When a C^*-algebra is a coefficient algebra for a given endomorphism

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The paper presents a criterion for a C^*-algebra to be a coefficient algebra associated with a given endomorphism.

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Contents

1 Introduction. Coefficient algebras and transfer operators

The notion of a coefficient algebra was introduced in [1] in connection with the study of extensions of C^*-algebras by partial isometries. In this paper the authors investigated the C^*-algebra $C^*(\mathcal{A}, V)$ generated by a C^*-algebra $\mathcal{A} \subset L(H)$ and a partial isometry $V \in L(H)$ such that the mapping $\mathcal{A} \ni a \mapsto VaV^*$ is an endomorphism of $\mathcal{A}$. It was uncovered that this investigation can be carried out successfully if $\mathcal{A}$ and V satisfy the following three conditions

$$Va = VaV^*V, \qquad a \in \mathcal{A}; \tag{1.1}$$

$$VaV^* \in \mathcal{A}, \qquad a \in \mathcal{A} \tag{1.2}$$

and

$$V^*aV \in \mathcal{A}, \qquad a \in \mathcal{A}. \tag{1.3}$$

In [1] the algebras possessing these properties were called the *coefficient algebras* (for $C^*(\mathcal{A}, V)$). This term was justified by the observation that any element in $C^*(\mathcal{A}, V)$ can be presented as a Fourier like series with coefficients from $\mathcal{A}$ ([1], Theorems 2.7, 2.13).

It was also shown ([1], Proposition 2.2) that if $\mathcal{A}$ contains the identity of $L(H)$ then the coefficient algebra can be equivalently defined by conditions (1.2), (1.3) along with the condition

$$V^*V \in Z(\mathcal{A}) \tag{1.4}$$

where $Z(\mathcal{A})$ is the center of $\mathcal{A}$ (it is worth mentioning that if $\mathcal{A}$ contains the identity then conditions (1.1), (1.2) and (1.3) automatically imply that V is a partial isometry and the mapping $\mathcal{A} \ni a \mapsto VaV^*$ is an endomorphism ([1], Proposition 2.2)).

In [1] there was also presented a certain procedure of constructing the coefficient algebras starting from certain initial algebras that are not coefficient ones. Following this construction in [2] the maximal ideal space of the arising commutative coefficient algebras (in the situation when the initial algebra is commutative) was described. The development of this and other constructions, ideas and methods led to the general construction of covariance (crossed product type) C^*-algebra for partial dynamical system [3].

The interrelation between the coefficient algebras and various crossed product type structures is not incidental. This is due to the fact that these algebras can also be described by means of the so-called transfer operators. Here is the description. Using (1.4) and recalling that V is a partial isometry one obtains for any $a, b \in \mathcal{A}$ the following relations

$$V^*VaV^*bV = aV^*VV^*bV = aV^*bV. \tag{1.5}$$

If we introduce the mappings

$$\delta\colon \mathcal{A} \to \mathcal{A}, \quad \delta(a) = VaV^* \tag{1.6}$$

and

$$\delta_*\colon \mathcal{A} \to \mathcal{A}, \quad \delta_*(a) = V^*aV \tag{1.7}$$

then as it has been already observed δ is an endomorphism of $\mathcal{A}$ and (1.5) means that δ_* is a linear positive mapping satisfying the condition

$$\delta_*(\delta(a)b) = a\delta_*(b), \qquad a, b \in \mathcal{A}. \tag{1.8}$$

The mappings δ_* possessing the latter property were called by R. Exel [4] the *transfer operators* (for a given endomorphism δ). It was shown in [4] that these objects play a significant role (they belong to the key constructive elements) in the theory of crossed products of C^*-algebras by endomorphisms.

Note that if any operator V satisfies the condition

$$V^*VaV^*bV = aV^*bV, \qquad a, b \in \mathcal{A} \tag{1.9}$$

then by passage to the adjoint we also obtain

$$V^*bVaV^*V = V^*bVa, \qquad a, b \in \mathcal{A}. \tag{1.10}$$

Relations (1.9) and (1.10) imply in particular that

$$aV^*V = V^*VaV^*V = V^*Va \tag{1.11}$$

that is (1.4) is true. Thus the coefficient algebra can also be defined as an algebra satisfying conditions (1.2), (1.3) and (1.9). In other words a C^*-algebra $\mathcal{A} \subset L(H)$ containing the identity of $L(H)$ is the coefficient algebra for $C^*(\mathcal{A}, V)$ iff the mapping δ (1.6) is an endomorphism and the mapping δ_* (1.7) is a transfer operator for δ.

The foregoing reasoning shows on the one hand the importance of coefficient algebras in a number of fields and on the other hand one arrives at the natural problem: when a given C^*-algebra is a coefficient algebra? The precise formulation of the problem is the following.

Let $\mathcal{A}$ be an (abstract) C^*-algebra containing an identity and δ be an (abstract) endomorphism of $\mathcal{A}$. Does there exist a triple (H, π, U) consisting of a Hilbert space H, faithful non-degenerate representation $\pi : \mathcal{A} \to L(H)$ and a linear continuous operator $U\colon H \to H$ such that for every $a \in \mathcal{A}$ the following conditions are satisfied

$$\pi(\delta(a)) = U\pi(a)U^*, \qquad U^*\pi(a)U \in \pi(\mathcal{A}) \tag{1.12}$$

and

$$U\pi(a) = \pi(\delta(a))U? \tag{1.13}$$

That is $\pi(\mathcal{A})$ is the coefficient algebra for $C^*(\pi(\mathcal{A}), U)$ under the fixed endomorphism $U \cdot U^*$.

The answer to this problem is the theme of the present article.

Since δ is an endomorphism it follows that $\delta(1)$ is a projection and so (1.12) implies that UU^* is a projection, so U is a partial isometry. The above discussion means also that instead of condition (1.13) one can use equivalently condition (1.4) or (1.9) (for $\pi(a)$ and U respectively).

The article is organized as follows. In Section 2 we study some properties of transfer operators and in particular we establish the conditions of existence of the so-called complete transfer operators. By means of these operators in Section 3 we give the answer to the problem stated above.

2 Non-degenerate and complete transfer operators

Let $\mathcal{A}$ be a C^*-algebra with an identity 1 and $\delta\colon \mathcal{A} \to \mathcal{A}$ be an endomorphism of this algebra. We start with recalling some definitions and facts concerning transfer operators. Everything henceforth up to Proposition 2.3 is borrowed from [4].

A linear map $\delta_* : \mathcal{A} \to \mathcal{A}$ is called a *transfer operator* for the pair $(\mathcal{A}, \delta)$ if it is continuous and positive an such that

$$\delta_*(\delta(a)b) = a\delta_*(b), \qquad a, b \in \mathcal{A}. \tag{2.1}$$

Since δ_* is positive it is self-adjoint an hence we have the symmetrized version of (2.1)

$$\delta_*(b\delta(a)) = \delta_*(b)a. \tag{2.2}$$

In particular (2.1) and (2.2) imply that $\delta_*(\mathcal{A})$ is a two-sided ideal.

By (2.1) and (2.2) we also have that

$$a\delta_*(1) = \delta_*(\delta(a)1) = \delta_*(1\delta(a)) = \delta_*(1)a \tag{2.3}$$

for all $a \in \mathcal{A}$, so $\delta_*(1)$ is a positive central element in $\mathcal{A}$ and $\delta_*(1)\mathcal{A}$ is a two-sided ideal.

It is also worth mentioning that (2.1) and (2.2) imply

$$\delta_*(\,\cdot\,) = \delta_*(\delta(1)\,\cdot\,) = \delta_*(\,\cdot\,\delta(1)). \tag{2.4}$$

Proposition 2.1 ([4], Proposition 2.3) *Let δ_* be a transfer operator for the pair $(\mathcal{A}, \delta)$. Then the following are equivalent:*

(*i*) *the composition $E = \delta \circ \delta_*$ is a conditional expectation onto $\delta(\mathcal{A})$,*

(*ii*) *$\delta \circ \delta_* \circ \delta = \delta$,*

(*iii*) *$\delta(\delta_*(1)) = \delta(1)$.*

If the equivalent conditions of Proposition 2.1 hold then R. Exel calls δ_* a *non-degenerate* transfer operator.

Proposition 2.2 ([4], Proposition 2.5) *Let δ_* be a non-degenerate transfer operator. Then $\mathcal{A}$ may be written as the direct sum of ideals*

$$\mathcal{A} = \operatorname{Ker}\delta \oplus \operatorname{Im}\delta_*$$

Proposition 2.3 ([4], Proposition 4.1) *$\delta(\mathcal{A})$ is a hereditary subalgebra of $\mathcal{A}$ iff $\delta(\mathcal{A}) = \delta(1)\mathcal{A}\delta(1)$.*

Remark 2.4 If δ is given then in general a non-degenerate transfer operator δ_* (if it exists) is not defined in a unique way.

Example. Let $\mathcal{A} = C(X)$ where $X = \mathbf{R}\,(\mathrm{mod}\,1)$ and let $\delta : \mathcal{A} \to \mathcal{A}$ be given by the formula

$$\delta(a)(x) = a(2x\,(\mathrm{mod}\,1)).$$

Take any continuous function ρ on X having the properties

$$0 \le \rho(x) \le 1, \qquad x \in X,$$

$$\rho\left(x + \frac{1}{2}\right) = 1 - \rho(x), \qquad 0 \le x \le \frac{1}{2}.$$

Let us define the operator δ_* on $C(X)$ by the formula

$$\delta_*(a)(x) = a\left(\frac{x}{2}\right)\rho\left(\frac{x}{2}\right) + a\left(\left[\frac{x}{2} + \frac{1}{2}\right]\right)\rho\left(\left[\frac{x}{2} + \frac{1}{2}\right]\right),$$

where $[x] = x\,(\mathrm{mod}\,1)$. Clearly for any ρ chosen δ_* is a transfer operator for δ and since $\delta_*(1) = 1$ it is non-degenerate.

Now we note that the non-degeneracy of transfer operators also implies some additional properties.

Proposition 2.5 *Let δ_* be a non-degenerate transfer operator. Then*

1) *$\delta_*(1)$ is an orthogonal projection lying in the center of $\mathcal{A}$,*

2) *$\delta_*(\mathcal{A}) = \delta_*(1)\mathcal{A}$.*

3) *$\delta_* \colon \delta(\mathcal{A}) \to \delta_*(\mathcal{A})$ is a *-isomorphism (the inverse to the *-isomorphism $\delta \colon \delta_*(\mathcal{A}) \to \delta(\mathcal{A})$).*

Proof. 1) It has been already observed that $\delta_*(1)$ is a central element in $\mathcal{A}$ so it is enough to show that it is a projection.

By Proposition 2.2 we have that

$$\mathcal{A} = \operatorname{Ker}\delta \oplus \operatorname{Im}\delta_*.$$

Thus the mapping

$$\delta \colon \operatorname{Im}\delta_* \to \delta(\mathcal{A}) \tag{2.5}$$

is a *-isomorphism and so the mapping

$$\delta^{-1} \colon \delta(\mathcal{A}) \to \operatorname{Im}\delta_* \tag{2.6}$$

is a *-isomorphism as well.

Since δ_* is non-degenerate it follows from (iii) of Proposition 2.1 that

$$\delta(\delta_*(1)) = \delta(1). \tag{2.7}$$

Now (2.6) and (2.7) imply

$$\delta_*(1) = \delta^{-1}(\delta(1)).$$

Since $\delta(1)$ is a projection (as δ is an endomorphism) and δ^{-1} is a morphism it follows that $\delta^{-1}(\delta(1))$ is a projection as well. So 1) is proved.

2) Using (2.2) we obtain

$$\delta_*(1)\mathcal{A} = \delta_*(1\delta(\mathcal{A})) \subset \delta_*(\mathcal{A}). \tag{2.8}$$

In view of the non-degeneracy of δ_* and (iii) of Proposition 2.1 we have

$$\delta(\delta_*(1)\mathcal{A}) = \delta(\delta_*(1)) \cdot \delta(\mathcal{A}) = \delta(1)\delta(\mathcal{A}) = \delta(\mathcal{A}). \tag{2.9}$$

Now we conclude from (2.8), (2.5) and (2.9) that the mapping

$$\delta \colon \delta_*(1)\mathcal{A} \to \delta(\mathcal{A}) \tag{2.10}$$

is a *-isomorphism (we have already noticed that $\delta_*(1)\mathcal{A}$ is an ideal (recall (2.3))). Therefore (2.10) and (2.5) imply the equality $\delta_*(1)\mathcal{A} = \delta_*(\mathcal{A})$ and the proof of 2) is finished.

3) For any $a \in \mathcal{A}$ we have that $\delta(a) \in \delta(\mathcal{A})$ and $\delta_*\delta(a) \in \delta_*(\mathcal{A})$, and $\delta\delta_*\delta(a) = \delta(a)$ by Proposition 2.1 (ii). Therefore $\delta_* \colon \delta(\mathcal{A}) \to \delta_*(\mathcal{A})$ is a right inverse to $\delta \colon \delta_*(\mathcal{A}) \to \delta(\mathcal{A})$. But since it has been already observed that $\delta \colon \delta_*(\mathcal{A}) \to \delta(\mathcal{A})$ is an isomorphism it follows that δ_* is an isomorphism as well. $\square$

Proposition 2.6 *Let $\mathcal{A}$ be a C^*-algebra with an identity* 1, $\delta\colon \mathcal{A}\to\mathcal{A}$ *be an endomorphism of* $\mathcal{A}$ *and* δ_{*i}, $i=1,2$ *be two non-degenerate transfer operators for* $(\mathcal{A},\delta)$. *Then*

1) $\delta_{*1}(1)=\delta_{*2}(1)$,
2) $\delta_{*1}(\mathcal{A})=\delta_{*2}(\mathcal{A})$.

Proof. By 1) of Proposition 2.5 we have that $\delta_{*i}(1)$, $i=1,2$ are the orthogonal projections belonging to the center of $\mathcal{A}$. Set

$$P=\delta_{*1}(1)\delta_{*2}(1).$$

By 1) of Proposition 2.5 we have

$$P\in\delta_{*i}(\mathcal{A}),\qquad i=1,2, \tag{2.11}$$

and by the non-degeneracy of δ_{*i}, $i=1,\,2$ (see (iii) of Proposition 2.1) we obtain

$$\delta(P)=\delta(\delta_{*1}(1)\delta_{*2}(1))=\delta(\delta_{*1}(1))\delta(\delta_{*2}(1))=\delta(1)\delta(1)=\delta(1). \tag{2.12}$$

But in view of Proposition 2.2 the mappings

$$\delta\colon \delta_{*i}(\mathcal{A})\to\delta(\mathcal{A}),\quad i=1,2 \tag{2.13}$$

are *-isomorphisms. Now (2.11), (2.12) and (2.13) imply

$$P=\delta_{*1}(1)=\delta_{*2}(1).$$

Since by 1) $\delta_{*1}(1)=\delta_{*2}(1)$ we conclude by 2) of Proposition 2.5 that $\delta_{*1}(\mathcal{A})=\delta_{*2}(\mathcal{A})$. The proof is complete. $\square$

Remark 2.7 It has been observed that in general a non-degenerate transfer operator (for a given pair $(\mathcal{A},\delta)$) is not unique. Part 2) of Proposition 2.6 and part 3) of Proposition 2.5 tell us that nevertheless its restriction onto $\delta(\mathcal{A})$ is unique.

We shall call the transfer operator δ_* *complete,* if

$$\delta\delta_*(a)=\delta(1)a\delta(1),\qquad a\in\mathcal{A}. \tag{2.14}$$

Observe that a complete transfer operator is non-degenerate. Indeed, (2.14) implies

$$\delta\delta_*\delta(a)=\delta(1)\delta(a)\delta(1)=\delta(a)$$

and so condition (ii) of Proposition 2.1 is satisfied.

Note also that (2.14) implies

$$\delta_*\delta\delta_*=\delta_*. \tag{2.15}$$

The next result presents the criteria for the existence of a complete transfer operator.

Theorem 2.8 *Let $\mathcal{A}$ be a C^*-algebra with an identity* 1 *and $\delta\colon \mathcal{A} \to \mathcal{A}$ be an endomorphism of $\mathcal{A}$. The following are equivalent:*

1) *there exists a complete transfer operator δ_* (for $(\mathcal{A}, \delta)$),*

2) (*i*) *there exists a non-degenerate transfer operator δ_* and*
   (*ii*) *$\delta(\mathcal{A})$ is a hereditary subalgebra of $\mathcal{A}$;*

3) (*i*) *there exists a central orthogonal projection $P \in \mathcal{A}$ such that*
   a) $\delta(P) = \delta(1)$,
   b) the mapping $\delta\colon P\mathcal{A} \to \delta(\mathcal{A})$ is a *-isomorphism, and
   (*ii*) *$\delta(\mathcal{A}) = \delta(1)\mathcal{A}\delta(1)$.*

Moreover the objects in 1) – 3) *are defined in a unique way (i. e. the transfer operator δ_* in* 1) *and* 2) *is unique and the projection P in* 3) *is unique as well) and*

$$P = \delta_*(1) \tag{2.16}$$

and

$$\delta_*(a) = \delta^{-1}(\delta(1)a\delta(1)), \quad a \in \mathcal{A} \tag{2.17}$$

where $\delta^{-1}\colon \delta(\mathcal{A}) \to P\mathcal{A}$ is the inverse mapping to $\delta\colon P\mathcal{A} \to \delta(\mathcal{A})$.

Proof. 1) $\Rightarrow$ 2). By Proposition 2.3 2) (ii) and 3) (ii) are equivalent.

We have already noticed that if δ_* is complete then it is non-degenerate. Moreover (2.14) implies

$$\delta(\mathcal{A}) \supset \delta(1)\mathcal{A}\delta(1)$$

and on the other hand

$$\delta(\mathcal{A}) = \delta(1)\delta(\mathcal{A})\delta(1) \subset \delta(1)\mathcal{A}\delta(1).$$

Thus 1) implies 2).

2) $\Rightarrow$ 3). We have already noticed that 2) (ii) and 3) (ii) are equivalent.

Set $P = \delta_*(1)$. By 1) of Proposition 2.5 P is an orthogonal central projection, and by (iii) of Proposition 2.1 $\delta(P) = \delta(1)$. So 3) (i) (a) is true.

By 2) of Proposition 2.5 we have that $\delta_*(\mathcal{A}) = P\mathcal{A}$ and it follows from Proposition 2.2 that the mapping $\delta\colon P\mathcal{A} \to \delta(\mathcal{A})$ is a *-isomorphism. So 3) (i) (b) is true.

3) $\Rightarrow$ 1). Let $\delta^{-1}\colon \delta(\mathcal{A}) \to P\mathcal{A}$ be the inverse mapping to $\delta\colon P\mathcal{A} \to \delta(\mathcal{A})$. Define the operator δ_* by the formula $\delta_*(a) = \delta^{-1}(\delta(1)a\delta(1))$. Clearly δ_* is positive and satisfies (2.14). Note that

$$\delta\big(\delta_*(\delta(a)b)\big) = \delta(1)\delta(a)b\delta(1) = \delta(a)\delta(1)b\delta(1) = \delta(a\delta_*(b)).$$

But in view of the definition of δ_* the elements $\delta_*(\delta(a)b)$ and $a\delta_*(b)$ belong to the ideal $P\mathcal{A}$ where the endomorphism δ is injective. Therefore they coincide and thus (2.1) is proved. So δ_* is a complete transfer operator.

Now to finish the proof let us verify the uniqueness of the objects mentioned in 1) – 3).

The uniqueness of the projection P in 3) is in fact established in 1) of Proposition 2.6 (since here we have $P = \delta_*(1)$ for some non-degenerate δ_* which also proves (2.16)).

Recalling (2.4) we obtain

$$\delta_*(a) = \delta_*(\delta(1)a\delta(1)), \quad a \in \mathcal{A}. \tag{2.18}$$

But by 3) (ii) $\delta(1)a\delta(1) \in \delta(\mathcal{A})$. Therefore (2.18) and 3) of Proposition 2.5 imply the uniqueness of δ_*.

Finally we note that formula (2.17) has been already proven in the course of the proof of 3) $\Rightarrow$ 1).

The proof is complete. $\square$

In connection with Theorem 2.8 it makes sense to note the next useful observation which is given in Proposition 2.9. We recall that a *partial automorphism* of a C^*-algebra $\mathcal{A}$ is a triple (θ, I, J) where I and J are closed two-sided ideals of $\mathcal{A}$ and $\theta\colon I \to J$ is a *-isomorphism.

Proposition 2.9 *Let δ_* be a complete transfer operator for $(\mathcal{A}, \delta)$ and $\delta(1) \in Z(\mathcal{A})$, then*

1) *both the triples $(\delta, \delta_*(1)\mathcal{A}, \delta(1)\mathcal{A})$ and $(\delta_*, \delta(1)\mathcal{A}, \delta_*(1)\mathcal{A})$ are partial automorphisms (that are inverses to each other), and*

2) *$\delta_*\colon \mathcal{A} \to \mathcal{A}$ is an endomorphism.*

Proof. 1) By Proposition 2.5 we have that $\delta_*(1)$ is a central projection. Since by the condition of the lemma $\delta(1)$ is a central projection as well it follows that both $\delta_*(1)\mathcal{A}$ and $\delta(1)\mathcal{A}$ are the ideals. In addition by 3) (ii) of Theorem 2.8 we have that $\delta(\mathcal{A}) = \delta(1)\mathcal{A}\delta(1) = \delta(1)\mathcal{A}$. Now 1) follows from Proposition 2.5.

2) Using (2.17) and the condition that $\delta(1) \in Z(\mathcal{A})$ and recalling that $\delta_* : \delta(\mathcal{A}) \to \delta_*(\mathcal{A})$ is a *-isomorphism we obtain for any $a, b \in \mathcal{A}$

$$\begin{aligned}\delta_*(ab) &= \delta^{-1}(\delta(1)ab\delta(1)) = \delta^{-1}(\delta(1)a\delta(1)\delta(1)b\delta(1)) \\ &= \delta^{-1}(\delta(1)a\delta(1))\,\delta^{-1}(\delta(1)b\delta(1)) = \delta_*(a)\delta_*(b).\end{aligned}$$

Thus $\delta_*\colon \mathcal{A} \to \mathcal{A}$ is an endomorphism. $\square$

3 Coefficient algebras associated with given endomorphisms

Now we pass to the main result of the article.

Let $\mathcal{A}$ be a C^*-algebra containing an identity 1 and $\delta\colon \mathcal{A} \to \mathcal{A}$ be an endomorphism. We say that $\mathcal{A}$ is a *coefficient algebra associated with* δ if there exists a triple (H, π, U) consisting of a Hilbert space H, faithful non-degenerate representation $\pi : \mathcal{A} \to L(H)$ and a linear continuous operator $U\colon H \to H$ such that conditions (1.12) and (1.13) are satisfied.

Theorem 3.1 *$\mathcal{A}$ is a coefficient algebra associated with δ iff there exists a complete transfer operator δ_* for $(\mathcal{A}, \delta)$ (that is either of the equivalent conditions of Theorem 2.8 hold).*

Proof. *Necessity.* If conditions (1.12) and (1.13) are satisfied then (identifying $\mathcal{A}$ with $\pi(\mathcal{A})$) one can set

$$\delta_*(\cdot) = U^*(\cdot)U$$

and it is easy to verify that δ_* is a complete transfer operator.

Sufficiency. Let δ_* be a complete transfer operator. We shall construct the desired Hilbert space H by means of the lements of the initial algebra $\mathcal{A}$ in the following way. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a certain non-negative inner product on $\mathcal{A}$ (differing from a common inner product only in such a way that for certain non-zero elements $v \in \mathcal{A}$ the expression $\langle v, v\rangle$ may be equal to zero). For example this inner product may have the form $\langle v, u\rangle = f(u^*v)$ where f is some positive linear functional on $\mathcal{A}$. If one factorizes $\mathcal{A}$ by all the elements v such that $\langle v, v\rangle = 0$ then he obtains a linear space with a strictly positive inner product. We shall call the completion of this space with respect to the norm $\|v\| = \sqrt{\langle v, v\rangle}$ the *Hilbert space generated by the inner product* $\langle\,\cdot\,,\,\cdot\,\rangle$.

Let V be the set of all positive linear functionals on $\mathcal{A}$. The space H will be the completion of the direct sum $\bigoplus_{f\in V} H^f$ of some Hilbert spaces H^f. Every H^f will in turn be the completion of the direct sum of Hilbert spaces $\bigoplus_{n\in\mathbb{Z}} H^f_n$. The spaces H^f_n are generated by non-negative inner products $\langle\,\cdot\,,\,\cdot\,\rangle_n$ on the initial algebra $\mathcal{A}$ that are given by the following formulae

$$\langle v, u\rangle_0 = f(u^*v); \tag{3.1}$$

$$\langle v, u\rangle_n = f\big(\delta_*^n(u^*v)\big), \qquad n \ge 0; \tag{3.2}$$

$$\langle v, u\rangle_n = f\big(u^*\delta^{|n|}(1)v\big), \qquad n \le 0. \tag{3.3}$$

The properties of these inner products are described in the next

Lemma 3.2 *For any* $v, u \in \mathcal{A}$ *the following equalities are true*

$$\langle \delta(v), u\rangle_{n+1} = \langle v, \delta_*(u)\rangle_n, \qquad n \ge 0; \tag{3.4}$$

$$\langle \delta^{|n|}(1)v, u\rangle_{n+1} = \langle v, \delta^{|n|}(1)u\rangle_n, \qquad n < 0. \tag{3.5}$$

Proof. Indeed, the proof of (3.4) reduces to the verification of the equality

$$\delta_*^{n+1}(u^*\delta(v)) = \delta_*^n(\delta_*(u^*)v)$$

which follows from (2.2), and the proof of (3.5) reduces to the verification of the equality

$$u^*\delta^{|n|-1}(1)\delta^{|n|}(1)v = u^*\delta^{|n|}(1)\delta^{|n|}(1)v$$

which follows from the equalities

$$\delta^{|n|-1}(1)\delta^{|n|}(1) = \delta^{|n|-1}(1\cdot\delta(1)) = \delta^{|n|}(1) = (\delta^{|n|}(1))^2. \quad \Box$$

Now let us define the operators U and U^* on the space H constructed. These operators will leave invariant all the subspaces $H^f \subset H$. The action of U and U^* on every H^f is the same and its scheme is presented in the first line of the next diagram.

$$
\begin{array}{ccccccccccccccc}
\dots & \underset{\delta^3(1)\,\cdot}{\overset{\delta^3(1)\,\cdot}{\rightleftarrows}} & H^f_{-2} & \underset{\delta^2(1)\,\cdot}{\overset{\delta^2(1)\,\cdot}{\rightleftarrows}} & H^f_{-1} & \underset{\delta(1)\,\cdot}{\overset{\delta(1)\,\cdot}{\rightleftarrows}} & H^f_0 & \underset{\delta_*}{\overset{\delta}{\rightleftarrows}} & H^f_1 & \underset{\delta_*}{\overset{\delta}{\rightleftarrows}} & H^f_2 & \underset{\delta_*}{\overset{\delta}{\rightleftarrows}} & \dots & & \underset{U^*}{\overset{U}{\rightleftarrows}} \\
\dots & & \delta^2(a) & & \delta(a) & & a & & a & & a & & \dots & & \pi(a)
\end{array}
$$

Formally this action is defined in the following way. Consider any *finite* sum

$$h = \bigoplus_n h_n \in H^f, \quad h_n \in H^f_n.$$

Set

$$Uh = \bigoplus_n (Uh)_n \qquad \text{and} \qquad U^*h = \bigoplus_n (U^*h)_n$$

where

$$(Uh)_n = \begin{cases} \delta(h_{n-1}), & \text{if } \ n > 0, \\ \delta^{|n|+1}(1)h_{n-1}, & \text{if } \ n \le 0, \end{cases} \tag{3.6}$$

$$(U^*h)_n = \begin{cases} \delta_*(h_{n+1}), & \text{if } \ n \ge 0, \\ \delta^{|n|}(1)h_{n+1}, & \text{if } \ n < 0. \end{cases} \tag{3.7}$$

Lemma 3.2 guarantees that the operators U and U^* are well defined (i. e. they preserve factorization and completion by means of which the spaces H^f_n were built from the algebra $\mathcal{A}$) and U and U^* are mutually adjoint.

Now let us define the representation $\pi : \mathcal{A} \to L(H)$. For any $a \in \mathcal{A}$ the operator $\pi(a) : H \to H$ will leave invariant all the subspaces $H^f \subset H$ and also all the subspaces $H^f_n \subset H^f$. If $h_n \in H^f_n$ then we set

$$\pi(a)h_n = \begin{cases} ah_n, & n \ge 0, \\ \delta^{|n|}(a)h_n, & n \le 0. \end{cases} \tag{3.8}$$

The scheme of the action of the operator $\pi(a)$ is presented in the second line of the diagram given above.

Let us verify equalities (1.12) for the representation π. Take any $h_n \in H^f_n$. Then for $n < 0$ we have

$$U^*\pi(a)Uh_n = \delta^{|n|}(1)\delta^{|n|-1}(a)\delta^{|n|}(1)h_n,$$

$$\pi(\delta_*(a))h_n = \delta^{|n|}(\delta_*(a))h_n = \delta^{|n|}(1)\delta^{|n|-1}(a)\delta^{|n|}(1)h_n$$

(where the final equality follows from (2.14)); and for $n \ge 0$ we have

$$\pi(\delta_*(a))h_n = \delta_*(a)h_n,$$

$$U^*\pi(a)Uh_n = \delta_*(a\delta(h_n)) = \delta_*(a)h_n.$$

In addition for $n \leq 0$ one has

$$\pi(\delta(a))h_n = \delta^{|n|+1}(a)h_n,$$

$$U\pi(a)U^*h_n = \delta^{|n|+1}(1)\delta^{|n|+1}(a)\delta^{|n|+1}(1)h_n = \delta^{|n|+1}(a)h_n;$$

and for $n > 0$

$$\pi(\delta(a))h_n = \delta(a)h_n,$$

$$U\pi(a)U^*h_n = \delta(a\delta_*(h_n)) = \delta(a)h_n\delta(1),$$

and moreover (3.2) and (2.2) imply that for $n > 0$ the element $\delta(a)h_n\delta(1)$ coincides with $\delta(a)h_n$ in the space H_n^f.

Thus we have proved that $U\pi(a)U^* = \pi(\delta(a))$ and $U^*\pi(a)U = \pi(\delta_*(a))$ for any $a \in \mathcal{A}$.

To finish the proof it is enough to observe the faithfulness of the representation π. But this follows from the definition of the inner product in (3.1), the definition of π (see the second line in the diagram) and the standard Gelfand-Naimark faithful representation of a C^*-algebra. The proof is complete. $\square$

We shall say that an *endomorphism* $\delta : \mathcal{A} \to \mathcal{A}$ *is generated by an isometry* if there exists a triple (H, π, U) consisting of a Hilbert space H, faithful non-degenerate representation $\pi : \mathcal{A} \to L(H)$ and an *isometry* $U : H \to H$ such that for every $a \in \mathcal{A}$ the following conditions are satisfied

$$\pi(\delta(a)) = U\pi(a)U^*, \qquad U^*\pi(a)U \in \pi(\mathcal{A}). \tag{3.9}$$

Theorem 3.1 implies in particular an obvious

Corollary 3.3 *An endomorphism δ is generated by an isometry iff there exists a complete transfer operator δ_* for $(\mathcal{A}, \delta)$ such that $\delta_*(1) = 1$ or, equivalently iff δ is a monomorphism with hereditary range.*

Proof. Evidently δ is generated by an isometry iff $\mathcal{A}$ is a coefficient algebra associated with δ and we have $U^*U = 1$. So in this case $\delta_*(1) = 1$ and the projection P mentioned in 3) of Theorem 2.8 is equal to 1. $\square$

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