lattice F is said to be *an o-weakly compact* ([8], p. 310) whenever T maps order bounded subsets of E onto relatively weakly compact subsets of F. Clearly, every weakly compact operator and so every compact operator, is o-weakly compact.

For an operator $T \in \mathcal{L}_r(E,F)$, where E and F are real Banach lattices, its *r-norm* ([9]; [15], p. 27) is defined by

$$\|T\|_r = \inf\{\|S\| : 0 \le S \in \mathcal{L}(E,F),\ \ |Tx| \le S|x|,\ x \in E\}.$$

Under the r-norm the space $\mathcal{L}_r(E,F)$ is a Banach space. The inequality $\|T\| \le \|T\|_r$ is valid. If $|T|$ exists its r-norm is $\|T\|_r = \||T|\|$. An operator $T \in \mathcal{L}_r(E,F)$ is called *r-compact* [9] if it can be approximated in the r-norm by an operators of finite-rank. Every r-compact operator T possesses the modulus $|T|$ and $|T|$ is r-compact [9]. Thus, the space $\mathcal{K}_r(E,F)$ of all r-compact operators is a Banach lattice under the r-norm and the ordering induced by the canonical order of $\mathcal{L}(E,F)$.

For a (real or complex) Riesz space E we put $E^\sim = \mathcal{L}_r(E,\mathbb{R})$ and $E^\sim_n = \mathcal{L}_n(E,\mathbb{R})$. Through $(E^\sim_n)^\circ$ will be denoted the polar of the band $E^\sim_n$ with respect to the dual system $\langle E, E^*\rangle$, that is,

$$(E^\sim_n)^\circ = \{x \in E : x^*x = 0 \text{ for all } x^* \in E^\sim_n\}.$$

The band of all functionals in the Riesz space $E^\sim$ that are disjoint from the band $E^\sim_n$ will be denoted by $E^\sim_\sigma$.

A Banach lattice E has *order continuous norm* ([1], §2.3) if $E^\sim_n = E^*$; equivalently, E is an ideal of E^{**}. The Banach lattice E is called *a KB-space* ([8], p. 225-226) if $(E^*)^\sim_n = E$; equivalently, E is a band of E^{**}.

We now state for convenience the following result [7] which has a specific importance below.

Theorem 1. *Let T be a positive operator on a Banach lattice E such that the spectral radius $r(T) \notin \sigma_{\mathrm{f}}(T)$ and there exists a net of a compact operators K_α satisfying $0 \le K_\alpha x \uparrow Tx$ for all $x \ge 0$. If T is order continuous and the order continuous dual $E^\sim_n$ separates the points of E, then $r(T) \notin \sigma^-_{\mathrm{w}}(T)$.*

2 The statement of the main results.

It is well known that for an operator T on a Banach space Z the equality $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*)$ holds, where T^* is the adjoint of T. In Section 3 the question when an analogue holds for the lower Weyl spectrum $\sigma^-_{\mathrm{w}}(T)$, will be discussed. The main result of this section is the next theorem.

Theorem 2. *Each of the following conditions ensures that for a positive operator T on a Banach lattice E the equality $\sigma^-_{\mathrm{w}}(T) = \sigma^-_{\mathrm{w}}(T^*)$ holds*:

(a) *The equality $\sigma_{\mathrm{w}}(T) = \sigma^-_{\mathrm{w}}(T)$ is valid* (*in particular, $\sigma_{\mathrm{w}}(T) = \sigma(T)$*);

(b) *The equality $\sigma^-_{\mathrm{w}}(T) = \sigma^-_{\mathrm{w}}(T^{**})$ is valid*;

(c) *The Banach lattice E has order continuous norm*;

(d) *The operator T is order continuous and there exists a Banach lattice F such that $E = F^*$ and $F = E^\sim_n$, moreover $T^*(E^\sim_\sigma) \subseteq E^\sim_\sigma$.*

The equality $\sigma_{\rm w}^-(T) = \sigma_{\rm w}^-(T^*)$ does not hold for an arbitrary Banach lattice E. Namely, Section 4 will be devoted the construction of *a weakly compact positive operator T on the space ℓ_∞ for which $\sigma_{\rm w}^-(T) \neq \sigma_{\rm w}^-(T^*)$* (see, in particular, Example 14 below).

In Section 5, using this example, we will show that *in Theorem* 1 *the assumption about the order continuity of an operator T is essential* (see Example 15).

The assumption in Theorem 1 that $E_n^\sim$ separates E, will also be discussed. How important it is, is closely connected with the search conditions which guarantee the order continuity of the residue T_{-1} of the resolvent $R(.,T)$ of an order continuous positive operator T at the point $r(T)$. This is the aim of Section 6. The main result of this section is the next theorem.

Theorem 3. *Let T be a positive order continuous operator on a Banach lattice E, moreover $r(T) \notin \sigma_{\rm f}(T)$. Then each of the following conditions ensures that the residue T_{-1} of the resolvent $R(.,T)$ at $r(T)$ is order continuous*:

(a) *The operator T is o-weakly compact*;

(b) *The band $(E_n^\sim)^\circ$ is a projection band and Lorenz seminorm* (2) *on $(E_n^\sim)^\circ$ is a norm.*

Moreover, it turns out that the problem of the order continuity of the residue T_{-1} of $R(.,T)$ at $r(T)$ leads to a study of compact order continuous operators on spaces with trivial order continuous dual, and is connected with the approximation problem. This is a treatment of Section 7. We give here the basic result of it.

Theorem 4. *Let E and F be two Banach lattices, and let $T : E \to F$ be a non-zero order continuous operator. Then $T \notin \overline{E_\sigma^\sim \otimes F}$, where the closure in $\mathcal{L}_r(E,F)$ with the r-norm.*

In particular, if $E_n^\sim = \{0\}$, then T is not r-compact.

3 When does the equality $\sigma_{\rm w}^-(T) = \sigma_{\rm w}^-(T^*)$ hold?

First note that the inclusion $\sigma_{\rm w}^-(T^*) \subseteq \sigma_{\rm w}^-(T)$ always holds. Indeed, if $\lambda \notin \sigma_{\rm w}^-(T)$, then $\lambda \notin \sigma(T-K)$ for some $K \in \mathcal{K}(E)$, $0 \le K \le T$. So $\lambda \notin \sigma(T^* - K^*)$ and $0 \le K^* \le T^*$, that is, $\lambda \notin \sigma_{\rm w}^-(T^*)$. Below the conditions when the equality $\sigma_{\rm w}^-(T) = \sigma_{\rm w}^-(T^*)$ holds, will be proved (see Theorem 2 above).

For a Banach space Z, j_Z will denote the natural embedding $j_Z : Z \to Z^{**}$. We shall identity $j_Z(Z)$ with the space Z without any further explanations. When we do so, the identification will be clear from the context. The following lemma is known. We include here a short proof for the sake of completeness, and because the construction of the required operator is important later on.

Lemma 5. *Let Z be a Banach space and an operator $T \in \mathcal{L}(Z^*)$. The following assertions are equivalent*:

(a) *The subspace Z of Z^{**} is T^*-invariant*;

(b) *There exists a unique operator $S \in \mathcal{L}(Z)$ such that $S^* = T$*;

(c) *The operator T is $\sigma(Z^*, Z)$-continuous.*

In particular, $S \in \mathcal{K}(Z)$ iff $T \in \mathcal{K}(Z^*)$. If Z is a Banach lattice, then $S \ge 0$ iff $T \ge 0$.

Proof. (a) $\Rightarrow$ (b) For an arbitrary element $y \in Z$ there exists a unique element $x \in Z$ such that

$$j_Z(x) = T^*(j_Z(y)). \tag{3}$$

Put $Sy = x$. Fix $z \in Z$ and $z^* \in Z^*$. The relations

$$(S^*z^*)z = z^*(Sz) = j_Z(Sz)z^* = (T^*(j_Z(z)))z^* = j_Z(z)Tz^* = (Tz^*)z$$

hold, hence $S^* = T$.

(b) $\Rightarrow$ **(c)** If a net $z^*_\alpha \overset{\sigma(Z^*,Z)}{\longrightarrow} 0$, then we have $(Tz^*_\alpha)z = z^*_\alpha(Sz) \to 0$ for an arbitrary $z \in Z$, so $Tz^*_\alpha \overset{\sigma(Z^*,Z)}{\longrightarrow} 0$.

(c) $\Rightarrow$ **(a)** Fix $z \in Z$. Let $z^*_\alpha \overset{\sigma(Z^*,Z)}{\longrightarrow} 0$. From the relations $(T^*j_Z(z))z^*_\alpha = (Tz^*_\alpha)z \to 0$, it follows that the functional $T^*j_Z(z)$ is $\sigma(Z^*, Z)$-continuous, whence $T^*j_Z(z) \in Z$.

The last assertions follow at once from the equality $S^* = T$. $\square$

Lemma 6. *Let a Banach lattice E be the direct sum of projection bands B_i, that is, the equality $E = \bigoplus_{i=1}^{n} B_i$ holds. If T is a positive operator on E such that B_i is T-invariant for all $i = 1, ..., n$, then $\sigma^-_{\mathrm{w}}(T) = \bigcup_{i=1}^{n} \sigma^-_{\mathrm{w}}(T_i)$, where T_i is the restriction of T to B_i.*

Proof. If $\lambda \notin \sigma^-_{\mathrm{w}}(T_i)$ for all i, then there exist an operators $K_i \in \mathcal{K}(B_i)$ satisfying the relations $0 \le K_i \le T_i$ and $\lambda \notin \sigma(T_i - K_i)$. The operator $K = \bigoplus_{i=1}^{n} K_i$ is compact, $0 \le K \le T$ and $\lambda \notin \sigma(T - K) = \bigcup_{i=1}^{n} \sigma(T_i - K_i)$, so $\lambda \notin \sigma^-_{\mathrm{w}}(T)$.

For the converse, if $\lambda \notin \sigma^-_{\mathrm{w}}(T)$, then $\lambda \notin \sigma(T - K)$, where $K \in \mathcal{K}(E)$ and $0 \le K \le T$. Bands B_i are K-invariant therefore, K has a representation $K = \bigoplus_{i=1}^{n} K_i$ with $K_i \in \mathcal{K}(B_i)$ and $0 \le K_i \le T_i$. Clearly, $\lambda \notin \sigma(T_i - K_i)$ for all i, whence $\lambda \notin \bigcup_{i=1}^{n} \sigma^-_{\mathrm{w}}(T_i)$. $\square$

We proceed now to the proof of Theorem 2 (see Section 2) which collects the necessary conditions guaranteeing the validity of the equality $\sigma^-_{\mathrm{w}}(T) = \sigma^-_{\mathrm{w}}(T^*)$.

Proof of Theorem 2. (a) The desired equality follows from the relations

$$\sigma^-_{\mathrm{w}}(T^*) \subseteq \sigma^-_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*) \subseteq \sigma^-_{\mathrm{w}}(T^*).$$

(b) Sufficiently to observe that

$$\sigma^-_{\mathrm{w}}(T^{**}) \subseteq \sigma^-_{\mathrm{w}}(T^*) \subseteq \sigma^-_{\mathrm{w}}(T) = \sigma^-_{\mathrm{w}}(T^{**}).$$

(c) Let $\lambda \notin \sigma^-_{\mathrm{w}}(T^*)$. There exists a compact operator K on E^* such that $0 \le K \le T^*$ and $\lambda \notin \sigma(T^* - K)$. Clearly, $0 \le K^* \le T^{**}$ and E is K^*-invariant as E is a T^{**}-invariant ideal of E^{**}. By Lemma 5, there exists a compact operator S on E satisfying $0 \le S \le T$ and $S^* = K$. Finally, the operator $\lambda - (T - S)$ is invertible, that is, $\lambda \notin \sigma^-_{\mathrm{w}}(T)$ thus, the inclusion $\sigma^-_{\mathrm{w}}(T) \subseteq \sigma^-_{\mathrm{w}}(T^*)$ is valid, as required.

(d) First of all we remark that $F = (F^*)^\sim_n$. In particular, F has order continuous norm. From the order continuity of the operator T, we have $T^*(F) \subseteq F$. Define an operator T' as the restriction of T^* to $E^\sim_n = j_F(F)$. Since $E^\sim_\sigma$ is T^*-invariant, then Lemma 6 implies

$$\sigma^-_{\mathrm{w}}(T') \subseteq \sigma^-_{\mathrm{w}}(T^*). \tag{4}$$

By Lemma 5, there exists a positive operator S on F satisfying the following equalities $S^* = T$ and $j_F(Sx) = T^* j_F(x)$ for all $x \in F$ (see (3)). So the restriction T' to F coincides with S, it follows $\sigma_{\mathrm{w}}^-(S) = \sigma_{\mathrm{w}}^-(T')$. Using the assertion **(c)** and (4), we have

$$\sigma_{\mathrm{w}}^-(T) = \sigma_{\mathrm{w}}^-(S^*) = \sigma_{\mathrm{w}}^-(S) = \sigma_{\mathrm{w}}^-(T') \subseteq \sigma_{\mathrm{w}}^-(T^*) \subseteq \sigma_{\mathrm{w}}^-(T),$$

hence $\sigma_{\mathrm{w}}^-(T) = \sigma_{\mathrm{w}}^-(T^*)$. □

In the next section an example of an operator T such that $\sigma_{\mathrm{w}}^-(T) \neq \sigma_{\mathrm{w}}^-(T^*)$, will be given (Example 14).

The condition (b) of the previous theorem implies the necessity of the study of the connection between $\sigma_{\mathrm{w}}^-(T)$ and $\sigma_{\mathrm{w}}^-(T^{**})$. Recall that for an operator $T \in \mathcal{L}(Z)$, where Z is a Banach space, $\rho_\infty(T)$ denotes the unbounded component in $\mathbb{C}$ of the resolvent set $\rho(T)$ of T.

Theorem 7. *Let T be a positive operator on a Banach lattice E having order continuous norm. Then*:

(a) $r(T) \in \sigma_{\mathrm{w}}^-(T)$ *implies* $r(T) \in \sigma_{\mathrm{w}}^-(T^{**})$;

(b) *If for every $K \in \mathcal{K}(E^{**})$, $0 \leq K \leq T^{**}$, the equality*

$$\rho(T^{**} - K) = \rho_\infty(T^{**} - K) \tag{5}$$

holds, then $\sigma_{\mathrm{w}}^-(T) = \sigma_{\mathrm{w}}^-(T^{**})$.

Proof. **(a)** If $r(T) \notin \sigma_{\mathrm{w}}^-(T^{**})$, then for some $K \in \mathcal{K}(E^{**})$, $0 \leq K \leq T^{**}$, the operator $r(T) - (T^{**} - K)$ is invertible. Therefore, $r(T) > r(T^{**} - K) \geq r(T - K|_E)$, where $K|_E$ is the restriction of K to E, that is, $r(T) \notin \sigma_{\mathrm{w}}^-(T)$.

(b) If $\lambda \notin \sigma_{\mathrm{w}}^-(T^{**})$, then for some $K \in \mathcal{K}(E^{**})$, $0 \leq K \leq T^{**}$, we have ([1], p. 256)

$$\lambda \in \rho(T^{**} - K) = \rho_\infty(T^{**} - K) \subseteq \rho(T - K|_E),$$

whence $\lambda \notin \sigma_{\mathrm{w}}^-(T)$. □

The equality (5) holds for every bounded operator T on a Banach space Z with the spectrum of T is at most countable. Namely, in this case

$$\rho(T + S) = \rho_\infty(T + S) \tag{6}$$

for every $S \in \mathcal{I}(Z)$, where $\mathcal{I}(Z)$ is the set of all *inessential* operators on Z ([3], §7.1; in particular, p. 379), that is, $\mathcal{I}(Z)$ is a collection of an operators on Z defined by

$$\{S \in \mathcal{L}(Z) : T + S \text{ is Fredholm operator on Z whenever } T \text{ is Fredholm operator on Z}\}.$$

The inclusion ([1], p. 162; [3], p. 371) $\mathcal{K}(Z) \subseteq \mathcal{I}(Z)$ is valid. For the proof of (6) it is enough to show that for every operator $S \in \mathcal{I}(Z)$ the spectrum of $T + S$ is at most countable. Indeed, in this case there is a path joining an arbitrary point of $\rho(T + S)$ with a point of the circle $\{\lambda : |\lambda| = r(T+S)\}$ and lying inside of $\rho(T+S)$, whence the equality $\rho(T+S) = \rho_\infty(T+S)$ follows. So fix $\lambda \in \sigma(T + S) \setminus \sigma(T)$. The equality $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{f}}(T + S)$ implies $\lambda \notin \sigma_{\mathrm{f}}(T + S)$. There is a path lying outside $\sigma_{\mathrm{f}}(T+S)$ and joining λ with some point $\xi \in \rho(T+S)$, hence ([1], p. 300) λ is an isolated point of $\sigma(T+S)$ and so of $\sigma(T+S)\setminus\sigma(T)$. Therefore, $\sigma(T+S)\setminus\sigma(T)$ and so $\sigma(T + S)$ is at most countable.

In the case $T \in \mathcal{L}_n(E)$, where E is a Banach lattice, the band $E_n^\sim$ is T^*-invariant. Denote the restriction of T^* to $E_n^\sim$ by T'. The proof of the following assertion is analogous to the part **(b)** of Theorem 7: *If $\rho(T^* - K) = \rho_\infty(T^* - K)$ for every $0 \le K \in \mathcal{K}(E^*) \le T^*$, then $\sigma_{\mathrm{w}}^-(T') \subseteq \sigma_{\mathrm{w}}^-(T^*)$.*

The proof of the following assertion is quite similar to that of the part **(a)** of Theorem 7: *If $0 \le T \in \mathcal{L}(E)$, a closed ideal A of E is T-invariant, then the inclusion $r(T) \in \sigma_{\mathrm{w}}^-(T|_A)$, where $T|_A$ is the restriction of T to A, implies $r(T) \in \sigma_{\mathrm{w}}^-(T)$.*

We close this section with few remarks dealing with *Lozanovsky's spectrum* [6]

$$\sigma_1(T) = \bigcap_{\substack{0 \le Q \le T \\ Q \le K \in \mathcal{K}(E)}} \sigma(T - Q)$$

of a positive operator T on a Banach lattice E. The conditions when $\sigma_{\mathrm{w}}(T) \subseteq \sigma_1(T)$ holds, are given in [7]. Again $\sigma_1(T^*) \subseteq \sigma_1(T)$ is valid. The following theorem is similar to Theorem 2.

Theorem 8. *Each of the following conditions ensures that for a positive operator T on a Banach lattice E the equality $\sigma_1(T) = \sigma_1(T^*)$ holds*:

(a) *The equality $\sigma_1(T) = \sigma_1(T^{**})$ is valid*;

(b) *E and E^* are atomic with order continuous norms*;

(c) *E is a KB-space.*

Proof. **(a)** The proof is analogous to the proof of the part **(b)** of Theorem 2.

(b) The inequalities $0 \le Q \le K$, where K is a compact operator either on E or on E^*, imply [17] the compactness of Q, hence $\sigma_1(T) = \sigma_{\mathrm{w}}^-(T)$ and $\sigma_1(T^*) = \sigma_{\mathrm{w}}^-(T^*)$. By the part **(c)** of Theorem 2, we have $\sigma_{\mathrm{w}}^-(T) = \sigma_{\mathrm{w}}^-(T^*)$, so $\sigma_1(T) = \sigma_1(T^*)$.

(c) The Banach lattice E is a band of E^{**} as E is a KB-space. Define the real positive order projection from E^{**} onto E by P_E. Let $\lambda \notin \sigma_1(T^*)$. Then the operator $\lambda - (T^* - Q)$ is invertible, where $0 \le Q \le T^*$, $Q \le K \in \mathcal{K}(E^*)$. The space $E = j_E(E)$ is Q^*-invariant. By Lemma 5, there exists operator Q_0 on E satisfying $Q_0^* = Q$,

$$0 \le Q_0 \le T \tag{7}$$

and $j_E(Q_0 y) = Q^*(j_E(y))$ for all $y \in E$. The space E is also $P_E K^*$-invariant. There exists $K_0 \in \mathcal{K}(E)$ such that $j_E(K_0 y) = P_E K^*(j_E(y))$ for all $y \in E$. Then for $x \in E^+$ we have

$$j_E(Q_0 x) = Q^*(j_E(x)) = P_E Q^*(j_E(x)) \le P_E K^*(j_E(x)) = j_E(K_0 x),$$

hence

$$Q_0 \le K_0 \in \mathcal{K}(E). \tag{8}$$

Thus, according to the invertibility of the operator $\lambda - (T - Q_0)$ and the relations (7) and (8), we have $\lambda \notin \sigma_1(T)$, so $\sigma_1(T) = \sigma_1(T^*)$. $\square$

In the proof of the part **(c)** of the previous theorem the existence of (positive) projection from E^{**} onto the ideal E was only used. In fact, this implies [14] that E is a KB-space. Remark also that E^* has order continuous norm iff E^* is a KB-space. In particular, if E^{**} is a Banach lattice with an order continuous norm then E is a KB-space (see [8], p. 225).

It is not known if the equality $\sigma_1(T) = \sigma_1(T^*)$ holds for an arbitrary positive operator T on a Banach lattice E.

4 An example of an operator T such that $\sigma_{\mathrm{w}}^{-}(T) \neq \sigma_{\mathrm{w}}^{-}(T^*)$.

If a Banach lattice E has order continuous norm, then $\sigma_{\mathrm{w}}^{-}(T) = \sigma_{\mathrm{w}}^{-}(T^*)$ (Theorem 2, **(c)**). The main example of a Banach lattice which does not have order continuous norm, is the space ℓ_∞ of all bounded sequences with the sup norm. Below the example of a weakly compact positive operator T on the space ℓ_∞ such that $\sigma_{\mathrm{w}}^{-}(T) \neq \sigma_{\mathrm{w}}^{-}(T^*)$, will be obtained.

First of all we recall some definitions and results about ℓ_∞. The space ℓ_∞ is an AM-space with a unit $e = (1, 1, \ldots)$, so by Kakutani-Bohnenblust-M.Krein-S.Krein theorem ([8], p. 194), ℓ_∞ is lattice isometric onto a space $C(K)$ of all continuous functions on some Hausdorff compact topological space K; in fact, K is homeomorphic to the Stone-Čech compactification $\beta\mathbb{N}$ of the set of natural numbers $\mathbb{N}$. The Banach lattice ℓ_∞^* can be identified with the direct sum of ℓ_1 and $\ell_\infty^s = \{x^* \in \ell_\infty^* : x^*(c_0) = \{0\}\}$, where c_0 is the space of all sequences converging to zero, by this $(\ell_\infty)_n^\sim = \ell_1$ and $(\ell_\infty)_\sigma^\sim = \ell_\infty^s$, in particular, $\ell_1 \perp \ell_\infty^s$.

It is easily to see that an operator T on ℓ_∞ is norm bounded iff T has a representation $Tx = (x_1^* x, x_2^* x, \ldots)$, where $x_n^* \in \ell_\infty^*$ and $\sup\limits_n \|x_n^*\| < \infty$. In this case the order continuity of T is equivalent to the condition $x_n^* \in \ell_1$ for all n. The following result gives the conditions of the compactness of an operator T.

Lemma 9. *Let $Tx = (x_1^* x, x_2^* x, \ldots)$ be a bounded operator on ℓ_∞, $x_n^* \in \ell_\infty^*$. Then*:
(a) *An operator T is compact iff the set $\{x_n^*\}_{n=1}^\infty$ is relatively norm compact in ℓ_∞^**;
(b) *An operator T is weakly compact iff the set $\{x_n^*\}_{n=1}^\infty$ is relatively weakly compact in ℓ_∞^*.*
Proof. (a) Necessity. The set TU, where U is the closed unit ball of ℓ_∞, is relatively norm compact. Therefore ([11], p. 260), for an arbitrary $\epsilon > 0$ there exist a disjoint partition of the set of natural numbers $\mathbb{N} = \bigcup\limits_{i=1}^m N_i$ and elements $n_i \in N_i$ such that $\sup\limits_{n \in N_i} |(Tx)_{n_i} - (Tx)_n| \leq \epsilon$ for all $x \in U$ and $i = 1, \ldots, m$. Whence $\sup\limits_{n \in N_i} |x_{n_i}^* x - x_n^* x| \leq \epsilon$, so $\sup\limits_{n \in N_i} \|x_{n_i}^* - x_n^*\| \leq \epsilon$ for $i = 1, \ldots, m$, that is, $\{x_n^*\}_{n=1}^\infty \subseteq \bigcup\limits_{i=1}^m B(x_{n_i}^*, \epsilon)$, where $B(x_{n_i}^*, \epsilon)$ is the closed ball centered at $x_{n_i}^*$ with radius ϵ. The last inclusion means that the set $\{x_n^*\}_{n=1}^\infty$ is totally bounded, so is relatively norm compact.

The sufficiency contains in [5] (the proof of Theorem 2). For the sake of completeness we include the proof. Let the sequence $x_k \in \ell_\infty$, $\|x_k\| \leq M$, $M > 0$. By passing to a subsequence if needed, we can assume that $\lim\limits_{k \to \infty} x_n^* x_k = z_n$ for all n. The sequence Tx_k converges to the element $z = (z_1,\ z_2,\ \ldots) \in \ell_\infty$ in the norm. In fact, assuming by way of contradiction and passing to one more subsequence if necessary, we can find a subsequence n_k of $\mathbb{N}$ such that

$$|x_{n_k}^* x_k - z_{n_k}| \geq \epsilon_1 > 0 \tag{9}$$

for all k, $x_{n_k}^* \to x_0^*$ in the norm of ℓ_∞^* and $\lim\limits_{k \to \infty} z_{n_k} = z_0$. Then

$$\lim_{k \to \infty} x_0^* x_k = z_0. \tag{10}$$

Indeed, fix $\epsilon_2 > 0$ and choose k_1 such that $\|x_0^* - x_{n_{k_1}}^*\| \leq \frac{\epsilon_2}{3M}$, $|z_0 - z_{n_{k_1}}| \leq \frac{\epsilon_2}{3}$. There exists k_2 such that $|x_{n_{k_1}}^* x_k - z_{n_{k_1}}| \leq \frac{\epsilon_2}{3}$ for each $k \geq k_2$. So $|x_0^* x_k - z_0| \leq \epsilon_2$ for $k \geq k_2$, that is, (10) holds. Finally, $\lim\limits_{k \to \infty} (x_{n_k}^* x_k - z_{n_k}) = 0$, contrary to (9).

(b) The operator T on ℓ_∞ is weakly compact iff ([8], p. 335, 347) T is a Dunford-Pettis operator, that is, T carries weakly convergent sequences onto norm convergent sequences. Let $x_k \overset{\sigma(\ell_\infty,\ell_\infty^*)}{\longrightarrow} 0$. The sequence $Tx_k \to 0$ in the norm of ℓ_∞ iff $\lim\limits_{k\to\infty} x_n^* x_k = 0$ uniformly for n. By Grothendieck theorem ([16], p. 126), the letter is equivalent to the assertion that the set $\{x_n^*\}_{n=1}^\infty$ is relatively weakly compact. $\square$

Lemma 10. *Let $Tx = (x_1^* x, x_2^* x, \ldots)$ be a positive operator on ℓ_∞, moreover $\|T\| \le 1$, $x_n^* \in \ell_\infty^s$ and $\lim\limits_{n\to\infty} \|x_n^*\| = 1$. Then $\|T^k\| = 1$ for all k; in particular, $r(T) = 1$.*

Proof. From the equalities $\|T\| = \|Te\|$ and $\|x_n^*\| = x_n^* e$, we get $1 \ge \|T\| = \sup\limits_n \|x_n^*\| \ge 1$, hence $\|T\| = 1$. So $\|T^k\| \le 1$ for all k, that is,

$$0 \le T^k e \le e. \tag{11}$$

Next, if $x = (x_1, x_2, \ldots) \in \ell_\infty$, $0 \le x \le e$ and $\lim\limits_{n\to\infty} x_n = 1$, then $\lim\limits_{n\to\infty} (Tx)_n = 1$. Actually, we have

$$(Tx)_n = x_n^* x = x_n^* e - x_n^*(e - x) = \|x_n^*\| \to 1$$

as $e - x \in c_0$. From the last relations, the elementary induction and the inequalities (11), it is easy to see that $\lim\limits_{n\to\infty} (T^k e)_n = 1$ for all k. Hence, $\|T^k\| = 1$. Now by Gelfand formula ([1], p. 243), the equality $r(T) = 1$ is obvious. $\square$

On the other hand, the space ℓ_∞^* is an AL-space, so by Kakutani-Bohnenblust-Nakano theorem ([8], p. 192), there exists a lattice isometry $\Phi_{\ell_\infty^*}$ from ℓ_∞^* onto the space of all integrable functions $L_1(\Omega_\infty, \mu_\infty)$, moreover the measure μ_∞ is not σ-finite and is not purely atomic (if the functional $x^* \in \ell_\infty^*$ is a generalized limit, then [4] the restriction of μ_∞ to the support of the function $\Phi_{\ell_\infty^*} x^*$ is a non-atomic measure). By this, the band $\Phi_{\ell_\infty^*} \ell_1$ is a L_1-space associated with an atomic measure.

To continue our discussion, we need the following construction. Let (Ω, Σ, μ) be an arbitrary non-atomic probability measure space. Define the function $r_0 \equiv 1$. There exists disjoint measurable sets A_{11} and A_{12} such that $\Omega = A_{11} \cup A_{12}$ and $\mu(A_{11}) = \mu(A_{12}) = \frac{1}{2}$. Put $r_1 = \chi_{A_{12}} - \chi_{A_{11}}$. Sets A_{ni}, $1 \le i \le 2^n$, and the sequence r_n, $n \in \mathbb{N}$, will be constructed by induction. Assume that sets A_{ni}, $1 \le i \le 2^n$, with the properties

$$A_{ni} \cap A_{nj} = \emptyset, \ \ i \ne j, \ \ \Omega = \bigcup_{i=1}^{2^n} A_{ni}, \ \ \mu(A_{ni}) = \frac{1}{2^n}$$

and the function $r_n = \sum\limits_{i=1}^{2^n} (-1)^i \chi_{A_{ni}}$ have been constructed. Next, there exist sets $A_{n+1,i}$, $1 \le i \le 2^{n+1}$, such that

$$A_{n+1,i} \cap A_{n+1,j} = \emptyset, \ \ i \ne j, \ \ \Omega = \bigcup_{i=1}^{2^{n+1}} A_{n+1,i}, \ \ \mu(A_{n+1,i}) = \frac{1}{2^{n+1}},$$

moreover if i is odd, then $A_{n+1,i} \cup A_{n+1,i+1} = A_{n,\frac{i+1}{2}}$. Now the function r_{n+1} is defined by $r_{n+1} = \sum\limits_{i=1}^{2^{n+1}} (-1)^i \chi_{A_{n+1,i}}$. Any sequence constructed as above is called *a sequence of*

Rademacher functions (see [1], p. 496-497). Clearly, $|r_n| = 1$. The relation $r_n \overset{\sigma(L_1,L_\infty)}{\longrightarrow} 0$ is valid.

Lemma 11. *Let* $B_n = \bigcup_{i=1}^{2^{n-1}} A_{n,2i-1}$*, where sets* A_{ni} *were defined above. Then for every subsequence* B_{n_i} *the equality* $\mu(\bigcup_{i=1}^{m} B_{n_i}) = 1 - \frac{1}{2^m}$ *holds for all* m*. In particular,* $\mu(\bigcup_{i=1}^{\infty} B_{n_i}) = 1$*.*

Proof. It suffices to establish that the set $B_{n_1} \cup \ldots \cup B_{n_m}$ can be written as a union of $2^{n_m} - 2^{n_m - m}$ (different) sets of the form $A_{n_m,i}$. To verify this, we use induction on m. For $m = 1$ the set B_{n_1} consists of $2^{n_1-1} = 2^{n_1} - 2^{n_1-1}$ sets of the form $A_{n_1,i}$. For the induction step, suppose that the statement is true for some m. Then for every k the set $B_{n_1} \cup \ldots \cup B_{n_m}$ consists of $2^{n_m+k} - 2^{n_m-m+k}$ sets of the form $A_{n_m+k,i}$ and the set $\Omega \setminus (B_{n_1} \cup \ldots \cup B_{n_m})$ consists of 2^{n_m-m} sets of the form $A_{n_m,i}$. Therefore, the set $B_{n_m+k} \setminus (B_{n_1} \cup \ldots \cup B_{n_m})$ can be written as a union of $2^{n_m-m+k-1}$ sets of the form $A_{n_m+k,i}$, so the set $B_{n_1} \cup \ldots \cup B_{n_m} \cup B_{n_m+k}$ consists of $2^{n_m+k} - 2^{n_m-m+k} + 2^{n_m-m+k-1}$ sets of the form $A_{n_m+k,i}$. Taking $k = n_{m+1} - n_m$, we obtain that the set $B_{n_1} \cup \ldots \cup B_{n_{m+1}}$ can be written as a union of

$$2^{n_{m+1}} - 2^{n_{m+1}-m} + 2^{n_{m+1}-m-1} = 2^{n_{m+1}} - 2^{n_{m+1}-(m+1)}$$

sets of the form $A_{n_{m+1},i}$, as desired. □

If E is an ideal of the space of measurable functions $L_0(\nu)$, then for an arbitrary measurable set A the order projection on the space E defined by $P_A x = \chi_A x$, $x \in E$.

Lemma 12. *Let* E *be a Banach function space associated with* $L_0(\nu)$*. If the sequence* $z_n \in E^+$*, moreover* $z_n \to z \neq 0$ *in* E*, then there exist a subsequence* z_{n_k}*, a set* A *with* $\nu(A) > 0$ *and a number* $a > 0$ *such that* $z_{n_k} \geq a\chi_A$ *for all* k*.*

Proof. There exist a set B, $\nu(B) > 0$, and a number $a > 0$ satisfying $P_B z \geq a\chi_B$. Indeed, sets $B_i = \{s : z(s) \geq \frac{1}{i}\}$ have the property $\bigcup_{i=1}^{\infty} B_i = \{s : z(s) > 0\}$, hence $\nu(B_{i_0}) > 0$ for some i_0 as $z \neq 0$. Putting $B = B_{i_0}$, $a = \frac{1}{2i_0}$, we obtain $P_B z \geq 2a\chi_B$. The sequence z_n has ([1], p. 195) a subsequence z_{n_k} which is relatively uniformly convergent to z, that is, there exists a positive function $u \in L_0(\nu)$ such that for each $\epsilon > 0$ the inequality $|z_{n_k} - z| \leq \epsilon u$ holds for all $k \geq k_\epsilon$. For some $A \subseteq B$, $\nu(A) > 0$, the function $P_A u$ is bounded. Pick $\epsilon_1 > 0$ such that the inequality $\epsilon_1 P_A u \leq a\chi_A$ is valid. Then

$$\epsilon_1 P_A u \geq |P_A z_{n_k} - P_A z| \geq P_A z - P_A z_{n_k}$$

for $k \geq k_{\epsilon_1}$, whence

$$z_{n_k} \geq P_A z_{n_k} \geq 2a\chi_A - \epsilon_1 P_A u \geq 2a\chi_A - a\chi_A \geq a\chi_A,$$

as claimed. □

Lemma 13. *Let* r_n *be a sequence of Rademacher functions on a space* Ω *with a non-atomic probability measure* μ *defined above, and let* E *be a Banach function space associated with* $L_0(\mu)$*. If a sequence* $z_n \in E$ *such that the set* $\{z_n\}_{n=1}^{\infty}$ *is relatively norm compact in* E *and the inequalities* $r_0 + r_n \geq z_n \geq 0$ *hold in* $L_0(\mu)$ *for every* $n \in \mathbb{N}$*, then* $z_n \to 0$ *in* E*.*

Proof. The sequence of Rademacher functions r_n is equal $r_n = \sum_{i=1}^{2^n} (-1)^i \chi_{A_{ni}}$, where sets A_{ni} are defined above, $B_n = \bigcup_{i=1}^{2^{n-1}} A_{n,2i-1}$. Obviously, $0 \le P_{B_n} z_n \le P_{B_n}(r_0 + r_n) = 0$ hold in $L_0(\mu)$ thus,

$$P_{B_n} z_n = 0 \tag{12}$$

for all n. Assume that some subsequence z_{n_k} of z_n satisfies $z_{n_k} \to z \neq 0$ in E. By Lemma 12, we can find a measurable set A, $\mu(A) > 0$, and a number $a > 0$ such that the inequality $z_{n_k} \ge a\chi_A$ holds for all k. From (12), we get $P_{B_{n_k}} \chi_A = \chi_{A \cap B_{n_k}} = 0$, that is, $\mu(A \cap B_{n_k}) = 0$. This gives $\mu(A \cap \bigcup_{k=1}^{\infty} B_{n_k}) = 0$. By virtue of Lemma 11, we have $\mu(A) = 0$, a contradiction. So $z = 0$. Therefore, the zero function is the only accumulation point of the set $\{z_n\}_{n=1}^{\infty}$ in E, whence $z_n \to 0$ in E. $\square$

Now we are ready to give an example of an operator T such that $\sigma_{\mathrm{w}}^{-}(T) \neq \sigma_{\mathrm{w}}^{-}(T^*)$.

Example 14 (*a weakly compact positive operator T on ℓ_∞ satisfying $\sigma_{\mathrm{w}}^{-}(T) \neq \sigma_{\mathrm{w}}^{-}(T^*)$*).

Step 1. The construction of T.

Let $\Phi_{\ell_\infty^*}$ be a lattice isometry from ℓ_∞^* onto $L_1(\Omega_\infty, \mu_\infty)$. Fix an arbitrary measurable set A, $\mu_\infty(A) > 0$, such that the restriction μ_A of the measure μ_∞ to A is the non-atomic measure. We can assume that $\mu_\infty(A) = 1$. Put $x_0^* = \Phi_{\ell_\infty^*}^{-1} \chi_A$. Consider an arbitrary Rademacher sequence r_n, $n \in \mathbb{N}$, supported by A with the measure μ_A. In the case of the necessity, we can consider the sequence r_n as a sequence in $L_1(\Omega_\infty, \mu_\infty)$. Let $x_n^* = x_0^* + \Phi_{\ell_\infty^*}^{-1} r_n$. Clearly, $x_n^* \in \ell_\infty^s$ for $n \ge 0$. The relation $r_n \overset{\sigma(L_1, L_1^*)}{\longrightarrow} 0$ implies $x_n^* \overset{\sigma(\ell_\infty^*, \ell_\infty^{**})}{\longrightarrow} x_0^*$. Consider the operator T on ℓ_∞ defined by $Tx = (x_1^* x, x_2^* x, \ldots)$. Then $\|T\| = 1$ and $T \ge 0$ as $\|x_n^*\| = 1$ and $x_n^* \ge 0$ for all n. By Lemma 9, **(b)**, the operator T is weakly compact. Furthermore, the relation $T(c_0) = \{0\}$ implies

$$T^*(\ell_\infty^*) \subseteq \ell_\infty^s. \tag{13}$$

Step 2. The equalities

$$\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{w}}(T) = \{0\}, \quad \sigma(T) = \{0, 1\} \tag{14}$$

are valid.

Since T is weakly compact, by Dunford-Pettis theorem ([8], p. 337), the operator T^2 is compact, so T is a Riesz operator, i.e., $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{w}}(T) = \{0\}$. In particular, every non-zero point $\lambda \in \sigma(T)$ is an eigenvalue of T. The equality $Te = e$ implies $\{0, 1\} \subseteq \sigma(T)$. Fix a non-zero $\lambda \in \sigma(T)$. From the above, it follows that $Tx = \lambda x$ for some $x \neq 0$. The equalities $x_0^* x = \lim_{n\to\infty} x_n^* x = \lambda \lim_{n\to\infty} x_n$ give

$$\lim_{n\to\infty} x_n = \frac{1}{\lambda} x_0^* x. \tag{15}$$

So $\lambda x_n = x_n^* x = \frac{1}{\lambda} x_0^* x$, hence $x_n = \frac{1}{\lambda^2} x_0^* x$ for all n. In particular, $x_0^* x \neq 0$. Using (15) once more, we have $\frac{1}{\lambda} x_0^* x = \frac{1}{\lambda^2} x_0^* x$ therefore, $\lambda = 1$. Consequently, $\sigma(T) = \{0, 1\}$. Remark also that we have established the following equality for the null space of the operator $I - T$

$$N(I - T) = \{ae : a \in \mathbb{C}\}. \tag{16}$$

The proof of the equality

$$N(I - T^*) = \{ax_0^* : a \in \mathbb{C}\} \tag{17}$$

is similar.

Step 3. Let K be a compact operator on ℓ_∞ presenting in the form $Kx = (z_1^*x, z_2^*x, \ldots)$, $z_n^* \in \ell_\infty^*$, and satisfying the inequalities $0 \le K \le T$. Then $z_n^* \to 0$ in ℓ_∞^*.

Indeed, the inequalities $0 \le K \le T$ imply $0 \le z_n^* \le x_n^*$, On the other hand, according to Lemma 9, **(a)**, the set $\{z_n^*\}_{n=1}^\infty$ is relatively norm compact in ℓ_∞^*, hence $0 \le \Phi_{\ell_\infty^*} z_n^* \le \chi_A + r_n$ and $\{\Phi_{\ell_\infty^*} z_n^*\}_{n=1}^\infty$ is relatively norm compact in $L_1(\mu_A)$. Using Lemma 13, we conclude that $\Phi_{\ell_\infty^*} z_n^* \to 0$ in $L_1(\mu_A)$ thus, $z_n^* \to 0$ in ℓ_∞^*.

Step 4. The equality

$$\sigma_{\mathrm{w}}^-(T) = \{0, 1\} \tag{18}$$

is valid.

The inclusions $\sigma_{\mathrm{f}}(T) \subseteq \sigma_{\mathrm{w}}^-(T) \subseteq \sigma(T)$ and Step 2 give $\{0\} \subseteq \sigma_{\mathrm{w}}^-(T) \subseteq \{0, 1\}$. We will show that $1 \in \sigma_{\mathrm{w}}^-(T)$. In fact, let $0 \le K \le T$ hold, where $K \in \mathcal{K}(\ell_\infty)$, $Kx = (z_1^*x, z_2^*x, \ldots)$, $z_n^* \in \ell_\infty^*$. The relations $\|T - K\| \le \|T\| = 1$ are valid and functionals $x_n^* - z_n^*$ belong to ℓ_∞^s. Using Step 3, we have $z_n^* \to 0$ in ℓ_∞^* therefore, $\|x_n^* - z_n^*\| = (x_n^* - z_n^*)e = 1 - z_n^*e \to 1$ as $n \to \infty$. This implies, via Lemma 10, that $r(T-K) = 1$, so $1 \in \sigma(T-K)$. Hence, $1 \in \sigma_{\mathrm{w}}^-(T)$.

Step 5. The spectral radius $r(T) = 1$ is a simply pole of the resolvent $R(., T)$, moreover for the residue T_{-1} of $R(., T)$ at the point $\lambda = 1$ the equality $T_{-1} = x_0^* \otimes e$ holds.

According to Step 2, we have $\sigma_{\mathrm{f}}(T) = \{0\}$. This guarantees that the point $\lambda = 1$ is a pole of $R(., T)$ and T_{-1} is a finite-rank operator. For every element $x \in \ell_\infty$ the equalities $T^2x = T(x_1^*x, x_2^*x, \ldots) = (x_0^*x)e$ hold as $\lim\limits_{n\to\infty} x_n^*x = x_0^*x$. Hence, if $(I - T)^2x = 0$, then $x - 2Tx + T^2x = 0$ or $x_n - 2x_n^*x + x_0^*x = 0$ for all n. Therefore, $\lim\limits_{n\to\infty} x_n = x_0^*x$, so $x_n^*x = x_0^*x$. This implies $x_n = x_0^*x$. We get $N((I - T)^2) = \{ae : a \in \mathbb{C}\}$. Now a glance at (16) yields $N(I - T) = N((I - T)^2)$, so $\alpha(I - T) = 1$, where $\alpha(I - T)$ is the ascent of $I - T$. It follows ([1], p. 80, 267) that the point $\lambda = 1$ is a simply pole of $R(., T)$. Using (16), (17) and the equalities ([1], p. 266, 268) $R(T_{-1}) = N(I - T)$ and $R(T_{-1}^*) = N(I - T^*)$, we obtain a representation of the operator T_{-1} in the form $T_{-1} = x_0^* \otimes e$, as desired.

Step 6. Let x_0^{**} be a positive functional on ℓ_∞^{**} satisfying the relations $x_0^{**}(\ell_1) = \{0\}$ and $\|x_0^{**}\| \le 1$. Then for the operator $K = x_0^{**} \otimes x_0^*$ the inequalities $0 \le K \le T^*$ hold.

Indeed, if $0 \le x^* \in \ell_1$, then $Kx^* = 0 \le T^*x^*$. Let $0 \le x^* \in \ell_\infty^s$. The equality

$$T^*x^* = \|x^*\|x_0^* \tag{19}$$

is valid, so $Kx^* = (x_0^{**}x^*)x_0^* \le \|x^*\|x_0^* = T^*x^*$.

Step 7. Let x_0^{**} be a positive functional on ℓ_∞^{**} satisfying the relations $x_0^{**}(\ell_1) = \{0\}$ and $\|x_0^{**}\| \le 1$, moreover $x_0^{**}x_0^* = 1$. If $K = x_0^{**} \otimes x_0^*$, then $r(T^* - K) = 0$.

Assume by way of contradiction that $\lambda = r(T^* - K) > 0$. The number λ is an eigenvalue of the operator $T^* - K$ as $\sigma_{\mathrm{f}}(T^* - K) = \sigma_{\mathrm{f}}(T) = \{0\}$. So

$$T^*x^* - Kx^* = \lambda x^* \tag{20}$$

for some positive functional $x^* \in \ell_\infty^*$, $\|x^*\| = 1$. The equality $T_{-1}^*T^* = T_{-1}^*$ yields

$$\lambda T_{-1}^*x^* = T_{-1}^*T^*x^* - T_{-1}^*Kx^* = T_{-1}^*x^* - T_{-1}^*Kx^*.$$

Using Step 5, we have

$$(1-\lambda)x_0^* = (1-\lambda)T_{-1}^* x^* = T_{-1}^* K x^* = ((Kx^*)e)x_0^* = (x_0^{**}x^*)x_0^*.$$

Hence,

$$\lambda = 1 - x_0^{**}x^*. \tag{21}$$

According to (13) and (20), we have $x^* \in \ell_\infty^s$. It follows from (19) that

$$\lambda x^* = \|x^*\|x_0^* - Kx^* = \|x^*\|x_0^* - (x_0^{**}x^*)x_0^* = (1 - x_0^{**}x^*)x_0^*.$$

The equality (21) implies $x^* = x_0^*$. So $\lambda = 1 - x_0^{**}x^* = 1 - x_0^{**}x_0^* = 0$, which is impossible. Therefore, $r(T^* - K) = 0$.

Step 8. The equality $\sigma_{\mathrm{w}}^-(T^*) = \{0\}$ is valid.

There exists a positive functional x_0^{**} satisfying all condition of Step 7. In fact, consider the functional x^* on $L_1(\Omega_\infty, \mu_\infty)$ defined by $x^*x = \int\limits_A x\, d\mu_\infty$. Then the functional $x_0^{**} = \Phi_{\ell_\infty^*}^* x^*$ satisfies the desired conditions. Using Steps 6 and 7, we get the relations $0 \le K \le T^*$ and $r(T^* - K) = 0$, where $K = x_0^{**} \otimes x_0^*$, hence $1 \notin \sigma_{\mathrm{w}}^-(T^*)$. On the other hand,

$$\{0\} = \sigma_{\mathrm{w}}(T^*) \subseteq \sigma_{\mathrm{w}}^-(T^*) \subseteq \sigma_{\mathrm{w}}^-(T) = \{0, 1\}.$$

Thus, $\sigma_{\mathrm{w}}^-(T^*) = \{0\}$.

Finally, according to Steps 4 and 8 the relation $\sigma_{\mathrm{w}}^-(T) \neq \sigma_{\mathrm{w}}^-(T^*)$ holds. □

Nevertheless, remark that for the operator T from the previous example the equalities $\sigma_{\mathrm{l}}(T) = \sigma_{\mathrm{l}}(T^*) = \{0\}$ satisfy. Indeed, it suffices to observe that the operator T is dominated by the rank-one operator $2x_0^* \otimes e$ (we use the notations from Example 14).

It is easy to see that the operator T from Example 14 is not order continuous. In general, if T is an arbitrary bounded operator on ℓ_∞, then the order continuity of T is equivalent to the fact that the subspace ℓ_1 of ℓ_∞^* is T^*-invariant. If ℓ_∞^s is also T^*-invariant (equivalently, c_0 is T-invariant) and T is positive, then, using Theorem 2, **(d)**, we have $\sigma_{\mathrm{w}}^-(T) = \sigma_{\mathrm{w}}^-(T^*)$. *It is not known if the equality $\sigma_{\mathrm{w}}^-(T) = \sigma_{\mathrm{w}}^-(T^*)$ holds for an arbitrary positive order continuous operator T on a Banach lattice E.*

5 The order continuity is important in Theorem 1!

The objective of this section is to show the essentiality of the assumption about the order continuity of the operator T in Theorem 1 (see Section 1).

Example 15. We will use the notations and results from Example 14. The positive operator $Tx = (x_1^*x, x_2^*x, \dots)$ acts on ℓ_∞. The order continuous dual $(\ell_\infty)_n^\sim = \ell_1$ separates ℓ_∞. According to (14) and (18), we have $r(T) = 1 \notin \sigma_{\mathrm{f}}(T)$ and $1 \in \sigma_{\mathrm{w}}^-(T)$. Obviously, the sequence $K_n x = (x_1^*x, \dots, x_n^*x, 0, 0, \dots)$ of positive compact operators satisfies the property $K_n \uparrow T$ (such sequence exists for each positive operator on a Banach function space associated with a σ-finite atomic measure). □

It is not known if the assumption in Theorem 1 that $E_n^\sim$ separates E, is essential.

The main tool of the proof of Theorem 1 (see [7], the proof of Theorem 20) is the following theorem about the Frobenius normal form [7].

Let B be a projection band of E. Through P_B will be denoted the order projection onto B, i.e., if $x = x_1 + x_2$, where $x_1 \in B$, $x_2 \in B^{\mathrm{d}}$, then $P_B x = x_1$. Put $T_B = P_B T P_B$ and denote the restriction T_B to B by $\widetilde{T}_B$. Recall that in a Dedekind complete Riesz space every band is a projection band ([8], p. 33).

Theorem 16. *Let E be a Dedekind complete Banach lattice such that $E_n^{\sim}$ separates the points of E. Let T be a positive order continuous operator on E, moreover $r(T) \notin \sigma_{\mathrm{f}}(T)$. Then there exist T-invariant bands B_i, $E = B_n \supset B_{n-1} \supset \ldots \supset B_0 = \{0\}$, such that if the equality $r(T_{B_i \cap B_{i-1}^{\mathrm{d}}}) = r(T)$ holds for some $i = 1, ..., n$, then the operator $\widetilde{T}_{B_i \cap B_{i-1}^{\mathrm{d}}}$ is band irreducible.*

In fact (see [7], Section 2.2), the assertion that $E_n^{\sim}$ separates E, is only necessary as the condition which guarantees the order continuity of the residue T_{-1} of $R(.,T)$ at $r(T)$ (see Theorem 3 above). Thus, the question arises naturally: *Does the order continuity of the operator T imply the order continuity of the residue T_{-1} in the general case*? The affirmative answer will main the validity of Theorem 16 and so Theorem 1 by the additional assumption of the Dedekind completeness without the condition that $E_n^{\sim}$ separates E.

In the following section we will discuss this question in detail.

6 The order continuity of the residue.

The main task of this section is to discuss the conditions of the order continuity of the residue T_{-1} of the resolvent $R(.,T)$ at $r(T)$.

Recall that if G is some set of linear functionals on Y and $H \subseteq Z$, where Y and Z are vector spaces, then

$$G \otimes H = \{\sum_{i=1}^{k} y_i' \otimes z_i \ : \ y_i' \in G, \ z_i \in H, \ i = 1, ..., k, \ k \in \mathbb{N}\}.$$

In particular, if Y and Z are Banach spaces, then $Y^* \otimes Z$ is the set of all finite-rank operators from Y into Z.

Let T be a positive operator on some Banach lattice E. If the spectral radius $r(T)$ is a pole of the resolvent $R(.,T)$, then $R(.,T)$ has the Laurent expansion

$$R(\lambda, T) = \frac{1}{(\lambda - r(T))^m} T_{-m} + \ldots + \frac{1}{\lambda - r(T)} T_{-1} + T_0 + (\lambda - r(T)) T_1 + \ldots, \qquad (22)$$

around $r(T)$, where m is the order of the pole of $R(.,T)$ at $r(T)$. Mention that $T_{-m} \geq 0$ and all operators T_i are real. The spectral radius $r(T) \notin \sigma_{\mathrm{f}}(T)$ iff ([1], p. 300-302) $r(T) > 0$, the point $r(T)$ is a pole of $R(.,T)$ and the residue $T_{-1} \in E^* \otimes E$; by this $T_{-i} \in E^* \otimes E$, $i = 1, ..., m$.

Next, if an operator $T \geq 0$ is order continuous, then ([1], p. 256) $R(\lambda, T)$ is also order continuous for each $\lambda > r(T)$. Nevertheless, this fact and the relation (we assume that $r(T)$ is a pole of $R(.,T)$ of the order m)

$$T_{-m} = \lim_{\lambda \downarrow r(T)} (\lambda - r(T))^m R(\lambda, T) \qquad (23)$$

do not imply the order continuity of the operator T_{-m} in general. Actually, there exists [13] the example of a sequence of order continuous positive operators which converge in norm to a positive operator which is not order continuous (even σ-order continuous). Here the following result holds (see [13], Theorem 2.16, where the given fact was established for rather other class than the class of order continuous operators, while in the our case the proof of it is analogous and will be omitted).

Lemma 17. *Let E and F be Banach lattices and suppose that the Lorenz seminorm* (2) *on F is a norm, i.e., for $x \in F$ the equality $\|x\|_L = 0$ implies $x = 0$ (in particular, this is true when $F_n^\sim$ separates F or F is an AM-space with a unit). If a sequence $S_k \in \mathcal{L}_n(E,F)$ converges in $\mathcal{L}(E,F)$ to $S \geq 0$, then $S \in \mathcal{L}_n(E,F)$.*

Therefore, if $0 \leq T \in \mathcal{L}_n(E)$, the Lorenz seminorm on E is a norm and for $R(.,T)$ the expansion (22) holds, then, using (23), we have $T_{-m} \in \mathcal{L}_n(E)$. When E is a Dedekind complete AM-space with a unit, for the proof sufficiently to observe that ([1], p. 96-97) the space $\mathcal{L}_n(E)$ is a band of the Banach lattice $\mathcal{L}(E)$ and so is closed (for the operator norm). In fact, in this case it is easy to see from the equalities

$$T_i = \lim_{\lambda \downarrow r(T)} (\lambda - r(T))^{-i} \left(R(\lambda, T) - \frac{1}{(\lambda - r(T))^m} T_{-m} - \ldots - (\lambda - r(T))^{i-1} T_{i-1} \right), \quad (24)$$

$i \geq -(m-1)$, that the relations $T_i \in \mathcal{L}_n(E)$ hold.

Lemma 18. *Let E and F be two Riesz spaces, moreover $E_n^\sim = \{0\}$. Then the equality $(E^\sim \otimes F) \cap \mathcal{L}_n(E,F) = \{0\}$ is valid.*

Proof. An operator $K \in (E^\sim \otimes F) \cap \mathcal{L}_n(E,F)$ has a representation $K = \sum_{i=1}^{k} x_i^* \otimes x_i$ with $x_i^* \in E^\sim$ and elements $x_1, \ldots, x_k$ in F linearly independent. Let a net $z_\alpha \downarrow 0$ in E. The net $x_i^* z_\alpha$ converges to some number a_i, $i = 1, ..., k$, as every functional $x^* \in E^\sim$ has the decomposition $x^* = (x^*)^+ - (x^*)^-$. Then $K z_\alpha \xrightarrow{o} \sum_{i=1}^{k} a_i x_i$. Taking into account the order continuity of K, the last relation yields $\sum_{i=1}^{k} a_i x_i = 0$. Hence, $a_i = 0$. Thus, x_i^* are order continuous. Therefore, $x_i^* = 0$ for all i. So $K = 0$, as required. □

The following result which at once follows from the previous lemma, gives a necessary condition of the order continuity of the residue.

Theorem 19. *Let T be a positive operator on a Banach lattice E, $r(T) \notin \sigma_{\mathrm{f}}(T)$. If there is at least one an order continuous operator among of the operators $T_{-m}, \ldots, T_{-1}$ in the expansion* (22)*, then $E_n^\sim \neq \{0\}$.*

We start our discussion on a sufficient conditions of the order continuity of T_{-1} with the next auxiliary results.

Lemma 20. *Let E be a Riesz space, let E_0 be a finite dimensional vector subspace of E, and let Γ be a vector subspace of the space of linear functionals on E separating the points of E. If a net $z_\alpha \in E_0$ and $z_\alpha \xrightarrow{\sigma(E,\Gamma)} 0$, then $z_\alpha \xrightarrow{o} 0$ in E.*

Proof. The collection of restrictions of functionals from Γ to E_0 will be denoted by Γ_0. Clearly, Γ_0 separates E_0, so the topology $\sigma(E_0, \Gamma_0)$ is well defined and it coincides with every Hausdorff linear topology on E_0. In particular, it coincides with the topology generated by the norm $\|z\| = \max_{1 \leq i \leq n} |b_i|$, where $z = \sum_{i=1}^{n} b_i e_i$ and $e_1, \ldots, e_n$ is a basis of E_0. Then

$\|z_\alpha\| \to 0$ or $\max\limits_{1\le i\le n} |b_{\alpha i}| \to 0$, where $z_\alpha = \sum\limits_{i=1}^{n} b_{\alpha i} e_i$. Therefore, $\max\limits_{\beta\ge\alpha} \|z_\beta\| \downarrow_\alpha 0$, hence $|z_\alpha| \le (\max\limits_{\beta\ge\alpha} \|z_\beta\|) \sum\limits_{i=1}^{n} |e_i| \downarrow_\alpha 0$, that is, $z_\alpha \xrightarrow{o} 0$. □

The following lemma is similar to Lemma 17.

Lemma 21. *Let E and F be Banach lattices, moreover $F_n^\sim$ separates F. If a sequence $S_k \in \mathcal{L}_n(E,F)$ converges in $\mathcal{L}(E,F)$ to $S \in E^* \otimes F$, then $S \in \mathcal{L}_n(E,F)$.*

Proof. Let a non-zero functional $x^* \in F_n^\sim$. By Lemma 20, it is enough to show that if a net $x_\alpha \xrightarrow{o} 0$ in E, then $x^*(Sx_\alpha) \to 0$. There exists a net $z_\alpha \downarrow 0$ satisfying $|x_\alpha| \le z_\alpha$. Fix $\epsilon > 0$ and an index α_0. Pick k_0 with $\|S - S_{k_0}\| \le \frac{\epsilon}{2\|x^*\|\|z_{\alpha_0}\|}$ (we can assume $z_{\alpha_0} \ne 0$) and $\alpha_1 \ge \alpha_0$ with $|x^*(S_{k_0}x_\alpha)| \le \frac{\epsilon}{2}$ for $\alpha \ge \alpha_1$. Then

$$|x^*(Sx_\alpha)| \le |x^*((S - S_{k_0})x_\alpha)| + |x^*(S_{k_0}x_\alpha)| \le \frac{\epsilon}{2} + |x^*(S_{k_0}x_\alpha)| \le \epsilon$$

for $\alpha \ge \alpha_1$, as claimed. □

Lemma 22. *If a net z_α in a Banach lattice E is relatively weakly compact set and $z_\alpha \xrightarrow{o} 0$, then $z_\alpha \xrightarrow{\sigma(E,E^*)} 0$.*

Proof. Pick a net y_α with $|z_\alpha| \le y_\alpha \downarrow 0$. Let z be a weak cluster point of the set $\{z_\alpha\}$, that is ([11], p. 29), for each $\sigma(E,E^*)$-neighbourhood U of the point z and each α_0 there exists $\alpha \ge \alpha_0$ such that $z_\alpha \in U$. Fix β. For $\alpha \ge \beta$ we have $z - y_\beta \le z - y_\alpha \le z - z_\alpha$. For an arbitrary $x^* \in (E^*)^+$ and $\epsilon > 0$ pick $\alpha' \ge \beta$ with $x^*(z - z_{\alpha'}) \le \epsilon$. Hence $x^*(z - y_\beta) \le \epsilon$, so $z \le y_\beta$ for all β. Consequently, $z \le 0$. Analogously, $-z \le 0$. Finally, $z = 0$. We obtain that the zero is only a weak cluster point of $\{z_\alpha\}$.

Fix $x^* \in E^*$. If the net $x^* z_\alpha$ does not converge to zero, then for every α there exists $\beta_\alpha \ge \alpha$ such that

$$|x^* z_{\beta_\alpha}| \ge \epsilon > 0. \tag{25}$$

The set $\{z_{\beta_\alpha}\}$ is a net. Indeed, for indexes $\beta_{\alpha_1}, \dots, \beta_{\alpha_n}$ pick α_0 satisfying $\alpha_0 \ge \beta_{\alpha_i}$ for all $i = 1, ..., n$. Then $\beta_{\alpha_0} \ge \beta_{\alpha_i}$. Therefore ([11], p. 29), the net $\{z_{\beta_\alpha}\}$ has a weak cluster point z'. Obviously, z' is also a weak cluster point of $\{z_\alpha\}$. So, as showed above, $z' = 0$, which is impossible in view of (25). Thus, $\lim\limits_\alpha x^* z_\alpha = 0$, as desired. □

Lemma 23. *Let T be an o-weakly compact order continuous operator acting from a Banach lattice E onto a Banach lattice F. Then the inclusion $R(T^*) \subseteq E_n^\sim$ is valid.*

In particular, for $E = F$ and $\lambda \ne 0$ the inclusion $\mathcal{N}^\infty(\lambda - T^*) \subseteq E_n^\sim$ holds, where $\mathcal{N}^\infty(\lambda - T^*) = \bigcup\limits_{k=1}^{\infty} N((\lambda - T^*)^k)$.

Proof. We may suppose that the operator T is real. Consider a net $x_\alpha \xrightarrow{o} 0$ in E. Assume x_α is order bounded. By the order continuity of T, we have $Tx_\alpha \xrightarrow{o} 0$. On the other hand, by the o-weakly compactness of T, the set $\{Tx_\alpha\}$ is relatively weakly compact. Using Lemma 22, we obtain $Tx_\alpha \xrightarrow{\sigma(E,E^*)} 0$, hence $x^*(Tx_\alpha) \to 0$. So the relation $T^*x^* \in E_n^\sim$ holds, as desired.

In the case, when $E = F$ and $\lambda \ne 0$, it suffices to observe that ([3], p. 3) the inclusion $\mathcal{N}^\infty(\lambda - T^*) \subseteq R(T^*)$ is valid. □

Corollary 24. *Suppose that there exists a non-zero o-weakly compact order continuous operator $T : E \to F$, where E and F are Banach lattice. Then $E_n^\sim \ne \{0\}$.*

Lemma 25. *Let T be a positive operator on a Banach lattice E, moreover $r(T) \notin \sigma_{\rm f}(T)$. If B is a T-invariant projection band, then in a sufficiently small deleted neighbourhood of $r(T)$ we have*

$$P_B R(\lambda, T) P_{B^{\rm d}} = R(\lambda, T_B) P_B T P_{B^{\rm d}} R(\lambda, T_{B^{\rm d}}).$$

Proof. The relation [10] $r(T) \notin \sigma_{\rm f}(T_B) \cup \sigma_{\rm f}(T_{B^{\rm d}})$ is true. Therefore, in some sufficiently small deleted neighbourhood U of $r(T)$ the operators $R(., T_B)$ and $R(., T_{B^{\rm d}})$ are well defined. The band B is $R(\lambda, T)$-invariant for $\lambda \in U$. Then for $\lambda \in U$ we have

$$(\lambda - T_B) P_B R(\lambda, T) P_{B^{\rm d}} (\lambda - T_{B^{\rm d}}) = P_B(\lambda - TP_B) R(\lambda, T)(\lambda - P_{B^{\rm d}} T) P_{B^{\rm d}} =$$

$$= P_B(\lambda - T + TP_{B^{\rm d}}) R(\lambda, T)(\lambda - P_{B^{\rm d}} T) P_{B^{\rm d}} = P_B(I + TP_{B^{\rm d}} R(\lambda, T))(\lambda - P_{B^{\rm d}} T) P_{B^{\rm d}} =$$

$$= P_B(\lambda - P_{B^{\rm d}} T) P_{B^{\rm d}} + P_B T P_{B^{\rm d}} R(\lambda, T)(\lambda - T + P_B T) P_{B^{\rm d}} =$$

$$= P_B T P_{B^{\rm d}} (I + R(\lambda, T) P_B T) P_{B^{\rm d}} = P_B T P_{B^{\rm d}},$$

and the proof is finished. □

Our purpose here is to establish Theorem 3 (see Section 2) giving the necessary conditions of the order continuity of the residue.

Proof of Theorem 3. (a) The equality $R(T^*_{-1}) = N((r(T) - T)^m)$ is valid, where m is the order of the pole of $R(., T)$ at $r(T)$. By Lemma 23, $R(T^*_{-1}) \subseteq E^\sim_n$. The operator T_{-1} has a representation $T_{-1} = \sum\limits_{i=1}^{k} x^*_i \otimes x_i$ with elements $x_1, \ldots, x_k$ linearly independent. Pick functionals $z^*_1, \ldots, z^*_k$ with $z^*_j x_i = \delta_{ji}$, $i, j = 1, ..., k$. Since $T^*_{-1} z^*_j = (\sum\limits_{i=1}^{k} x_i \otimes x^*_i) z^*_j = x^*_j$, we have $x^*_j \in E^\sim_n$. Therefore, $T_{-1} \in \mathcal{L}_n(E)$.

(b) Step 1. The case $(E^\sim_n)^\circ = \{0\}$, that is, the band $E^\sim_n$ separates E.

In this case the validity of the given assertion was mentioned in [7]. The proof will be derived here because it was omitted in [7]. Moreover, for the completeness and by the reason of a significance of this assertion, we will give the proof in two different ways.

The first way. Since $r(T) \notin \sigma_{\rm f}(T)$, the operators $T_{-m}, \ldots, T_{-1}$ in (22) are of finite rank. Now the desired assertion follows at once from the relations (23), (24) and Lemma 21.

The second way. The idea of the proof is borrowed from [12], Propositions 4, 5, where the analogous statement was proved for the case $\sigma_{\rm f}(T) = \{0\}$. Let T' be the restriction of T^* to $E^\sim_n$. Then $r(T) = r(T') \notin \sigma_{\rm f}(T')$. If m be the order of the pole of $R(., T)$ at $r(T)$, then

$$\begin{gathered} N((r(T) - T')^m) \subseteq N((r(T) - T^*)^m), \\ \dim N((r(T) - T^*)^m) = \dim N((r(T) - T)^m) < \infty. \end{gathered} \tag{26}$$

Next, let T'' be the restriction of $(T')^*$ to $(E^\sim_n)^\sim_n$. The Banach lattice E can be considered as a subspace of $(E^\sim_n)^\sim_n$. If j_n is this natural embedding, then the equality $j_n(Tx) = T''(j_n(x))$ holds. So

$$N((r(T) - T)^m) \subseteq N((r(T) - T'')^m) \subseteq N((r(T) - (T')^*)^m). \tag{27}$$

Using

$$\dim N((r(T) - (T')^*)^m) = \dim N((r(T) - T')^m) < \infty$$

and the relations (26) and (27), we have

$$R(T_{-1}^*) = N((r(T) - T^*)^m) = N((r(T) - T')^m) \subseteq E_n^\sim.$$

Consequently, $T_{-1} \in \mathcal{L}_n(E)$.

Step 2. The general case.

The band $(E_n^\sim)^\circ$ is T-invariant. Put $B = (E_n^\sim)^\circ$. The relation $E = B^{\mathrm{d}} \oplus B$ is valid. The operators T_B and $\widetilde{T}_B$ are order continuous and [7] $r(T_B) = r(\widetilde{T}_B)$ holds. The band B is T_{-1}-invariant, whence

$$P_{B^{\mathrm{d}}} T_{-1} P_B = 0. \tag{28}$$

We wish to show that $r(\widetilde{T}_B) < r(T)$ holds. To see this, let $r(T_B) = r(T)$. Then [10] $r(T_B) \notin \sigma_{\mathrm{f}}(T_B)$, so $r(\widetilde{T}_B) \notin \sigma_{\mathrm{f}}(\widetilde{T}_B)$. The Lorenz seminorm on B is a norm. By remarks after Lemma 17, the non-zero finite-rank operator $(\widetilde{T}_B)_{-m_B} = (\widetilde{T}_B - r(\widetilde{T}_B))^{m_B-1}(\widetilde{T}_B)_{-1}$, where $(\widetilde{T}_B)_{-1}$ and m_B is the residue and the order of the pole of $R(.,\widetilde{T}_B)$ at $r(\widetilde{T}_B)$, respectively, is order continuous. According to the relation $B_n^\sim = \{0\}$ and Lemma 18, which is impossible. Thus, $r(\widetilde{T}_B) < r(T)$.

So [7]

$$(T_B)_{-1} = (T_{-1})_B = 0, \tag{29}$$

moreover $r(\widetilde{T}_{B^{\mathrm{d}}}) = r(T)$. Since the band $(B^{\mathrm{d}})_n^\sim$ separates B^{d}, using Step 1, we get the order continuity of the residue $(\widetilde{T}_{B^{\mathrm{d}}})_{-1}$ of $R(.,\widetilde{T}_{B^{\mathrm{d}}})$ at $r(\widetilde{T}_{B^{\mathrm{d}}})$. If $(T_{B^{\mathrm{d}}})_{-1}$ is the residue of $R(.,T_{B^{\mathrm{d}}})$ at $r(T_{B^{\mathrm{d}}})$, then the band B^{d} is $(T_{B^{\mathrm{d}}})_{-1}$-invariant and [7] the restriction $(T_{B^{\mathrm{d}}})_{-1}$ to B^{d} coincides with $(\widetilde{T}_{B^{\mathrm{d}}})_{-1}$. Therefore, the operator

$$(T_{-1})_{B^{\mathrm{d}}} \in \mathcal{L}_n(E), \tag{30}$$

so the operators

$$(T_{B^{\mathrm{d}}})_{-i} = (T - r(T_{B^{\mathrm{d}}}))^{i-1}(T_{B^{\mathrm{d}}})_{-1}, \ \ i \geq 1,$$

are also order continuity. Using Lemma 25, we have

$$P_B R(\lambda, T) P_{B^{\mathrm{d}}} = R(\lambda, T_B) P_B T P_{B^{\mathrm{d}}} R(\lambda, T_{B^{\mathrm{d}}}),$$

where functions $R(\lambda, T_B)$ and $R(\lambda, T_{B^{\mathrm{d}}})$ are analytic on some deleted neighbourhood of the point $r(T)$ and have representations

$$R(\lambda, T_B) = \sum_{i=0}^{\infty} (-1)^i (\lambda - r(T))^i R(r(T), T_B)^{i+1},$$

$$R(\lambda, T_{B^{\mathrm{d}}}) = \frac{1}{(\lambda - r(T))^m}(T_{B^{\mathrm{d}}})_{-m} + \ldots + \frac{1}{\lambda - r(T)}(T_{B^{\mathrm{d}}})_{-1} + (T_{B^{\mathrm{d}}})_0 + \ldots$$

(here m is the order of the pole of $R(.,T_{B^{\mathrm{d}}})$ at $r(T)$). Using the order continuity of $R(r(T), T_B)$ and of $(T_{B^{\mathrm{d}}})_{-i}$, $i = 1, ..., m$, we get $P_B T_{-1} P_{B^{\mathrm{d}}} \in \mathcal{L}_n(E)$. So according (28), (29) and (30)

$$T_{-1} = (T_{-1})_B + P_B T_{-1} P_{B^{\mathrm{d}}} + P_{B^{\mathrm{d}}} T_{-1} P_B + (T_{-1})_{B^{\mathrm{d}}} = P_B T_{-1} P_{B^{\mathrm{d}}} + (T_{-1})_{B^{\mathrm{d}}} \in \mathcal{L}_n(E),$$

and the proof is finished. □

In the case of **(a)** the previous theorem is true if instead of the point $r(T)$ an arbitrary non-zero isolated point λ_0 of the spectrum $\sigma(T)$, $\lambda_0 \notin \sigma_{\mathrm{f}}(T)$, is considered.

In the general case it is not known if the order continuity of $T \geq 0$, $r(T) \notin \sigma_{\mathrm{f}}(T)$, *implies the order continuity of the residue* T_{-1} *of* $R(.,T)$ *at* $r(T)$. Moreover, the author does not know an example of a Banach lattice E such that the band $(E_n^\sim)^\circ$ is not a projection band.

If $0 \leq T \in \mathcal{L}(E)$, where E is a Banach lattice, and $r(T) \notin \sigma_{\mathrm{f}}(T)$, then the residue T_{-1} of $R(.,T)$ at $r(T)$ is a non-zero finite-rank operator. If, in addition to this, $E_n^\sim = \{0\}$ (most important examples of Banach lattices satisfying the property $E_n^\sim = \{0\}$ are an AM-space $C[0,1]$ and its a Dedekind completion), then by Lemma 18, the operator T_{-1} can not be order continuous. Thus, a question arises naturally: *Can a positive order continuous operator* T, $r(T) \notin \sigma_{\mathrm{f}}(T)$, *on a Banach lattice* E *with the property* $E_n^\sim = \{0\}$ *exist*? The affirmative answer to this question means that the order continuity of the operator T does not imply the order continuity of the residue T_{-1} of $R(.,T)$ at $r(T)$. As it see from the proof of Theorem 3 (see the case of **(b)**, Step 2), the negative answer should have meant the order continuity of T_{-1} if $(E_n^\sim)^\circ$ is a projection band. By Corollary 24, such operator T can not be o-weakly compact, so it can not be compact. In particular, *there exists no a non-zero order continuity compact operator on* $C[0,1]$. In the next section this result will be derived (see Corollaries 33, 34 below) from other a more general result (see Theorem 4 above) which allows looking at the reason of the absence of a non-zero order continuous compact operator on $C[0,1]$ in a new fashion, namely with the point of view of the approximation problem. Remark also that *the conditions* $0 \leq T \in \mathcal{L}_n(C[0,1])$ *imply* $r(T) \in \sigma_{\mathrm{f}}(T)$ as the Lorenz seminorm on $C[0,1]$ is a norm.

7 Compact order continuous operators.

Let E and F be two Banach lattices. If a Banach lattice F has the approximation property (in particular, is an AM-space), then every compact operator $K : E \to F$ is the limit in the operator norm of a sequence of finite-rank operators. On the other hand, every operator $S \in E^* \otimes F$, where E and F are arbitrary Banach lattices, has a decomposition $S = S_1 + S_2$ with $S_1 \in E_n^\sim \otimes F \subseteq \mathcal{L}_n(E,F)$ and $S_2 \in E_\sigma^\sim \otimes F$, moreover $E_n^\sim \otimes F \perp E_\sigma^\sim \otimes F$ in $\mathcal{L}(E,F)$ (see Lemma 28). Below the conditions when an order continuous operator $T : E \to F$ can be approximated by an operators from $E_\sigma^\sim \otimes F$, will be considered.

Lemma 26. *Let* T *and* K *be an operators from a Riesz space* E *into a Riesz space* F (*not necessarily Dedekind complete*) *such that* $T \in \mathcal{L}_n(E,F)$ *and* $K \in (E_\sigma^\sim)^+ \otimes F^+$. *If an operator* $S : E \to F$ *satisfies the inequalities* $T + S \geq K$ *and* $S \geq 0$, *then* $S \geq K$.

In particular, if the modulus $|T - K|$ *exists, then* $|T - K| \geq K$.

Proof. The operator K has a representation $K = \sum_{i=1}^{k} x_i^* \otimes x_i$, $x_i^* \in (E_\sigma^\sim)^+$, $x_i \in F^+$. Obviously, the relation $\sum_{i=1}^{k} x_i^* \perp E_n^\sim$ is valid. Fix $\epsilon > 0$ and $z \in E^+$. The equality ([8], p. 46) $\inf\{\sup_\alpha \sum_{i=1}^{k} x_i^* z_\alpha : 0 \leq z_\alpha \uparrow z\} = 0$ holds. Consequently, there exists a net z_α, $0 \leq z_\alpha \uparrow z$, satisfying $\sum\limits_{i=1}^{k} x_i^* z_\alpha \le \epsilon$ for all α. Then

$$T(z - z_\alpha) \ge K(z - z_\alpha) - S(z - z_\alpha) =$$

$$= \sum_{i=1}^{k} (x_i^*(z - z_\alpha))x_i - S(z - z_\alpha) \ge \sum_{i=1}^{k} (x_i^* z - \epsilon)x_i - Sz = Kz - \epsilon \sum_{i=1}^{k} x_i - Sz.$$

Using $z - z_\alpha \downarrow 0$ and $T \in \mathcal{L}_n(E, F)$, we infer $0 \ge Kz - \epsilon \sum\limits_{i=1}^{k} x_i - Sz$. Since ϵ is arbitrary, this implies $0 \ge Kz - Sz$. Hence, $S \ge K$, and we are done.

If $|T - K|$ exists, then the inequality $|T - K| \ge K - T$ implies $T + |T - K| \ge K$, whence $|T - K| \ge K$. □

Recall that *a locally convex-solid topology* on a Riesz space F is a locally convex topology generated by a family of lattice seminorms $\{p_i : i \in A\}$ on F, that is, seminorms having the property: $|x| \le |y|$ in F implies $p_i(x) \le p_i(y)$ for every $i \in A$ (for details, see [8], §11).

Lemma 27. *Let $0 \le T \in \mathcal{L}_n(E, F)$, where E and F are Riesz spaces. Then for every element $z \in E^+$ satisfying $Tz > 0$ there exist no a collection of an operators $K_i \in (E_\sigma^\sim)^+ \otimes F^+$, $i \in A$, and a locally convex-solid topology τ_z on F such that zero belongs to the τ_z-closures of the set $\{|T - K_i|z : i \in A\}$.*

Proof. Assuming by way of contradiction, we find a net $K_\alpha \in (E_\sigma^\sim)^+ \otimes F^+$ and a locally convex-solid topology τ_z on F such that moduli $|T - K_\alpha|$ exist and

$$|T - K_\alpha|z \xrightarrow{\tau_z} 0. \tag{31}$$

The inequality $|Tz - K_\alpha z| \le |T - K_\alpha|z$ implies $|Tz - K_\alpha z| \xrightarrow{\tau_z} 0$, so

$$K_\alpha z \xrightarrow{\tau_z} Tz. \tag{32}$$

On the other hand, Lemma 26 guarantees $|T - K_\alpha| \ge K_\alpha$, hence $|T - K_\alpha|z - K_\alpha z \ge 0$. Since the cone F^+ is τ_z-closed, it follows from the relations (31) and (32) that $-Tz \ge 0$. This implies $Tz \le 0$, which is a contradiction. □

Clearly, if an operator $K \in E^* \otimes F$, where E and F are Banach lattices, then the modulus of K exists and is r-compact (see Section 1). In fact, $|K|$ belongs to the closure of $(E^*)^+ \otimes F^+$ in $\mathcal{L}_r(E, F)$ with the r-norm ([16], p. 253-254, the proof of Theorem IV.4.6).

Lemma 28. *Let E and F be two Banach lattices. If $K_1 \in E_n^\sim \otimes F$ and $K_2 \in E_\sigma^\sim \otimes F$, then* $\inf\{|K_1|, |K_2|\} = 0$ *in an ordered vector space $\mathcal{L}(E, F)$.*

Proof. From the preceding discussion there exist moduli $|K_1|$ and $|K_2|$. Obviously, if $K_1 = \sum\limits_{i=1}^{k} x_i^* \otimes x_i$ with $x_i^* \in E_n^\sim$, $x_i \in F$, then

$$|K_1| \le \sum_{i=1}^{k} |x_i^* \otimes x_i| = \sum_{i=1}^{k} |x_i^*| \otimes |x_i| \le \sum_{i=1}^{k} |x_i^*| \otimes \sum_{i=1}^{k} |x_i| = x^* \otimes x,$$

where $x^* = \sum\limits_{i=1}^{k} |x_i^*| \in (E_n^\sim)^+$, $x = \sum\limits_{i=1}^{k} |x_i|$. Analogously, $|K_2| \le y^* \otimes y$, $y^* \in (E_\sigma^\sim)^+$, $y \in F^+$. Now remain to notice that

$$(x^* \otimes x) \wedge (y^* \otimes y) \le (x^* \otimes (x + y)) \wedge (y^* \otimes (x + y)) = (x^* \wedge y^*) \otimes (x + y) = 0$$

as $x^* \perp y^*$. $\square$

Lemma 29. *Let $K \in E_\sigma^\sim \otimes F$, where E and F are two Banach lattices. Then we have $|K| \in \overline{(E_\sigma^\sim)^+ \otimes F^+}$, where the closure in $\mathcal{L}_r(E,F)$ with the r-norm.*

Proof. There exists a sequence $K_n \in (E^*)^+ \otimes F^+$ converging to the operator $|K|$ in the r-norm. For an arbitrary n the operator K_n has a decomposition $K_n = K_{n1} + K_{n2}$ with $K_{n1} \in (E_n^\sim)^+ \otimes F^+$ and $K_{n2} \in (E_\sigma^\sim)^+ \otimes F^+$. Consider the band $B_{E_\sigma^\sim \otimes F}$ generated by the set $\mathcal{K}_r(E,F)$ of $E_\sigma^\sim \otimes F$. By the previous lemma, $K_{n1} \perp B_{E_\sigma^\sim \otimes F}$ in $\mathcal{K}_r(E,F)$. Obviously, $|K| \in B_{E_\sigma^\sim \otimes F}$, so $K_{n1} \perp K_{n2} - |K|$ in $\mathcal{K}_r(E,F)$. Then

$$|K_{n1}| + |K_{n2} - |K|| = |K_{n1} + K_{n2} - |K|| = |K_n - |K|| \to 0$$

in $\mathcal{K}_r(E,F)$. The inequality $|K_{n1}| + |K_{n2} - |K|| \geq |K_{n2} - |K||$ implies $K_{n2} \to |K|$ in the r-norm, as claimed. $\square$

Lemma 30. *Let E and F be two Riesz spaces. If an operator $T \in \mathcal{L}_n(E,F)$ possesses a modulus $|T|$, moreover for every $x \in E^+$ the equality*

$$|T|x = \sup\{Ty : |y| \leq x\} \tag{33}$$

holds, then also $|T| \in \mathcal{L}_n(E,F)$.

For the case of a Dedekind complete Riesz space F the proof of this assertion can be found in [15], p. 29-30, Proposition 1.3.9 (see also [8], p. 43, Theorem 4.3). In the our case the proof of it is analogous, but for the sake of completeness we include the proof.

Proof. Consider a net $x_\alpha \downarrow 0$ in E. Let $|T|x_\alpha \geq z \geq 0$ in F. Fix an index β. For every $|y| \leq x_\beta$ and $\alpha \geq \beta$ the inequality

$$|y - (y^+ \wedge x_\alpha - y^- \wedge x_\alpha)| \leq x_\beta - x_\alpha$$

holds. Therefore,

$$Ty + z - |T|x_\beta \leq |T(y^+ \wedge x_\alpha)| + |T(y^- \wedge x_\alpha)|.$$

Hence, using the relations $y^+ \wedge x_\alpha \downarrow 0$, $y^- \wedge x_\alpha \downarrow 0$ and the order continuity of T, we obtain $Ty + z \leq |T|x_\beta$. It follows from (33) that $z \leq 0$. $\square$

Now we are in a position to prove Theorem 4 (see Section 2).

Proof of Theorem 4. Assume by way of contradiction that the operator $T \in \mathcal{L}_r(E,F)$ and there exists a sequence $K_n \in E_\sigma^\sim \otimes F$ which is convergent to T in the r-norm. Then the operator T is r-compact. In particular [9], the modulus of T exists and the equality (33) is valid. By Lemma 30, $|T|$ is order continuous. Next, $|K_n| \to |T|$ in the r-norm. By Lemma 29, we can assume that $K_n \in (E_\sigma^\sim)^+ \otimes F^+$. Obviously, $|K_n - |T||z \to 0$ in F for each $z \in E^+$. According to Lemma 27, we have $|T|z = 0$ and hence $T = 0$, a contradiction. $\square$

Lemma 31. *For a Banach lattice E and an AM-space F the next statements hold*:

(a) *If a sequence $K_n \in E^* \otimes F$ converges in $\mathcal{L}(E,F)$ to an operator T, then $T \in \mathcal{L}_r(E,F)$ and $K_n \to T$ in the r-norm*;

(b) *The equality $\mathcal{K}(E,F) = \mathcal{K}_r(E,F)$ is valid.*

Proof. (a) Since F is an AM-space, by Krengel theorem ([8], p. 271), the space $\mathcal{K}(E,F)$ of all compact operators from E into F is a Banach lattice under the r-norm. The sequence K_n is a $\|\cdot\|_r$-Cauchy sequence. Indeed,

$$\|K_n - K_m\|_r = \||K_n - K_m|\| = \||K_n - K_m|^{**}\|. \tag{34}$$

Next, the relation $K_n - K_m \in E^* \otimes F$ implies ([16], p. 296)

$$|K_n - K_m|^{**} = |K_n^{**} - K_m^{**}|. \tag{35}$$

The space F^{**} is ([8], p. 188, 193) a Dedekind complete AM-space with a unit, so ([1], p. 96)

$$\| |K_n^{**} - K_m^{**}| \| = \|K_n^{**} - K_m^{**}\| = \|K_n - K_m\|.$$

Therefore, using (34) and (35), we have $\|K_n - K_m\|_r = \|K_n - K_m\| \to 0$ as $n, m \to \infty$. Thus, K_n converges in the r-norm. Obviously, $K_n \to T$ in the r-norm.

(b) By Grothendieck's results ([16], p. 239; [1], p. 125-129), an AM-space F has the approximation property, that is, every operator $K \in \mathcal{K}(E, F)$ can be approximated in the operator norm by an operators of finite rank. It remains to use of **(a)**. □

Now we are ready to derive a number of corollaries of Theorem 4.

Corollary 32. *Let E and F be two Banach lattices, moreover F is an AM-space. If a non-zero operator T belongs to $\mathcal{L}_n(E, F)$, then $T \notin \overline{E_\sigma^\sim \otimes F}$, where the closure in $\mathcal{L}(E, F)$ with the operator norm.*

Proof. If there exists a sequence $K_n \in E_\sigma^\sim \otimes F$ which is convergent to the operator T in $\mathcal{L}(E, F)$, then by previous lemma, $K_n \to T$ in the r-norm. It is a contradiction in view of Theorem 4. □

Corollary 33. *If E and F are Banach lattices, F is an AM-space, $E_n^\sim = \{0\}$, then $\mathcal{K}(E, F) \cap \mathcal{L}_n(E, F) = \{0\}$.*

Corollary 34. *If E is an AM-space with $E_n^\sim = \{0\}$ (for example, $E = C[0, 1]$), then $\mathcal{K}(E) \cap \mathcal{L}_n(E) = \{0\}$.*

In the case of a Dedekind complete Banach lattice F the space $\mathcal{L}_r(E, F)$ is a Banach lattice under the r-norm ([8], p. 248), so the band $\mathcal{L}_n(E, F)$ of order continuous operators is closed in the r-norm, that is, the relations $S_k \in \mathcal{L}_n(E, F)$, $S_k \to S$ in the r-norm, imply $S \in \mathcal{L}_n(E, F)$. The next theorem improves this fact and Theorem 4.

Theorem 35. *Let E and F be two Banach lattices with F Dedekind complete, and let $S_i \in \mathcal{L}_n(E, F)$ and $K_i \in E_\sigma^\sim \otimes F$ be two arbitrary collections of an operators, $i \in A$. If $\inf_{i \in A} \|S_i - K_i\|_r = 0$, then $\inf_{i \in A} \|K_i\|_r = 0$.*

Proof. Clearly, $\inf_{i \in A} \| |S_i| - |K_i| \|_r = 0$. Fix $\epsilon > 0$. Using Lemma 29, we find an operators $Q_i \in (E_\sigma^\sim)^+ \otimes F^+$ such that

$$\inf_{i \in A} \| |S_i| - Q_i \|_r = 0, \quad \sup_{i \in A} \| |K_i| - Q_i \|_r \leq \epsilon. \tag{36}$$

The inequality $|S_i| + ||S_i| - Q_i| \geq |Q_i|$ is valid. By Lemma 26, $||S_i| - Q_i| \geq |Q_i|$, it follows from (36) that $\inf_{i \in A} \|Q_i\|_r = 0$. Using (36) once more, we have $\|K_i\|_r \leq \epsilon + \|Q_i\|_r$ for all i, so $\inf_{i \in A} \|K_i\|_r \leq \epsilon$. Letting $\epsilon \downarrow 0$ yields $\inf_{i \in A} \|K_i\|_r = 0$, as desired. □

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