SOME PROPERTIES OF ESSENTIAL SPECTRA OF A POSITIVE OPERATOR, II

E. A. ALEKHNO (Belarus, Minsk)

Abstract. Let T be a positive operator on a Banach lattice E. Some properties of Weyl essential spectrum $\sigma_{\mathrm{ew}}(T)$, in particular, the equality $\sigma_{\mathrm{ew}}(T) = \bigcap\limits_{0\le K\in\mathcal{K}(E)} \sigma(T+K)$, where $\mathcal{K}(E)$ is the set of all compact operators on E, are established. If $r(T)$ does not belong to Fredholm essential spectrum $\sigma_{\mathrm{ef}}(T)$, then $r(T) \notin \sigma(T+a|T_{-1}|)$ for every $a \neq 0$, where T_{-1} is a residue of the resolvent $R(.,T)$ at $r(T)$. The new conditions for which $r(T) \notin \sigma_{\mathrm{ef}}(T)$ implies $r(T) \notin \sigma^{-}_{\mathrm{ew}}(T) = \bigcap\limits_{0\le K\in\mathcal{K}(E)\le T} \sigma(T-K)$, are derived. The question when the relation $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$ holds, where $\sigma_{\mathrm{el}}(T) = \bigcap\limits_{\substack{0\le Q\le T\\ Q\le K\in\mathcal{K}(E)}} \sigma(T-Q)$ is Lozanovsky's essential spectrum, will be considered. Lozanovsky's order essential spectrum is introduced. A number of auxiliary results are proved. Among them the following generalization of Nikol'sky's theorem: if T is an operator of index zero, then $T = R + K$, where R is invertible, $K \ge 0$ is of finite rank. Under the natural assumptions (one of them is $r(T) \notin \sigma_{\mathrm{ef}}(T)$) a theorem about the Frobenius normal form is proved: there exist T-invariant bands $E = B_n \supseteq B_{n-1} \supseteq \ldots \supseteq B_0 = \{0\}$ such that if $r(P_{D_i}TP_{D_i}) = r(T)$, where $D_i = B_i \cap B^{\mathrm{d}}_{i-1}$, then an operator $P_{D_i}TP_{D_i}$ on D_i is band irreducible.

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1 Introduction.

This note is a continuation of research which was begun by the author in the note [4] and is devoted to special subsets of the spectrum of a positive operator T on some Banach lattice E.

Let Z be a Banach space, T be an (linear, bounded) operator on Z. As usual, the spectrum of an operator T will be denoted by $\sigma(T)$. Recall that *the Fredholm essential spectrum* of an operator T is the set

$$\sigma_{\mathrm{ef}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not a Fredholm operator}\},$$

and *the Weyl essential spectrum* is the set

$$\sigma_{\mathrm{ew}}(T) = \{\lambda \in \mathbb{C} : \mathrm{ind}(\lambda - T) \neq 0\} = \bigcap_{K\in\mathcal{K}(Z)} \sigma(T+K) = \bigcap_{K\in\mathcal{F}(Z)} \sigma(T+K),$$

where $\mathcal{K}(Z)$ and $\mathcal{F}(Z)$ are sets of all compact operators and of all finite-rank operators on Z, respectively.

In the case, when E is a Banach lattice, T is an operator on E, we define [4]

$$\sigma^+_{\mathrm{ew}}(T) \;=\; \bigcap_{0\leq K\in\mathcal{K}(E)} \sigma(T+K),$$

and when T is a positive operator

$$\sigma^-_{\mathrm{ew}}(T) \;=\; \bigcap_{0\leq K\in\mathcal{K}(E)\leq T} \sigma(T-K),\;\; \sigma_{\mathrm{el}}(T)=\bigcap_{\substack{0\leq Q\leq T\\ Q\leq K\in\mathcal{K}(E)}} \sigma(T-Q).$$

It will be show below that the equality $\sigma^+_{\mathrm{ew}}(T)=\sigma_{\mathrm{ew}}(T)$ always holds hence, if the spectral radius $r(T)\notin\sigma_{\mathrm{ef}}(T)$, then there exists a compact operator $K\geq 0$ such that $r(T)\notin\sigma(T+K)$. The question about the concrete operators K satisfying the last relation, will be considered. The discussion of the question when $r(T)\notin\sigma_{\mathrm{ef}}(T)$ implies $r(T)\notin\sigma^-_{\mathrm{ew}}(T)$, will be continued. The conditions such that the inclusion $\sigma_{\mathrm{ew}}(T)\subseteq\sigma_{\mathrm{el}}(T)$ is true, will be given. Two auxiliary results which are of independent interest, will be proved. Namely, an analog for the case of a Banach lattice of the classical Nikol'sky's theorem and a theorem about the Frobenius normal form of a positive operator.

For terminology, notions, and properties on the theory of Banach lattices and operators on them not explained or proved in this note, we refer to [2, 5]; see also [9, 11]. Throughout the note, unless otherwise stated, a Banach lattice E will be assumed to be complex and infinite dimensional and an operator T on E will be assumed linear and bounded.

2 Auxiliary results.

2.1 Nikol'sky's theorem for the case of a Banach lattice.

Nikol'sky's theorem [10] asserts that an operator T on a Banach space Z is a Fredholm operator of index zero iff $T=R+K$, where the operator R is invertible and K is a finite-rank operator. For an operators on a Banach lattice this result can be made more precisely (Theorem 3 below). We need the next lemma.

Lemma 1. *Let R be an invertible operator on a Banach space Z, $K\in\mathcal{K}(Z)$, $\lambda\in\mathbb{C}$. Then there exists an invertible operator R_1 and a number $a\geq 0$ such that $R+\lambda K=R_1+aK$.*

Proof. The operator $R^{-1}K$ is compact therefore, $\frac{1}{a-\lambda}\notin\sigma(R^{-1}K)$ for some $a\geq 0$. Then

$$R+\lambda K=R-(a-\lambda)K+aK=R(I-(a-\lambda)R^{-1}K)+aK.$$

It remains to notice that $R_1=R(I-(a-\lambda)R^{-1}K)$ is invertible. □

Recall that if K is a finite-rank operator between two Banach lattices E and F, then the modulus of K exists and is a compact operator. Moreover, $|K|$ can be approximated in $\mathcal{L}(E,F)$ by a finite-rank positive operators ([11], p. 253-254, Theorem IV.4.6). The next lemma improves this result. The proof of it is analogous, but for the sake of completeness we include the proof.

Lemma 2. *Let E and F be Banach lattices, an operator $K\in\mathcal{F}(E,F)$. Then the operators K^+ and K^- can be approximated in $\mathcal{L}(E,F)$ by a finite-rank positive operators.*

Proof. It suffices to consider K^+. There exists $z \in F^+$ such that $|Kx| \leq z$ for all $x \in U$, where U is the closed unit ball of E. The ideal F_z is an AM-space with the unit z. Clearly, $K(E) \subseteq F_z$. The restriction of K to F_z is denoted by K_z. Then K_z^+ exists and is compact. For every $\epsilon > 0$ there exist $y_i \in F_z^+$ and $y_i^* \in (F_z^*)^+$ such that $\|y - \sum\limits_{i=1}^{n}(y_i^* y)y_i\|_\infty \leq \epsilon$ for all $y \in K_z^+(U)$. Putting $x_i^* = (K_z^+)^* y_i^*$, we obtain $\|K^+x - \sum\limits_{i=1}^{n}(x_i^* x)y_i\| \leq \epsilon\|z\|\|x\|$, as desired.□

Theorem 3. *An operator T on a Banach lattice E is a Fredholm operator of index zero iff $T = R + K$, where R is invertible and K is a positive finite-rank operator.*

Proof. Only the necessity needs to be proved. By Nikol'sky's theorem there exist an invertible operator R and a finite-rank operator K such that $T = R + K$. The operator K is presented in the form $K = K_1 + iK_2$, where K_1 and K_2 are a real finite-rank operators thus,

$$T = (K_1)^+ + R - (K_1)^- + i(K_2)^+ - i(K_2)^-.$$

Lemma 1 guarantees the existence of a number $a_1 \geq 0$ and an invertible operator R_1 with

$$T = (K_1)^+ + a_1(K_1)^- + R_1 + i(K_2)^+ - i(K_2)^-.$$

Using Lemma 1 again, we find numbers $a_2 \geq 0, a_3 \geq 0$ and an invertible operator R_3 such that

$$T = (K_1)^+ + a_1(K_1)^- + a_2(K_2)^+ + a_3(K_2)^- + R_3.$$

It remains to use of Lemma 2 for the completion of the proof. □

2.2 The Frobenius normal form of a positive operator.

A classical result about the Frobenius normal form is next; a simultaneous permutation of rows and columns can convert a nonnegative matrix to lower block triangular form

$$\begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix},$$

where the matrices A_{ii}, $i = 1, m$, are irreducible. The main purpose of this section is a proof of an analog of this result for the case of a positive operator on a Banach lattice (Theorem 13 below).

Lemma 4. *Let E be a Riesz space, P_α be a net of a band projections on E such that $P_\alpha x \downarrow 0$ for all $x \geq 0$ and for some $x_\alpha > 0$ $P_\alpha x_\alpha = x_\alpha$. Then the linear span of $\{x_\alpha\}$ is infinite dimensional.*

Proof. Fix α_1. Then for some index $\alpha_2 \geq \alpha_1$ the inequality $x_{\alpha_1} - P_{\alpha_2} x_{\alpha_1} > 0$ holds. Next, some index $\alpha_3 \geq \alpha_2$ satisfies $x_{\alpha_2} - P_{\alpha_3} x_{\alpha_2} > 0$. Continuing the construction inductively in the obvious manner, we build the sequence α_i with the next properties: $x_{\alpha_i} - P_{\alpha_{i+1}} x_{\alpha_i} > 0$ and $\alpha_{i+1} \geq \alpha_i$ for all i. The proof would be finished if we show that elements $x_{\alpha_1}, \dots, x_{\alpha_n}$ are linearly independent for an arbitrary n. Let $\sum_{i=1}^{n} b_i x_{\alpha_i} = 0$ be an equality which holds for some scalars b_i. We have

$$0 = \sum_{i=1}^{n} b_i (P_{\alpha_1} - P_{\alpha_2}) x_{\alpha_i} = b_1 (P_{\alpha_1} - P_{\alpha_2}) x_{\alpha_1} = b_1 (x_{\alpha_1} - P_{\alpha_2} x_{\alpha_1}),$$

whence $b_1 = 0$. Applying the operator $P_{\alpha_2} - P_{\alpha_3}$ to the equality $\sum_{i=2}^{n} b_i x_{\alpha_i} = 0$, we get $b_2 = 0$. As a result $b_i = 0$, $i = 1, n$, and the proof is finished. □

Lemma 5. *Let Z be a Banach space and $T \in \mathcal{L}(Z)$ such that $r(T)$ belongs to the point spectrum $\sigma_p(T)$. If Z_0 is a closed T^*-invariant subspace of Z^* which separates the points of Z, then $r(T^*|_{Z_0}) = r(T)$, where $T^*|_{Z_0}$ is a restriction of T^* to Z_0.*

Proof. Put $T' = T^*|_{Z_0}$. The inequality $r(T') \leq r(T)$ is obvious. There exists a non-zero x such that $Tx = r(T)x$. For an arbitrary functional $x^* \in Z_0$ we have

$$0 = x^*(r(T)x - Tx) = ((r(T) - T')x^*)x$$

so $r(T) - T'$ is not invertible, i.e., $r(T') \geq r(T)$. □

Recall that if λ_0 is an isolated point of the spectrum $\sigma(T)$ of an operator T on a Banach space Z, then the resolvent $R(., T)$ of T has the Laurent expansion $R(\lambda, T) = \sum_{i=-\infty}^{+\infty} (\lambda - \lambda_0)^i T_i$ around λ_0. This expansion holds also when λ_0 belongs to the resolvent set $\rho(T)$. In this case, of course, $T_i = 0$, $i < 0$, moreover the converse is valid. There exists a path lying outside of $\sigma_{\mathrm{ef}}(T)$ and joining λ_0 with a point in $\rho(T)$ (*of course, it is true for $\lambda_0 = r(T)$, when T is a positive operator on a Banach lattice E and $r(T) \notin \sigma_{\mathrm{ef}}(T)$*) iff ([2], p. 300-302) λ_0 is a pole of $R(., T)$ and the residue T_{-1} is a finite-rank operator. If $T \geq 0$ and $r(T) \notin \sigma_{\mathrm{ef}}(T)$, then the operators T_i are real, $T_{-m} \geq 0$, where m is the order of the pole of $R(., T)$ at $r(T)$, and T_i are of finite rank for $i < 0$. Remark that T_0 is not of finite rank.

Lemma 6. *Let E be a Banach lattice and $T \geq 0$ an operator on E. Assume λ_0 belongs to the boundary of the unbounded component in $\mathbb{C}$ of $\rho(T)$, i.e., $\lambda_0 \in \partial\rho_\infty(T)$, and is an isolated point of $\sigma(T)$. The residue T_{-1} of $R(., T)$ at λ_0 fails to be an order continuous operator if there exists a net of T-invariant projection bands B_α such that $P_{B_\alpha} x \uparrow x$ for all $x \geq 0$ and $\lambda_0 \notin \sigma(T|_{B_\alpha})$ for all α.*

Proof. Assume by way of contradiction that T_{-1} is order continuous. Then the set $\rho_\infty(T)$ contains a deleted neighbourhood of λ_0 so ([2], p. 256) B_α are T_i-invariant for all i, where T_i are coefficients of the Laurent series expansion of $R(., T)$ around λ_0. The equality ([2], p. 256) $R(\lambda, T)|_{B_\alpha} = R(\lambda, T|_{B_\alpha})$ for λ sufficiently close to λ_0 implies

$$\sum_{i=-\infty}^{+\infty} (\lambda - \lambda_0)^i T_i|_{B_\alpha} = \sum_{i=0}^{+\infty} (\lambda - \lambda_0)^i (T|_{B_\alpha})_i$$

therefore, $T_{-1}|_{B_\alpha} = (T|_{B_\alpha})_{-1} = 0$, hence $T_{-1} P_{B_\alpha} = 0$ for all α so $T_{-1} = 0$, a contradiction. □

The next lemma gives the conditions under which the residue T_{-1} is order continuous. When T is Riesz operator, this result was established in [8] (Propositions 4, 5). In our case the proof of it is analogous and will be omitted.

Lemma 7. *Let E be a Banach lattice separated by $E_n^\sim$ and $T \geq 0$ an order continuous operator on E such that $r(T) \notin \sigma_{\mathrm{ef}}(T)$. Then the residue T_{-1} of $R(.,T)$ at $r(T)$ is order continuous.*

Recall the next important result [6] which repeatedly will be used in the future.

Lemma 8. *Let S and T be an operators on a Banach lattice E such that $0 \leq S \leq T$. Then $r(T) \notin \sigma_{\mathrm{ef}}(T)$ implies $r(T) \notin \sigma_{\mathrm{ef}}(S)$.*

If B is a projection band in E, P_B is the band projection on B, then put $T_B = P_B T P_B$ and denote the restriction T_B to B by $\widetilde{T}_B$.

Lemma 9. *If a projection band B is invariant under an operator $T \geq 0$ on a Banach lattice E, then $r(T_B) = r(T|_B)$.*

In particular, for every projection band B $r(T_B) = r(\widetilde{T}_B)$.

Proof. The equalities $(T^n)_B = (T_B)^n$ and $(T|_B)^n = T^n|_B$ hold. Consequently, by Gelfand formula ([2], p. 243), it suffices to establish the equality $\|T_B\| = \|T|_B\|$. For arbitrary $x \in E$ and $y \in B$ we have

$$\|T_B x\| = \|P_B T P_B x\| = \|T|_B P_B x\| \leq \|T|_B\| \|x\|,$$
$$\|T|_B y\| = \|Ty\| = \|T_B y\| \leq \|T_B\| \|y\|. \qquad \square$$

A similar result holds if in place of B an arbitrary closed complemented T-invariant subspace of a Banach space Z is considered.

The second statement of the previous lemma can be made more precise.

Lemma 10. *Let B be a projection band in a Banach lattice E and $T \geq 0$ an operator on E. Then $\sigma(\widetilde{T}_B) \subseteq \sigma(T_B) \subseteq \sigma(\widetilde{T}_B) \cup \{0\}$.*

Proof. Assume that B is non-trivial. Show the first inclusion. Let $\lambda \notin \sigma(T_B)$ so $\lambda \neq 0$. If $\widetilde{T}_B x = \lambda x$, then $x \in B$ therefore, $T_B x = \lambda x$ hence $x = 0$. Fix $z \in B$. There exists y satisfying $\lambda y - T_B y = z$. The element $y \in B$, it follows that $(\lambda - \widetilde{T}_B)y = z$. As a result $\lambda \notin \sigma(\widetilde{T}_B)$.

For a proof of the second inclusion we consider a non-zero $\lambda \notin \sigma(\widetilde{T}_B)$. If $T_B x = \lambda x$ so $x \in B$, hence $x = 0$. Fix $z \in E$. There exists $y \in B$ such that $(\lambda - \widetilde{T}_B)y = P_B z$ or $(\lambda - T_B)y = P_B z$. Then $(\lambda - T_B)(y + \frac{1}{\lambda} P_{B^{\mathrm{d}}} z) = P_B z + P_{B^{\mathrm{d}}} z = z$ and we get $\lambda \notin \sigma(T_B)$. $\square$

A simply ordered set of projection bands $\{B_n, \ldots, B_1\}$ is called a *T-invariant chain* if $B_n \supseteq \ldots \supseteq B_1$ and all B_i are T-invariant. Notice that $\{E, \{0\}\}$ is, of course, a T-invariant chain for every T, but the set $\{\{0\}, E\}$ is not a T-invariant chain.

Lemma 11. *Assume that $\{E = B_n, B_{n-1}, \ldots, B_1, B_0 = \{0\}\}$ is a T-invariant chain for a positive operator T on a Banach lattice E. Then we have the inclusion $\sigma(T) \subseteq \bigcup\limits_{i=1}^{n} \sigma(T_{B_i \cap B_{i-1}^{\mathrm{d}}})$.*

Proof. We can suppose that inclusions $B_n \supseteq B_{n-1} \supseteq \ldots \supseteq B_0$ are proper and $n > 1$ so $0 \in \bigcup\limits_{i=1}^{n} \sigma(T_{D_i})$, where $D_i = B_i \cap B_{i-1}^{\mathrm{d}}$. We will show that if $\lambda \notin \bigcup\limits_{i=1}^{n} \sigma(T_{D_i})$, then $\lambda \notin \sigma(T)$. The equality $\sum\limits_{j=1}^{n} P_{D_j} = I$ implies

$$\lambda - T = \lambda \sum_{j=1}^{n} P_{D_j} - \sum_{j=1}^{n} \sum_{i=1}^{n} P_{D_j} T P_{D_i} = \sum_{j=1}^{n} \left(\lambda P_{D_j} - \sum_{i=j}^{n} P_{D_j} T P_{D_i} \right)$$

so the existence and the uniqueness of a solution of the equation $\lambda x - Tx = z$ are equivalent to the existence and the uniqueness of a solution of the system

$$\begin{cases} \lambda x_n - P_{D_n} T P_{D_n} x_n = P_{D_n} z \\ \lambda x_{n-1} - (P_{D_{n-1}} T x_n + P_{D_{n-1}} T P_{D_{n-1}} x_{n-1}) = P_{D_{n-1}} z \\ \quad \dots \qquad\qquad \dots \qquad\qquad \dots \\ \lambda x_1 - (P_{D_1} T x_n + P_{D_1} T x_{n-1} + \ldots + P_{D_1} T P_{D_1} x_1) = P_{D_1} z. \end{cases}$$

The first equation of the given system has the unique solution. Therefore, also the second equation has unique solution. Next, with the help of the elementary induction, we easy obtain the desired solubility of the system. □

Now we are ready to prove the main lemma.

Lemma 12. *Let E be a Dedekind complete Banach lattice such that the order continuous dual $E_n^\sim$ separates the points of E. Let T be a positive order continuous operator on E such that $r(T) \notin \sigma_{\mathrm{ef}}(T)$. If $\{B_2, B_1\}$ is a T-invariant chain, $r(T_{B_2 \cap B_1^{\mathrm{d}}}) = r(T)$, then either*

(a) *there exists a T-invariant chain $\{B_2, B'', B', B_1\}$ with the properties: the operator $\widetilde{T}_{B'' \cap (B')^{\mathrm{d}}}$ is band irreducible,*

$$r(T_{B'' \cap (B')^{\mathrm{d}}}) = r(T),\ r(T_{B_2 \cap (B'')^{\mathrm{d}}}) < r(T),\ r(T_{B' \cap B_1^{\mathrm{d}}}) < r(T);$$

or

(b) *there exists a T-invariant chain $\{B_2, B''', B_1\}$ with the properties:*

$$r(T_{B_2 \cap (B''')^{\mathrm{d}}}) = r(T_{B''' \cap B_1^{\mathrm{d}}}) = r(T).$$

Proof. Introduce the set

$$\mathcal{E}_1 = \{B : B \text{ is a band},\ B_1 \subseteq B,\ B \perp B_2^{\mathrm{d}},\ T(B) \subseteq B,\ r(T_{B \cap B_1^{\mathrm{d}}}) < r(T)\}.$$

We have $\mathcal{E}_1 \neq \emptyset$ as $B_1 \in \mathcal{E}_1$. Let $\mathcal{E}_1$ be ordered by inclusion. We will show that $\mathcal{E}_1$ has a maximal element. Let $\{B_\alpha\}$ be a chain in $\mathcal{E}_1$. The corresponding real projections $P_{B_\alpha} \uparrow P_0$. Then P_0 is a band projection. Indeed, fix an index α_0. For $\alpha \geq \alpha_0$ we have $P_{B_{\alpha_0}} = P_{B_\alpha} P_{B_{\alpha_0}} \uparrow P_0 P_{B_{\alpha_0}}$ hence $P_{B_{\alpha_0}} = P_0 P_{B_{\alpha_0}}$ therefore, $P_{B_\alpha} = P_0 P_{B_\alpha} \uparrow P_0^2$ so $P_0^2 = P_0$. Put $B_0 = P_0(E)$. Clearly, B_0 is a band. Moreover B_0 is T-invariant. In fact, if $x \in B_0$, then $TP_\alpha x \uparrow TP_0 x = Tx$ and $TP_\alpha x \in B_\alpha \subseteq B_0$ so $Tx \in B_0$. Show that $r(T_{B_0 \cap B_1^{\mathrm{d}}}) < r(T)$. Assume by way of contradiction. By Lemma 8 the relation $r(T) \notin \sigma_{\mathrm{ef}}(T_{B_0 \cap B_1^{\mathrm{d}}})$ is satisfied. The equality ([2], p. 256) $R(\lambda, T_{B_0 \cap B_1^{\mathrm{d}}})|_{B_0 \cap B_1^{\mathrm{d}}} = R(\lambda, \widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}})$ holds for λ sufficiently close to $r(T)$, it follows that $r(T) \notin \sigma_{\mathrm{ef}}(\widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}})$. Bands $B_\alpha \cap B_1^{\mathrm{d}}$ are subsets of $B_0 \cap B_1^{\mathrm{d}}$ and are $\widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}}$-invariant. Actually, if an element $x \in B_\alpha \cap B_1^{\mathrm{d}}$, then $\widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}} x = P_{B_0 \cap B_1^{\mathrm{d}}} Tx \in B_1^{\mathrm{d}}$. Moreover, $Tx \in B_\alpha$ hence $P_{B_0 \cap B_1^{\mathrm{d}}} Tx \in B_\alpha$. Next,

$$\widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}}|_{B_\alpha \cap B_1^{\mathrm{d}}} x = \widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}} x = P_{B_0 \cap B_1^{\mathrm{d}}} Tx = P_{B_\alpha \cap B_1^{\mathrm{d}}} Tx = T_{B_\alpha \cap B_1^{\mathrm{d}}} x = \widetilde{T}_{B_\alpha \cap B_1^{\mathrm{d}}} x.$$

Then according to Lemma 9 we have

$$r(\widetilde{T}_{B_0 \cap B_1^{\mathrm{d}}}|_{B_\alpha \cap B_1^{\mathrm{d}}}) = r(\widetilde{T}_{B_\alpha \cap B_1^{\mathrm{d}}}) = r(T_{B_\alpha \cap B_1^{\mathrm{d}}}) < r(T).$$

In view of the obvious relation $P_{B_\alpha\cap B_1^{\mathrm{d}}} \uparrow P_{B_0\cap B_1^{\mathrm{d}}}$ and Lemma 7 this contradicts Lemma 6. Thus, $r(T_{B_0\cap B_1^{\mathrm{d}}}) < r(T)$ so $B_0 \in \mathcal{E}_1$. By Zorn's lemma $\mathcal{E}_1$ has a maximal element, say B_{m_1}.

Introduce the set

$$\mathcal{E}_2 = \{B : B \text{ is a band},\ B_2^{\mathrm{d}} \subseteq B,\ B \perp B_{m_1},\ T(B^{\mathrm{d}}) \subseteq B^{\mathrm{d}},\ r(T_{B\cap B_2}) < r(T)\}.$$

We have $\mathcal{E}_2 \neq \emptyset$ as $B_2^{\mathrm{d}} \in \mathcal{E}_2$. Show that $\mathcal{E}_2$ has a maximal (by inclusion) element. Again, let $\{B_\alpha\}$ be a chain in $\mathcal{E}_2$. The projections P_{B_α} order converges to the band projection P_{B_0} for some band B_0. Clearly, $B_2^{\mathrm{d}} \subseteq B_0$ and $B_0 \perp B_{m_1}$. Next, if $x \in B_0^{\mathrm{d}}$, then $x \in B_\alpha^{\mathrm{d}}$ so $Tx \in B_\alpha^{\mathrm{d}}$, hence $Tx = (I - P_{B_\alpha})Tx \downarrow (I - P_{B_0})Tx$ or $Tx \in B_0^{\mathrm{d}}$. Show that $r(T_{B_0\cap B_2}) < r(T)$. Assume by way of contradiction. Using Lemma 8 once more, we have $r(T) \notin \sigma_{\mathrm{ef}}(T_{B_0\cap B_2})$. Then $r(T_{B_\alpha^{\mathrm{d}}\cap B_0}) = r(T)$ for all α. Indeed, $\{B_2, B_\alpha^{\mathrm{d}}, B_0^{\mathrm{d}}\}$ is a $T_{B_0\cap B_2}$-invariant chain. By Lemma 11 the inclusion

$$\sigma(T_{B_0\cap B_2}) \subseteq \sigma(T_{B_2\cap B_\alpha}) \cup \sigma(T_{B_\alpha^{\mathrm{d}}\cap B_0}) \cup \{0\}$$

is valid. It follows that the inequality $r(T_{B_\alpha^{\mathrm{d}}\cap B_0}) < r(T)$ implies $r(T_{B_0\cap B_2}) < r(T)$, which is a contradiction. Thus, $r(T) \notin \sigma_{\mathrm{ef}}(T_{B_\alpha^{\mathrm{d}}\cap B_0})$. In particular, $r(T)$ is an eigenvalue of the operators $T_{B_\alpha^{\mathrm{d}}\cap B_0}$, i.e., $T_{B_\alpha^{\mathrm{d}}\cap B_0}x_\alpha = r(T)x_\alpha$ for some $x_\alpha > 0$. The equality $B_0 = B_\alpha + B_\alpha^{\mathrm{d}} \cap B_0$ implies

$$B_0 \cap B_2 = B_\alpha \cap B_2 + B_\alpha^{\mathrm{d}} \cap B_0 \cap B_2 = B_\alpha \cap B_2 + B_\alpha^{\mathrm{d}} \cap B_0,$$

hence, using the inclusion $Tx_\alpha \in B_\alpha^{\mathrm{d}}$, we have

$$P_{B_0\cap B_2}TP_{B_0\cap B_2}x_\alpha = (P_{B_0\cap B_2} - P_{B_\alpha^{\mathrm{d}}\cap B_0})Tx_\alpha + r(T)x_\alpha = P_{B_\alpha\cap B_2}Tx_\alpha + r(T)x_\alpha = r(T)x_\alpha.$$

So x_α are eigenvectors of the operator $T_{B_0\cap B_2}$. Notice that $P_{B_\alpha^{\mathrm{d}}\cap B_0} = P_{B_0} - P_{B_\alpha}P_{B_0} \downarrow 0$. According to Lemma 4 we have $\dim N(r(T) - T_{B_0\cap B_2}) = +\infty$, which is a contradiction. Thus, $r(T_{B_0\cap B_2}) < r(T)$ so $B_0 \in \mathcal{E}_2$. By Zorn's lemma $\mathcal{E}_2$ has a maximal element, say B_{m_2}.

Put $B' = B_{m_1}^{\mathrm{d}} \cap B_{m_2}^{\mathrm{d}}$. The set $\{B_2, B_{m_2}^{\mathrm{d}}, B_{m_1}, B_1\}$ is a $T_{B_2\cap B_1^{\mathrm{d}}}$-invariant chain. By Lemma 11, it follows that

$$\sigma(T_{B_2\cap B_1^{\mathrm{d}}}) \subseteq \{0\} \cup \sigma(T_{B_2\cap B_{m_2}}) \cup \sigma(T_{B'}) \cup \sigma(T_{B_{m_1}\cap B_1^{\mathrm{d}}}),$$

hence $r(T_{B'}) = r(T)$ (in particular, $B' \neq \{0\}$). If $\widetilde{T}_{B'}$ is band irreducible, then the T-invariant chain $\{B_2, B_{m_2}^{\mathrm{d}}, B_{m_1}, B_1\}$ of **(a)** is obtained.

Consider the situation when the operator $\widetilde{T}_{B'}$ is not band irreducible, i.e., there exists a non-trivial $\widetilde{T}_{B'}$-invariant band $B'' \subset B'$. The band $B''' = B'' \oplus B_{m_1}$ is T-invariant. In fact, let $x \in B''$, $y \in B_{m_1}$. Then $Ty \in B_{m_1}$ and

$$Tx = TP_{B'}x = T_{B'}x + P_{B_{m_1}}TP_{B'}x + P_{B_{m_2}}TP_{B'}x \in B'''$$

as $P_{B_{m_2}}TP_{B'}x = 0$. The last equality is valid a via of $P_{B'}x \in B_{m_2}^{\mathrm{d}}$, hence $TP_{B'}x \in B_{m_2}^{\mathrm{d}}$. Thus, we obtain the T-invariant chain $\{B_2, B''', B_1\}$ satisfying $r(T_{B_2\cap(B''')^{\mathrm{d}}}) = r(T_{B'''\cap B_1^{\mathrm{d}}}) = r(T)$. Actually, if $r(T_{B'''\cap B_1^{\mathrm{d}}}) < r(T)$, then the band $B''' = B'' \oplus B_{m_1} \in \mathcal{E}_1$, which is impossible in view of the maximality of B_{m_1}. Let us verify the equality $r(T_{B_2\cap(B''')^{\mathrm{d}}}) = r(T)$. The relations

$$(B_{m_2} \oplus (B' \cap (B'')^{\mathrm{d}}))^{\mathrm{d}} = B_{m_2}^{\mathrm{d}} \cap (B' \cap (B'')^{\mathrm{d}})^{\mathrm{d}} =$$

$$= B_{m_2}^{\mathrm{d}} \cap ((B')^{\mathrm{d}} \oplus B'') = (B_{m_2}^{\mathrm{d}} \cap (B')^{\mathrm{d}}) \oplus (B_{m_2}^{\mathrm{d}} \cap B'') =$$

$$= (B_{m_2}^{\mathrm{d}} \cap (B_{m_1}^{\mathrm{d}} \cap B_{m_2}^{\mathrm{d}})^{\mathrm{d}}) \oplus B'' = (B_{m_2}^{\mathrm{d}} \cap (B_{m_1} + B_{m_2})) \oplus B'' = B'''$$

hold. Therefore, if $r(T_{B_2 \cap (B''')^{\mathrm{d}}}) < r(T)$ then the band $B_{m_2} \oplus (B' \cap (B'')^{\mathrm{d}}) \in \mathcal{E}_2$, which is impossible in view of the maximality of B_{m_2}.

Finally, $\{B_2, B'' \oplus B_{m_1}, B_1\}$ is the desired T-invariant chain of **(b)**. $\square$

Notice that in the case of (a) the situation when $B'' = B_2$, $B' = B_1$, is possible. It is equivalent to the band irreducibility of the operator $\widetilde{T}_{B_2 \cap B_1^{\mathrm{d}}}$.

Further, throughout this section, we will assume that the assumptions of Lemma 12 hold.

A pair of T-invariant projection bands (B_2, B_1), $B_2 \supseteq B_1$, is called *irreducible*, if either $\widetilde{T}_{B_2 \cap B_1^{\mathrm{d}}}$ is band irreducible or $r(T_{B_2 \cap B_1^{\mathrm{d}}}) < r(T)$. The pair (B_2, B_1) which is not irreducible, is called (a)-*reducible* if for the T-invariant chain $\{B_2, B_1\}$ the condition (a) of Lemma 12 holds, and is called (b)-*reducible* if it is not (a)-reducible.

Let $\mathcal{T} = \{B_n, \dots, B_0\}$ be a T-invariant chain. Define a new T-invariant chain $\mathcal{T}_1$ by a next rule. Consider the pair (B_1, B_0). If this pair is (a)-reducible, then there exist bands B'', B' such that $\{B_1, B'', B', B_0\}$ is a T-invariant chain, moreover $\widetilde{T}_{B'' \cap (B')^{\mathrm{d}}}$ is band irreducible and the inequalities $r(T_{B_1 \cap (B'')^{\mathrm{d}}}) < r(T)$, $r(T_{B' \cap B_0^{\mathrm{d}}}) < r(T)$ hold. Consequently, the T-invariant chain $\mathcal{T}_1 = \{B_n, \dots, B_2, B_1, B'', B', B_0\}$ which is different from $\mathcal{T}$, is defined. Remark that pairs (B_1, B''), (B'', B'), (B', B_0) are irreducible. If (B_1, B_0) is (b)-reducible, then there exists B''' such that $\{B_1, B''', B_0\}$ is a T-invariant chain, $r(T_{B_1 \cap (B''')^{\mathrm{d}}}) = r(T_{B''' \cap B_0^{\mathrm{d}}}) = r(T)$ (we call this procedure by a (b)-*decomposition* of the pair (B_1, B_0) into pairs (B_1, B'''), (B''', B_0)). In this case, we define $\mathcal{T}_1 = \{B_n, \dots, B_2, B_1, B''', B_0\}$. Clearly, $\mathcal{T}_1 \neq \mathcal{T}$. If the pair (B_1, B_0) is irreducible, then letting $\mathcal{T}_1 = \mathcal{T}$. So, $\mathcal{T}_1$ is obtained. Consider $\mathcal{T}_1$ and the pair (B_2, B_1) corresponding to $\mathcal{T}_1$. Performing with (B_2, B_1) actions which are described above, we obtain a T-invariant chain $\mathcal{T}_2$. Remark that if the pair (B_2, B_1) is irreducible, then put $\mathcal{T}_2 = \mathcal{T}_1$. Consider $\mathcal{T}_2$ and the pair (B_3, B_2) corresponding to $\mathcal{T}_2$. We continue this process and obtain, as a result, a T-invariant chain $\mathcal{T}_n$ which is called *generated* by $\mathcal{T}$ and will be denoted by $[\mathcal{T}]$. Notice that $[\mathcal{T}]$ is not unique determinate.

Theorem 13. *Let the assumptions of Lemma 12 be satisfied. There exists a T-invariant chain $\{E = B_n, B_{n-1}, \dots, B_1, B_0 = \{0\}\}$ such that if the equality $r(T_{B_i \cap B_{i-1}^{\mathrm{d}}}) = r(T)$ holds for some $i = \overline{1, n}$, then the operator $\widetilde{T}_{B_i \cap B_{i-1}^{\mathrm{d}}}$ is band irreducible.*

Proof. Consider the T-invariant chain $\mathcal{T} = \{B_1'',\ B_1'\}$, where $B_1'' = E$, $B_1' = \{0\}$. We generate by $\mathcal{T}$ a T-invariant chain $[\mathcal{T}]$. If $[\mathcal{T}] = \mathcal{T}$, then the theorem is proved, otherwise we consider $[[\mathcal{T}]]$. Again, if $[[\mathcal{T}]] = [\mathcal{T}]$, then the proof is finished, otherwise we will continue this process further. Show that we stop on some step, i.e., we obtain $\mathcal{T}'$ such that $[\mathcal{T}'] = \mathcal{T}'$. It will be the desired T-invariant chain. Assume by way of contradiction. The pair (B_1'', B_1') is $(b)_\infty$-*reducible* that is, after the (b)-decomposition of it at least one from two new pairs also will be (b)-reducible, and after the (b)-decomposition of this new pair, we obtain at least one (b)-reducible pair again, moreover continuing this process further, we will be obtain at least one (b)-reducible pair every time. Thus, if B_2 is a band such that $\{B_1'', B_2, B_1'\}$ is a T-invariant chain satisfying $r(T_{B_1'' \cap B_2^{\mathrm{d}}}) = r(T_{B_2 \cap (B_1')^{\mathrm{d}}}) = r(T)$, then one of pairs (B_1'', B_2) or (B_2, B_1') is $(b)_\infty$-reducible again. Denote it by (B_2'', B_2') and the second pair by $({}''B_2, {}'B_2)$. Remark that $B_2'' \cap (B_2')^{\mathrm{d}} \perp ({}''B_2) \cap ({}'B_2)^{\mathrm{d}}$. There exists a band B_3 such that $\{B_2'', B_3, B_2'\}$ is a T-invariant chain satisfying $r(T_{B_2''\cap B_3^{\mathrm{d}}}) = r(T_{B_3\cap(B_2')^{\mathrm{d}}}) = r(T)$, moreover

$$(B_2''\cap B_3^{\mathrm{d}}) \cup (B_3\cap(B_2')^{\mathrm{d}}) \subseteq B_2''\cap(B_2')^{\mathrm{d}}, \quad B_2''\cap B_3^{\mathrm{d}} \perp B_3\cap(B_2')^{\mathrm{d}}.$$

One of (B_2'', B_3) or (B_3, B_2') is $(b)_\infty$-reducible again. Denote it by (B_3'', B_3'), the second one by $({''B_3}, {'B_3})$. Continuing this process further, as a result, we obtain a sequence of pairs of bands $({''B_n}, {'B_n})$ which have the next properties:

$$r(T_{({''B_n})\cap({'B_n})^{\mathrm{d}}}) = r(T), \quad ({''B_i})\cap({'B_i})^{\mathrm{d}} \perp ({''B_j})\cap({'B_j})^{\mathrm{d}}, \ \ i\neq j, \ \ i,j>1.$$

Put $D_n = ({''B_{n+1}}) \cap ({'B_{n+1}})^{\mathrm{d}}$. Then ([5], p. 76) $T_{D_i} \perp T_{D_j}$, $i\neq j$, and there exists the operator T_∞ such that $\sum_{i=1}^{n} T_{D_i} \uparrow T_\infty$. Clearly, $r(T_\infty) = r(T) \notin \sigma_{\mathrm{ef}}(T_\infty)$. On the other hand, $r(T_{D_n}) = r(T) \notin \sigma_{\mathrm{ef}}(T_{D_n})$, moreover the equality $T_{D_n}x = r(T)x$ implies $T_\infty x = r(T)x$. Whence $\dim N(r(T)-T_\infty) = +\infty$, which is a contradiction. □

When T is Riesz operator (that is, $\sigma_{\mathrm{ef}}(T) = \{0\}$) the theorem about the Frobenius normal form, in some other view, was proved in [8].

Remark that the assumption $r(T)\notin\sigma_{\mathrm{ef}}(T)$ is essential for the conclusion of the previous theorem. For example, if $T: L_p \to L_p$, $1<p\leq\infty$, is Cesaro operator $Tx = \frac{1}{t}\int\limits_0^t x(s)\,ds$, then it is easy to see that the assertion of theorem does not hold for T (of course, $r(T)\in\sigma_{\mathrm{ef}}(T)$). In fact ([7], p. 99),

$$\sigma_{\mathrm{ef}}(T) = \{\lambda : |\lambda - \frac{q}{2}| = \frac{q}{2}\}, \ \ \sigma_{\mathrm{ew}}(T) = \{\lambda : |\lambda - \frac{q}{2}| \leq \frac{q}{2}\},$$

where $\frac{1}{p}+\frac{1}{q} = 1$ for $1<p<\infty$ and $q=1$ for $p=\infty$.

In the next lemmas the connection between the residue T_{-1} of $R(.,T)$ at $r(T)$ and residues at $r(T)$ of resolvents of a "diagonal" operators in the Frobenius normal form of T, is shown.

Lemma 14. *Let E be a Banach lattice and $T\geq 0$ an operator on E such that $r(T)\notin\sigma_{\mathrm{ef}}(T)$. Let B be a T-invariant projection band. Then $(T_{-1})_B = (T_B)_{-1}$, $(T_{-1})_{B^{\mathrm{d}}} = (T_{B^{\mathrm{d}}})_{-1}$, where residues of resolvents of the corresponding operators at $r(T)$ are considered.*

Proof. Assume B is non-trivial. According to Lemma 8 in a sufficiently small deleted neighbourhood U of $r(T)$ the operator $\lambda - T_B$ is invertible. Moreover we can suppose that for $\lambda\in U$ the band B is $R(\lambda,T)$- and $R(\lambda,T_B)$-invariant. Then, it is easily to verify for $\lambda\in U$ the equality $R(\lambda,T)_B|_B = R(\lambda,T_B)|_B$ holds, hence $(T_{-1})_B|_B = (T_B)_{-1}|_B$. This with help of the equality $R(\lambda,T_B)P_{B^{\mathrm{d}}} = \frac{1}{\lambda}P_{B^{\mathrm{d}}}$ and so the equality $(T_B)_{-1}P_{B^{\mathrm{d}}} = 0$, gives $(T_{-1})_B = (T_B)_{-1}$.

The band B°, where the polar is taken in the dual system $\langle E, E^*\rangle$, is T^*-invariant. As showed above, $(T^*_{-1})_{B^\circ} = ((T^*)_{B^\circ})_{-1}$. So from the equality $P_{B^\circ} = (P_{B^{\mathrm{d}}})^*$ we have

$$((T_{-1})_{B^{\mathrm{d}}})^* = (P_{B^{\mathrm{d}}})^* T^*_{-1} (P_{B^{\mathrm{d}}})^* = (P_{B^\circ} T^* P_{B^\circ})_{-1} = ((T_{B^{\mathrm{d}}})^*)_{-1} = ((T_{B^{\mathrm{d}}})_{-1})^*,$$

it follows that $(T_{-1})_{B^{\mathrm{d}}} = (T_{B^{\mathrm{d}}})_{-1}$. □

Lemma 15. *Let E be a Banach lattice and $T\geq 0$ an operator on E such that $r(T)\notin\sigma_{\mathrm{ef}}(T)$. Let $\{B_2, B_1\}$ be a T-invariant chain. Then for $B = B_2\cap B_1^{\mathrm{d}}$ the equalities $(T_{-1})_B = (T_B)_{-1}$,*

$(T_B)_{-1}|_B = (\widetilde{T}_B)_{-1}$ *hold, where residues of resolvents of the corresponding operators at* $r(T)$ *are considered.*

In particular, $r(T_B) < r(T)$ *iff* $(T_{-1})_B = 0$.

Proof. By Lemma 14 we have

$$(T_{-1})_{B_1 \cap B_2^{\mathrm{d}}} = P_{B_1 \cap B_2^{\mathrm{d}}} T_{-1} P_{B_1 \cap B_2^{\mathrm{d}}} = P_{B_1} (T_{B_2^{\mathrm{d}}})_{-1} P_{B_1} = (T_{B_1 \cap B_2^{\mathrm{d}}})_{-1}.$$

For the verification of the second equality it suffices to observe that by λ from a sufficiently small deleted neighbourhood of $r(T)$ we have $R(\lambda, T_B)|_B = R(\lambda, \widetilde{T}_B)$. $\square$

3 Weyl spectrum $\sigma_{\mathrm{ew}}(T)$ for operators on Banach lattices.

We begin with the simply corollaries of Theorem 3.

Theorem 16. *Let* E *be a Banach lattice and* T *a bounded linear operator on* E*. Then*

$$\sigma_{\mathrm{ew}}(T) = \sigma_{\mathrm{ew}}^{+}(T) = \bigcap_{0 \le K \in \mathcal{F}(E)} \sigma(T+K) = \bigcap_{0 \le K \in \mathcal{K}(E)} \sigma(T-K) = \bigcap_{0 \le K \in \mathcal{F}(E)} \sigma(T-K).$$

Corollary 17. *Let* E *be a Banach lattice and* $T \in \mathcal{L}(E)$*. Then each of the following conditions ensures that* $\lambda_0 \notin \sigma(T+K)$ *for some positive operator* $K \in \mathcal{F}(E)$:

(a) $\lambda_0 \in \sigma(T)$ *and there is path lying outside of* $\sigma_{\mathrm{ef}}(T)$ *and joining* λ_0 *with a point in* $\rho(T)$;

(b) $\lambda_0 \in \partial\sigma(T)$*, range of* $\lambda_0 - T$ *is closed and either* $\mathrm{nul}(\lambda_0 - T) < \infty$ *or* $\mathrm{def}(\lambda_0 - T) < \infty$.

Proof. Both conditions **(a)** ([2], p. 300) and **(b)** ([7], p. 76) are equivalent to the fact that the point λ_0 is an isolated point of $\sigma(T)$ and $\lambda_0 \notin \sigma_{\mathrm{ef}}(T)$. Whence $\lambda_0 \notin \sigma_{\mathrm{ew}}(T)$. Indeed, if T_{-1} is a residue of $R(., T)$ at λ_0, then by spectral mapping theorem ([2], p. 260) the equality $\sigma(T + aT_{-1}) = (\sigma(T) \setminus \{\lambda_0\}) \cup \{\lambda_0 + a\}$ holds for all a. Now the desired assertion follows from Theorem 16. $\square$

A glance at the proof of Theorem 3 guarantees that if $\lambda_0 \notin \sigma(T + K_1 + iK_2)$, where the real operators $K_i \in \mathcal{K}(E)$, then as K, $0 \le K \in \mathcal{K}(E)$, satisfies $\lambda_0 \notin \sigma(T+K)$, we can take an operator of a form $K_1^+ + a_1 K_1^- + a_2 K_2^+ + a_3 K_3^-$ for some $a_i \ge 0$, $i = 1,3$. A question arises naturally, namely, what are concrete K (depending upon λ_0 and T) satisfying the relation $\lambda_0 \notin \sigma(T+K)$? The next theorem gives an answer to this question for the case $\lambda_0 = r(T)$.

Theorem 18. *Let* T *be a positive operator on a Banach lattice* E *such that* $r(T) \notin \sigma_{\mathrm{ef}}(T)$ *and* $R(\lambda, T) = \sum_{i=-\infty}^{+\infty} (\lambda - r(T))^i T_i$ *for* λ *close to* $r(T)$*. Then*:

(a) $r(T) \notin \sigma(T + a|T_{-1}|)$ *for an arbitrary non-zero number* a;

(b) *if* $r(T)$ *is a simply eigenvalue of the operator* T*, i.e.,* $\dim N(r(T) - T) = 1$*, then the relation* $r(T) \notin \sigma(T + z^* \otimes z)$ *holds for an arbitrary functional* $z^* \ge 0$ *and element* $z \ge 0$ *such that* $z^*(T_{-m} z) > 0$*, where* m *is the order of a pole of* $R(., T)$ *at* $r(T)$;

(c) *if* E *is either AM- or AL-space, the point* $r(T)$ *is a pole of* $R(., T)$ *the order two, then for every* $a > 0$ $r(T) \notin \sigma(T + (aT_{-1} + nT_{-2})^+)$ *for sufficiently large* n (*depending upon* a).

Proof. **(a)** Fix a number $a \ne 0$. With help of a passage to the dual space we can assume that all conditions of Lemma 12 are satisfied. Let $\{E = B_n, B_{n-1}, \ldots, B_0 = \{0\}\}$ be a T-invariant chain from Theorem 13. Put $D_i = B_i \cap B_{i-1}^{\mathrm{d}}$, $i = 1,n$. If for some i $r(T_{D_i}) < r(T)$, then by Lemma 15 $|T_{-1}|_{D_i} = |(T_{-1})_{D_i}| = 0$ so $r(T) \notin \sigma(T_{D_i} + a|T_{-1}|_{D_i})$. Let $r(T_{D_i}) = r(T)$ for some index i thus, $\widetilde{T}_{D_i}$ is band irreducible hence [3] $(\widetilde{T}_{D_i})_{-1} \geq 0$. Again according to Lemma 15 we have $(T_{D_i})_{-1}|_{D_i} = (\widetilde{T}_{D_i})_{-1}$ and

$$(T_{D_i})_{-1} = (T_{D_i})_{-1}P_{D_i} + (T_{-1})_{D_i}P_{D_i^{\mathrm{d}}} = (T_{D_i})_{-1}P_{D_i} \geq 0$$

therefore,

$$(T + a|T_{-1}|)_{D_i} = T_{D_i} + a|T_{-1}|_{D_i} = T_{D_i} + a|(T_{D_i})_{-1}| = T_{D_i} + a(T_{D_i})_{-1}.$$

The relation (see the proof of Corollary 17) $r(T) \notin \sigma(\widetilde{T}_{D_i} + a(\widetilde{T}_{D_i})_{-1})$ and Lemma 10 imply $r(T) \notin \sigma(T_{D_i} + a(T_{D_i})_{-1})$. Thus, $r(T) \notin \bigcup_{i=1}^{n} \sigma((T + a|T_{-1}|)_{D_i})$. By Lemma 11 we have $r(T) \notin \sigma(T + a|T_{-1}|)$.

The assertions $(\mathbf{b})$ and $(\mathbf{c})$ are proved in [4] (Theorem 4, proofs of (b), (c)). $\square$

4 Some properties of $\sigma^-_{\mathrm{ew}}(T)$.

As the following example shows, in general the equality $\sigma^-_{\mathrm{ew}}(T) = \sigma_{\mathrm{ew}}(T)$ does not hold.

Example 19 (*an operator* $T \geq 0$ *such that* $\sigma_{\mathrm{ef}}(T) \subset \sigma_{\mathrm{ew}}(T) \subset \sigma^-_{\mathrm{ew}}(T) \subset \sigma(T)$, *where all inclusions are proper*). Consider the Banach lattice $E = \ell_2 \oplus \ell_2 \oplus \ell_2$. Let T_1 be the forward shift operator on ℓ_2, T_2 be the backward shift operator on ℓ_2, define the operator T_3 by $T_3 = \frac{1}{2}T_1$, and let K_1 be an arbitrary compact positive operator on ℓ_2 satisfies $r(K_1) > 1$. Recall that ([7], p. 72-73) $\sigma_{\mathrm{ef}}(T_1) = \sigma_{\mathrm{ef}}(T_2) = \{\lambda : |\lambda| = 1\}$ and $\sigma_{\mathrm{ew}}(T_1) = \sigma_{\mathrm{ew}}(T_2) = \{\lambda : |\lambda| \leq 1\}$, moreover $\mathrm{ind}(\lambda - T_1) = -1$ and $\mathrm{ind}(\lambda - T_2) = 1$ for $|\lambda| < 1$. Consider the operator $T : E \to E$ defined by $T = (T_1 + K_1) \oplus T_2 \oplus T_3$. The operator $\lambda - T$ is a Fredholm operator, i.e., it belongs to $\mathcal{F}_{red}(E)$, iff the operators $\lambda - (T_1 + K_1)$, $\lambda - T_i \in \mathcal{F}_{red}(\ell_2)$, where $i = 2, 3$, by this $\lambda - (T_1 + K_1) \in \mathcal{F}_{red}(\ell_2)$ iff $\lambda - T_1 \in \mathcal{F}_{red}(\ell_2)$, hence $\sigma_{\mathrm{ef}}(T) = \{\lambda : |\lambda| = 1\} \cup \{\lambda : |\lambda| = \frac{1}{2}\}$. Next, for $\lambda \notin \sigma_{\mathrm{ef}}(T)$ the equalities

$$\mathrm{ind}(\lambda - T) = \mathrm{ind}(\lambda - (T_1 + K_1)) + \sum_{i=2}^{3} \mathrm{ind}(\lambda - T_i), \;\; \mathrm{ind}(\lambda - (T_1 + K_1)) = \mathrm{ind}(\lambda - T_1)$$

hold so $\sigma_{\mathrm{ew}}(T) = \{\lambda : |\lambda| \leq \frac{1}{2}\} \cup \{\lambda : |\lambda| = 1\}$. The inclusion

$$\sigma^-_{\mathrm{ew}}(T) \subseteq \sigma^-_{\mathrm{ew}}(T_1 \oplus T_2 \oplus T_3) \tag{*}$$

is valid. Indeed, if for some $K \in \mathcal{K}(E)$, $0 \leq K \leq T_1 \oplus T_2 \oplus T_3$,

$$\lambda \notin \sigma(T_1 \oplus T_2 \oplus T_3 - K) = \sigma(T - (K + K_1 \oplus 0 \oplus 0)),$$

then $\lambda \notin \sigma^-_{\mathrm{ew}}(T)$ as $0 \leq K + K_1 \oplus 0 \oplus 0 \leq T$. From (*) we have

$$\sigma^-_{\mathrm{ew}}(T) \subseteq \sigma^-_{\mathrm{ew}}(T_1 \oplus T_2 \oplus T_3) \subseteq \sigma(T_1 \oplus T_2 \oplus T_3) = \{\lambda : |\lambda| \leq 1\}. \tag{**}$$

Next,

$$\sigma^-_{\mathrm{ew}}(T_2) \subseteq \sigma^-_{\mathrm{ew}}(T). \tag{***}$$

Actually, let $\lambda \notin \sigma(T-K)$, where $K \in \mathcal{K}(E)$ and $0 \le K \le T$. Then the restriction of K to the band $0 \oplus \ell_2 \oplus 0$ defines the compact operator $K_2 \ge 0$ on ℓ_2 satisfying $0 \le K_2 \le T_2$. Clearly, $\lambda \notin \sigma(T_2 - K_2)$ and (***) is proved. From inclusions $\sigma_{\mathrm{ew}}(T_2) \subseteq \sigma^-_{\mathrm{ew}}(T_2)$, (***), (**) and the equality $\sigma_{\mathrm{ew}}(T_2) = \{\lambda : |\lambda| \le 1\}$ we have $\sigma^-_{\mathrm{ew}}(T) = \{\lambda : |\lambda| \le 1\}$. Remain to observe that the inclusion $\sigma^-_{\mathrm{ew}}(T) \subset \sigma(T)$ is proper as $r(K_1) > 1$. $\square$

How it has shown in the previous section if $r(T) \notin \sigma_{\mathrm{ef}}(T)$, then $r(T) \notin \sigma_{\mathrm{ew}}(T)$. The following theorem gives the conditions under which $r(T) \notin \sigma^-_{\mathrm{ew}}(T)$.

Theorem 20. *Let T be a positive operator on a Banach lattice on E such that $r(T) \notin \sigma_{\mathrm{ef}}(T)$ and there exists a net of a compact operators K_α satisfying*

$$0 \le K_\alpha x \uparrow Tx \tag{A}$$

for all $x \ge 0$. Then each of the following conditions ensures that $r(T - K_\alpha) < r(T)$ for some α so $r(T) \notin \sigma^-_{\mathrm{ew}}(T)$:

(a) *the point $r(T)$ is a simply pole of the resolvent $R(.,T)$, moreover the residue at this point is a strictly positive operator*;

(b) *the order continuous dual $E^\sim_n$ separates the points of E, T is order continuous.*

Proof. Under the assumptions of **(a)** the desired assertion is proved in [4] (Theorem 5, (a)). Suppose **(b)** is true. First of all we consider the case when E is Dedekind complete. Let $\{E = B_n, B_{n-1}, \dots, B_0 = \{0\}\}$ be a T-invariant chain from Theorem 13. Put $D_i = B_i \cap B^{\mathrm{d}}_{i-1}$, $i = 1, n$. If for some index i $r(T_{D_i}) < r(T)$, then, obviously, $r((T - K_\alpha)_{D_i}) < r(T)$. If $r(T_{D_i}) = r(T)$, then a glance at the part **(a)** guarantees that the band irreducibility of the operator $\widetilde{T}_{D_i}$ and the relation $(\widetilde{K_\alpha})_{D_i} \uparrow \widetilde{T}_{D_i}$ imply $r(\widetilde{T}_{D_i} - (\widetilde{K_\alpha})_{D_i}) < r(T)$ for some α (all conditions of **(a)** for the operator $\widetilde{T}_{D_i}$ are valid [3]). According to Lemmas 9 and 11 we get the desired conclusion.

In the general case, the band $E^\sim_n$ is T^*- and K^*_α-invariant. Restrictions of these operators to $E^\sim_n$ we denote by T' and K'_α, respectively. By this $K'_\alpha \uparrow T'$. Let $r(T - K_\alpha) = r(T)$ for all α. From Lemma 5 we know that $r(T - K_\alpha) = r(T' - K'_\alpha)$ and $r(T') = r(T)$, but, as showed above, $r(T' - K'_{\alpha_0}) < r(T')$ for some α_0, a contradiction. $\square$

It is easy to see from the proof, the condition $K_\alpha \in \mathcal{K}(E)$ is only playing the role for the conclusion $r(T) \notin \sigma^-_{\mathrm{ew}}(T)$. In others words, *under the assumptions of Theorem 20 for every net of an operators T_α, $0 \le T_\alpha x \uparrow Tx$ for all $x \ge 0$, the inequality $r(T - T_\alpha) < r(T)$ holds for some α*. It suffices to observe that by Lemma 8 the assertion of the part (a) of Theorem 5 from [4] is true without the assumption $K_\alpha \in \mathcal{K}(E)$.

Corollary 21. *Let E be a Banach lattice and $T \ge 0$ an operator on E such that $r(T) \notin \sigma_{\mathrm{ef}}(T)$ and there exists an increasing net $K_\alpha \in \mathcal{K}(E)$ satisfying*

$$0 \le K_\alpha x \to Tx \tag{A$_\mathrm{s}$}$$

for all $x \ge 0$, where the convergence is in the norm. Then $r(T - K_\alpha) < r(T)$ for some α so $r(T) \notin \sigma^-_{\mathrm{ew}}(T)$.

Proof. The desired assertion follows from the part (b) of the previous theorem as $K^*_\alpha \uparrow T^*$. $\square$

For a validity of the inequality $r(T - K_\alpha) < r(T)$ in Theorem 20 the assumption about the order continuity of the operator T is essential. Actually, consider the space ℓ_∞, the sequence

$z_n = (\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots) \in \ell_\infty$ and a functional $x^* \in \ell_\infty^*$ such that x^* is positive, $x^* \perp \ell_1$ and $\|x^*\| = 1$. Then $K_n = x^* \otimes z_n \uparrow x^* \otimes e = T$, where the element $e = (1, 1, \ldots)$. By this $r(T - K_n) = r(T) = 1$ for all n. Nevertheless, remark that $r(T) \notin \sigma_{\mathrm{ew}}^-(T) = \{0\}$.

5 When is the inclusion $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$ true?

The fact that $r(T) \in \sigma_{\mathrm{ef}}(T)$ implies $r(T) \in \sigma_{\mathrm{el}}(T)$ for an operator $T \geq 0$ on a Banach lattice E, was shown in [4] (Theorem 7). In fact (as can easily be seen from the proof) it is true with $\sigma_{\mathrm{ew}}(T)$ instead of $\sigma_{\mathrm{ef}}(T)$. Below the conditions when the more general inclusion $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$ is true, will be given.

First of all, note that in next cases the relations $0 \leq Q \leq K$, $K \in \mathcal{K}(E)$, imply $Q \in \mathcal{K}(E)$ and so $\sigma_{\mathrm{ew}}^-(T) = \sigma_{\mathrm{el}}(T)$, hence $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$:

(a) E and E^* have order continuous norms ([5], p. 279);

(b) either E or E^* is atomic with an order continuous norm [12].

Below for a regular operator T on E through $\sigma_{\mathrm{o}}(T)$ will be denoted the *order spectrum* of T, i.e., ([9], §4.5; see also [2], §7.4) the set

$$\sigma_{\mathrm{o}}(T) = \{\lambda : \lambda - T \text{ does not have a regular inverse on } E\};$$

by this $r_{\mathrm{o}}(T) = \max\limits_{\lambda \in \sigma_{\mathrm{o}}(T)} |\lambda|$. Recall that the *pure order spectrum* of an operator T is the set $\sigma_{\mathrm{po}}(T) = \sigma_{\mathrm{o}}(T) \setminus \sigma(T)$ (the inclusion $\sigma(T) \subseteq \sigma_{\mathrm{o}}(T)$ always holds). A positive operator T is said to be an *operator with an almost d-empty pure spectrum*, if at least for one natural n, the set $\sigma_{\mathrm{po}}(S) = \emptyset$ for all $0 \leq S \leq T^{*(n)}$ (where $T^{*(n)}$ denotes the n^{th} adjoint to T).

Recall also ([11], p. 244) that an operator T on E is called *cone absolutely summing* if for every an unconditionally convergent series $\sum\limits_{n=1}^{\infty} x_n$, $x_n \geq 0$, the series $\sum\limits_{n=1}^{\infty} Tx_n$ is absolutely convergent, and is called *majorizing* if for every $x_n \to 0$, the sequence Tx_n is order bounded. If E is an AL-space, then every $T \in \mathcal{L}(E)$ is cone absolutely summing, and if E is an AM-space, then every $T \in \mathcal{L}(E)$ is majorizing ([11], p. 248).

Example 22 (*the examples of an operators with an almost d-empty pure spectrum*):

(a) A positive cone absolutely summing operator T. Indeed, E^* has ([11], p. 299) the property (P) (i.e., there exists a positive, contractive projection $E^{***} \to E^*$), and every operator $S \geq 0$ which is dominated by T^*, is ([11], p. 249) majorizing so ([9], p. 303) $\sigma_{\mathrm{po}}(S) = \emptyset$.

(b) A positive majorizing operator. Arguments are similar to given in the part **(a)**.

(c) A positive operator T on a Dedekind complete Banach lattice E having the next properties: for all $x \geq 0$ there exists $z \geq x$ such that $T(E_z) \subseteq E_z$ and the restriction $T|_{E_z}$ of T to an AM-space E_z with an unit z is weakly compact. Indeed, if $0 \leq S \leq T$, then by Wickstead theorem ([5], p. 289) the operator $S|_{E_z}$ is also weakly compact so ([9], p. 303) $\sigma_{\mathrm{po}}(S) = \emptyset$.

(d) A positive orthomorphism T. If $0 \leq S \leq T^*$, then S is also orthomorphism and ([9], p. 309) $\sigma_{\mathrm{po}}(S) = \emptyset$.

(e) Let $\mathcal{I}$ be an (not necessarily closed) algebraic ideal in $\mathcal{L}(E)$ such that $\mathcal{I} \subseteq \mathcal{L}_r(E)$ and the relations $0 \leq S \leq T$, $T \in \mathcal{I}$, imply $S \in \mathcal{I}$ (for example, the ideal of the Hilbert-Schmidt operators on a Hilbert lattice). Then every positive operator from $\mathcal{I}$ has an almost d-empty pure spectrum. Indeed, if $\mathcal{L}_r(E) = \mathcal{L}(E)$, then [1] E is order isomorphic either to an AL- or

AM-space so from parts **(a)** and **(b)** the desired assertion follows. Let $\mathcal{L}_r(E) \neq \mathcal{L}(E)$. For an arbitrary positive operator $T \in \mathcal{I}$ and $\lambda \in \rho(T)$ we have $R(\lambda, T) = \frac{1}{\lambda} I + \frac{1}{\lambda} T R(\lambda, T) \in \mathcal{L}_r(E)$ as $\lambda \neq 0$ and $TR(\lambda, T) \in \mathcal{L}_r(E)$, therefore $\sigma_{\mathrm{po}}(T) = \emptyset$. □

Theorem 23. *Let T be a positive operator with an almost d-empty pure spectrum on a Banach lattice E. Then $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$.*

Proof. Let $\lambda \notin \sigma_{\mathrm{el}}(T)$, that is, the operator $R = \lambda - (T - Q)$ is invertible, where $0 \leq Q \leq T$, $Q \leq K \in \mathcal{K}(E)$. Then for some n sets $\sigma_{\mathrm{po}}(S) = \emptyset$ if $0 \leq S \leq T^{*(n)}$. Therefore, we have $\lambda \notin \sigma_{\mathrm{o}}(T^{*(n)} - Q^{*(n)}) = \sigma(T^{*(n)} - Q^{*(n)})$, i.e., the operator $(R^{*(n)})^{-1}$ is presented in the form $(R^{*(n)})^{-1} = R_1 + iR_2$, where the real operators R_1 and R_2 are regular. So the operators $R_1 Q^{*(n)}$ and $R_2 Q^{*(n)}$ are dominated by a compact operators. By Aliprantis-Burkinshaw theorem ([2], p. 90) $((R^{*(n)})^{-1} Q^{*(n)})^3$ is compact. Finally, the operator

$$\lambda - T^{*(n)} = R^{*(n)} - Q^{*(n)} = R^{*(n)}(I - (R^{*(n)})^{-1} Q^{*(n)})$$

is a Fredholm operator of index zero, hence $\lambda \notin \sigma_{\mathrm{ew}}(T)$. □

Thus, it follows from previous results that for all classical Banach lattices the inclusion $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$ holds. In particular, $\sigma_{\mathrm{el}}(T) \neq \emptyset$. *Nevertheless, it is not known if the inclusion $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{el}}(T)$ is true for an arbitrary Banach lattice E and a positive operator T.*

It turns out, however that a similar inclusion holds for a *Lozanovsky's order essential spectrum* of a positive operator T on a Banach lattice E:

$$\sigma_{\mathrm{oel}}(T) = \bigcap_{\substack{0 \leq Q \leq T \\ Q \leq K \in \mathcal{K}(E)}} \sigma_{\mathrm{o}}(T - Q).$$

Theorem 24. *Let E be a Banach lattice and $T \geq 0$ an operator on E. Then*:

(a) *the inclusion $\sigma_{\mathrm{ew}}(T) \subseteq \sigma_{\mathrm{oel}}(T)$ holds, in particular, $\sigma_{\mathrm{oel}}(T) \neq \emptyset$;*

(b) *if $r(T) \in \sigma_{\mathrm{oel}}(T)$, then $r(T) \in \sigma_{\mathrm{el}}(T)$.*

Proof. The part **(a)** can be check analogously to Theorem 23. Show the validity of **(b)**. For Q, $0 \leq Q \leq T$, $Q \leq K \in \mathcal{K}(E)$ the inclusion $r(T) \in \sigma_{\mathrm{o}}(T - Q)$ holds. So

$$r_{\mathrm{o}}(T - Q) \geq r(T) = r_{\mathrm{o}}(T) \geq r_{\mathrm{o}}(T - Q) = r(T - Q),$$

hence $r(T) \in \sigma(T - Q)$. □

Importantly to observe that by analogy of $\sigma_{\mathrm{oel}}(T)$ "order" spectra $\sigma_{\mathrm{oew}}(T)$ and $\sigma^{\pm}_{\mathrm{oew}}(T)$ can be considered.

Under the assumptions of Theorem 20 $r(T) \notin \sigma_{\mathrm{ef}}(T)$ implies $r(T) \notin \sigma_{\mathrm{el}}(T)$ as the inclusion $\sigma_{\mathrm{el}}(T) \subseteq \sigma^{-}_{\mathrm{ew}}(T)$ is true. It remains valid after the replacement of (A) by the weaker condition: there exist nets of a positive operators Q_α and a compact operators K_α such that

$$0 \leq Q_\alpha x \uparrow Tx, \ \ Q_\alpha \leq K_\alpha, \tag{A$_1$}$$

for every $x \geq 0$ (see the remarks after the proof of Theorem 20). By Lozanovsky's theorem ([2], p. 199) the condition (A_1) holds for every positive integral operator on a Banach function space.

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Egor A. Alekhno
Belarussian State University
Faculty of Mechanics and Mathematics
Minsk, Belarus
e-mail: Alekhno@bsu.by