# Cramér Asymptotics in the Averaging Method for Systems with Fast Hyperbolic Motions

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**Abstract**—A dynamical system $w' = S(w, z, \varepsilon)$, $z' = z + \varepsilon v(w, z, \varepsilon)$ is considered. It is assumed that slow motions are determined by the vector field $v(w, z, \varepsilon)$ in the Euclidean space and fast motions occur in a neighborhood of a topologically mixing hyperbolic attractor. For the difference between the true and averaged slow motions, a limit theorem is proved and sharp asymptotics for the probabilities of large deviations (that do not exceed $\varepsilon^{\delta}$) are calculated; the exponent $\delta$ depends on the smoothness of the system and approaches zero as the smoothness increases.

We consider a cascade system with fast and slow motions in which the slow motions occur in the Euclidean space and the fast motions occur in a neighborhood of a topologically mixing hyperbolic attractor. We assume that the fast hyperbolic motions depend on a slow variable, while the slow motions are determined by a vector field $v$ depending on both fast and slow variables. For each value of the slow variable $z$, an invariant probability measure $\mu_z$ on the attractor is constructed. It is used to define an averaged vector field $\overline{v}(z) = \int v \, d\mu_z$ on the space of slow motions. We prove that the measure $\mu_z$ smoothly depends on $z$ and that the field $\overline{v}$ is smooth as well. The dynamical system determined by this field is said to be *averaged*. The averaging method consists in replacing the slow motions by the trajectories of the averaged system. The method is substantiated by proving that the difference between the true and averaged slow motions is small with overwhelming probability.

In the case when the fast motion does not depend on the slow motion, the following results have been obtained [9, 7]. First, the difference mentioned above satisfies a central limit theorem with the mean-square deviation of order $\sqrt{\varepsilon}$ (where $\varepsilon$ is a small parameter characterizing the speed of the slow motion). Second, the probabilities of deviations of orders higher than $\sqrt{\varepsilon}$ are exponentially small. Third, rough (up to logarithmic equivalence) probability estimates have been obtained for various types of behavior of the slow components of trajectories deviating from the averaged trajectory by a finite distance.

In this paper, we prove a central limit theorem in the case when the fast motions depend on the slow motions. Moreover, we calculate sharp asymptotics for the probabilities of large deviations that do not exceed $\varepsilon^{\delta}$. The exponent $\delta$ depends on the smoothness of the system and approaches zero as the smoothness increases. These asymptotics are similar to the classical Cramér asymptotics for a sequence of independent identically distributed random variables. They can be obtained both for the invariant measure $\mu_z$ and for noninvariant measures of the following form. Let $\Gamma$ be a smooth submanifold of the phase space that is contained in the layer $z = \text{const}$ in a small neighborhood of the attractor and has the dimension of the expanding foliation and a direction close to that of the expanding foliation. Let $\mu$ be a smooth compactly supported probability measure on $\Gamma$. We denote the slow component of the trajectory with initial condition $z$ by $z(t)$ and the averaged trajectory

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with the same initial condition by $\overline{z}(t)$. For the difference $z(t/\varepsilon) - \overline{z}(t/\varepsilon)$, the Cramér asymptotics with respect to the measure $\mu$ can be calculated. It turns out that, in the first approximation, they do not depend on the pair $(\Gamma, \mu)$ but depend only on the initial condition $z$. If the dynamical system under consideration is generated by a mapping $\Sigma_\varepsilon$ and $J$ is a smooth positive function on the phase space, then similar results can be obtained for the measures of the form $\mu \times J \times J \circ \Sigma_\varepsilon \times \ldots \times J \circ \Sigma_\varepsilon^{t/\varepsilon}$.

The proof uses two independent techniques. The first is the apparatus of *foliated functions* developed in [1, 2, 5] (they were called regular functions in the first two papers). The second is the method of asymptotic expansions for semigroups of operators of weighted conditional mathematical expectation, which is described in [4] for the case of a cascade system with slow and fast Markov motions.

This paper is organized as follows. In Section 1, the system under examination is specified, nonformal definitions of foliated functions and averaged weighted shift operator are given (their precise definitions, which are rather cumbersome, are given in [5]), the basic properties of these functions and operators are described, and invariant measures on attractors are constructed. In Section 2, we state a theorem on an asymptotic expansion of the semigroup of averaged weighted shift operators (Theorem 2.6). It is proved in Section 3. Finally, in Section 4, we apply the asymptotic theorem to derive Cramér asymptotics for the probabilities of large deviations from the averaged motion. The main result about these asymptotics is Theorem 4.4.

## 1. FOLIATED FUNCTIONS

In what follows, we use the terminology and notation from [5]. Let $M$ be a convex domain in the standard Euclidean space and $W$ be a Riemannian manifold. We consider a family of $N$ times continuously differentiable self-mappings of the direct product $W \times M$ of the form

$$\begin{cases} w' = S(w, z, \varepsilon), \\ z' = z + \varepsilon v(w, z, \varepsilon), \end{cases} \tag{1}$$

where $\varepsilon$ is a small positive parameter, $w \in W$, and $z \in M$. In abbreviated notation, this family is written as $(w', z') = \Sigma_\varepsilon(w, z)$. We assume that $\Sigma_\varepsilon(W \times M) \subset W \times M$ for all sufficiently small values of $\varepsilon$. This can easily be achieved, e.g., by requiring that the vector field $v$ be zero in a neighborhood of the boundary of $M$. In this case, $\Sigma_\varepsilon$ generates a dynamical system with discrete time (cascade) on $W \times M$ with fast motions on $W$ and slow motions on $M$ (at a velocity of order $\varepsilon$). We assume that the mapping $S_z(w) = S(w, z, 0)$ has a mixing hyperbolic attractor for each $z \in M$ which continuously depends on $z$ (see the definition in [6, 8]). Suppose also that system (1) has a uniformly hyperbolic mixing atlas (the definition is given in [5]). The last condition is lightly restrictive because it holds locally (in a neighborhood of each point $z \in M$) and can always be satisfied by decreasing the domain $M$.

In [5], the notions of leaves, traces, foliated functions, and averaged weighted shift operators on spaces of foliated functions were introduced for such systems. In short, each leaf $\Gamma$ is a smooth submanifold in $W \times M$ that lies in a neighborhood of the attractor and has the dimension of the expanding foliation and a direction close to that of the expanding foliation. A trace is a pair $(\Gamma, \Phi)$ consisting of a leaf $\Gamma$ and a smooth function $\Phi$ on it, which is called a density. A standard trace is a triple $(\Gamma, \alpha, \Phi)$ consisting of a trace $(\Gamma, \Phi)$ and a point $\alpha \in \Gamma$ belonging to the $R_0$-interior of $\Gamma$ (where the positive number $R_0$ does not depend on the trace). The point $\alpha$ is called the center of the leaf $\Gamma$. A foliated function is a function on the set of standard traces whose value linearly depends on the density $\Phi$ and smoothly changes under a smooth deformation of the trace. In [5], various norms for foliated functions were defined (depending on their order of smoothness) and, accordingly, Banach spaces $\mathcal{F}^{pq}$ of foliated functions were introduced (here, $p$ and $q$ are positive

integers characterizing the smoothness of foliated functions and satisfying the conditions $p \geq 1$, $q \geq 4$, and $p + q \leq N - 1$, where $N$ is the order of smoothness of system (1)).

The simplest example of a foliated function is given by $g(\Gamma, \alpha, \Phi) = \Phi(\alpha)$. A less trivial example is a function of the form

$$g(\Gamma, \alpha, \Phi) = \int_{\Gamma} \xi(\rho(\alpha, \beta)) \Phi(\beta) \, d\mu(\beta), \tag{2}$$

where $\mu$ is the Riemannian volume on $\Gamma$, $\rho(\alpha, \beta)$ is the distance between the points $\alpha, \beta \in \Gamma$, and $\xi$ is a smooth nonnegative function on the real axis supported in a small neighborhood of zero.

Let $u$ be the dimension of the expanding foliation on the attractor. A weight function is an arbitrary smooth function $J$ defined on the manifold of $u$-dimensional subspaces tangent to $W \times M$. In particular, the canonical weight function is the reciprocal of the expansion coefficient of $u$-dimensional volumes under the mapping $\Sigma_{\varepsilon}$. The weight functions have natural restrictions to each leaf. It was proved in [5] that the foliated functions form a Banach module over the algebra of weight functions with multiplication $Jg(\Gamma, \alpha, \Phi) = g(\Gamma, \alpha, J\Phi)$.

The image of a trace $(\Gamma, \Phi)$ under the mapping $\Sigma_{\varepsilon}$ is the trace $(\Gamma'_{\varepsilon}, \Phi'_{\varepsilon}) = \Sigma_{\varepsilon}(\Gamma, \Phi)$ such that $\Gamma'_{\varepsilon} = \Sigma_{\varepsilon}(\Gamma)$ and $\Phi = \Phi'_{\varepsilon} \circ \Sigma_{\varepsilon}$. Take a family of weight functions $J_{\varepsilon}$. The corresponding averaged weighted shift operator $A_{\varepsilon, n}$ on the space of foliated functions is defined by the formula

$$[A_{\varepsilon, n} g](\Gamma, \alpha, \Phi) = \int_{\Gamma'_{\varepsilon}} \xi_{\Gamma}(\alpha, \beta) g(\Gamma'_{\varepsilon}, \beta', \Phi'_{\varepsilon}) \, d\mu(\beta'), \tag{3}$$

in which $(\Gamma'_{\varepsilon}, \Phi'_{\varepsilon}) = (\Sigma_{\varepsilon} J_{\varepsilon})^n (\Gamma, \Phi)$, $\beta \in \Gamma$, $\beta' = \Sigma_{\varepsilon}^n(\beta)$, the function $\xi_{\Gamma}(\alpha, \beta)$ has the form

$$\xi_{\Gamma}(\alpha, \beta) = \frac{\xi_0(\rho(\alpha, \beta))}{\int_{\Gamma} \xi_0(\rho(\alpha, \beta)) \, d\mu(\alpha)}, \tag{4}$$

and $\xi_0$ is a fixed smooth nonnegative function on the real axis supported in a small neighborhood of zero. This definition implies the homological identity

$$A_{\varepsilon, n}(fg) = f \circ \Sigma_{\varepsilon}^n \cdot A_{\varepsilon, n} g \tag{5}$$

for all foliated functions $g$ and smooth functions $f \colon W \times M \to \mathbb{R}$.

In [5, Theorems 3.2 and 3.3], the following assertion was proved.

**Theorem 1.1.** *The operator $A_{0, n}$ continuously maps each space $\mathcal{F}^{pq}$ to itself. For every $n$, there exists a small positive number $\varepsilon_0$ such that, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $l \leq q - 8$, the operator $d^l A_{\varepsilon, n} / d\varepsilon^l$ continuously maps the space $\mathcal{F}^{pq}$ to $\mathcal{F}^{p+1+l, q-4-l}$ (and, hence, to $\mathcal{F}^{p+2l, q-5l}$ if $l \neq 0$).*

A foliated function is said to be *positive* if it takes nonnegative values on the standard traces with nonnegative densities. A positive foliated function $g$ is *strongly positive* if there exists $c > 0$ such that, for any standard leaf $\Gamma$ centered at $\alpha$, $g(\Gamma, \alpha, 1) \geq c$. A linear functional on the space of foliated functions is positive if it takes nonnegative values on positive functions.

Now, suppose that the family $J_{\varepsilon}$ of weight functions is positive and bounded away from zero. For example, this is so for the canonical weight function. Then, the operator $A_{\varepsilon, n}$ maps positive and strongly positive foliated functions to positive and strongly positive foliated functions, respectively. Under these conditions, the following assertion is valid [5, Theorem 3.3].

**Theorem 1.2.** *If $N \geq 6$, then, for any sufficiently large $n \in \mathbb{N}$, there exist a function $\lambda = \lambda_n \in C^{N-4}(M)$, a strongly positive foliated function $h = h_n \in \mathcal{F}^{1, N-2}$, and a positive $C^{\infty}(M)$-linear functional $\nu = \nu_n \colon \mathcal{F}^{N-5, 4} \to C^2(M)$ such that, whenever $p \geq 1$, $q \geq 4$, and $p + q \leq N - 1$, the following assertions hold*:

(a) *the functional $\nu$ continuously maps the space $\mathcal{F}^{pq}$ to $C^{q-2}(M)$;*

(b) $A_{0,n}h = e^\lambda h, \ \nu \circ A_{0,n} = e^\lambda \nu, \ and \ \nu(h) \equiv 1;$

(c) *the sequence of operators $[e^{-\lambda}A_{0,n}]^m \colon \mathcal{F}^{pq} \to \mathcal{F}^{pq}$ converges to the projector $\overline{A}_{0,n}g = \nu(g)h$ in the uniform operator norm as $m \to \infty$.*

Consider a $C^\infty(M)$-linear functional $\mu_n \colon C^\infty(W \times M) \to C^{N-4}(M)$ defined by $\mu_n(f) = \nu(fh)$. Theorem 1.2 implies that this functional is positive (i.e., it takes nonnegative functions to nonnegative functions) and normalized (i.e., $\mu_n(1) \equiv 1$). In addition, it satisfies the identity

$$\mu_n(f \circ \Sigma_0^n) = \mu_n(f). \tag{6}$$

Indeed, by virtue of (5), we have

$$\mu_n(f \circ \Sigma_0^n) = \nu(f \circ \Sigma_0^n \cdot h) = \nu(f \circ \Sigma_0^n \cdot e^{-\lambda}A_{0,n}h) = e^{-\lambda}\nu\big(A_{0,n}(fh)\big) = \nu(fh) = \mu_n(f).$$

**Proposition 1.3.** *If two functions $f, f' \in C^\infty(W \times M)$ coincide at some $z = z_0$, then the values of the functions $\mu_n(f)$ and $\mu_n(f')$ at the point $z_0$ also coincide.*

**Proof.** Suppose that $z = (z_1, \ldots, z_k)$ and $z_0 = (z_{01}, \ldots, z_{0k})$. By the Hadamard lemma, the difference $f - f'$ can be represented in the form

$$f(z, w) - f'(z, w) = \sum_{i=1}^k (z_i - z_{0i})\varphi_i(z, w), \qquad \varphi_i \in C^\infty(W \times M).$$

Therefore, the difference $\mu_n(f) - \mu_n(f') = \sum_{i=1}^k (z_i - z_{0i})\mu_n(\varphi_i)$ vanishes at $z = z_0$.

Let us define a family of linear functionals $\mu_{n,z} \colon C^\infty(W \times M) \to \mathbb{R}$ by the formula $\mu_{n,z}(f) = [\mu_n(f)](z)$. Each of these functionals is positive and normalized. The Riesz theorem implies that each of them can be identified with a Borel probability measure on $W \times M$. According to Proposition 1.3, the measure $\mu_{n,z}$ is concentrated on the fiber $W \times \{z\}$. Finally, we set

$$\mu_z(f) = \frac{1}{n}\mu_{n,z}\big(f + f \circ \Sigma_0 + \ldots + f \circ \Sigma_0^{n-1}\big).$$

Then, $\mu_z$ is also a Borel probability measure on $W \times \{z\}$. By virtue of (6), it satisfies the identity $\mu_z(f) = \mu_z(f \circ \Sigma_0)$. Therefore, the measure $\mu_z$ is $\Sigma_0$-invariant. If $f \in C^{1+q}(W \times M)$, then, by Theorem 1.2, $\mu_z(f) \in C^{q-2}(M)$. This means that *the invariant measure $\mu_z$ smoothly depends on the parameter $z$.*

## 2. AN ASYMPTOTIC THEOREM

For our purposes, it is convenient to normalize the averaged weighted shift operator $A_{0,n}$ in such a way that its maximum eigenvalue is 1 rather than $e^{\lambda_n}$. To this end, it suffices to replace the weight function $J_\varepsilon$ in the definition of the operator $A_{\varepsilon,n}$ by the *normalized* weight function $\widehat{J}_\varepsilon = J_\varepsilon e^{-\lambda_n/n}$. We call the operator $B_\varepsilon$ thus obtained a *normalized averaged weighted shift operator*. In Section 4, we prove that the canonical weight function is normalized and $B_\varepsilon = A_{\varepsilon,n}$ for this function. The homological identity (5) implies that $B_0 = e^{-\lambda_n}A_{0,n}$. Theorem 1.2 states that the sequence $B_0^m$ converges to the projector $\overline{B} = \nu_n \otimes h_n$ in the space of linear continuous operators on $\mathcal{F}^{pq}$. We refer to this property as the *ergodicity* of the semigroup $B_0^m$. It implies, in particular, the existence of a large $C_0$ and $\Lambda_0 \in (0, 1)$ such that

$$\|B_0^m g\|_{pq} \le C_0 \Lambda_0^m \|g\|_{pq} \qquad \text{for} \quad g \in \mathcal{F}^{pq} \cap \ker \nu. \tag{7}$$

For an arbitrary function $F \in C^\infty(M)$, consider the family of operators $B_\varepsilon[F]$ defined by the equalities

$$B_\varepsilon[F]\, g = e^{-F/\varepsilon}B_\varepsilon\big(e^{F/\varepsilon}g\big), \qquad B_0[F]\, g = \lim_{\varepsilon \to 0} B_\varepsilon[F]\, g. \tag{8}$$

The homological identity (5) implies

$$B_\varepsilon(Fg) = F \circ \Sigma_\varepsilon^n \cdot B_\varepsilon g, \qquad B_0(Fg) = FB_0 g, \tag{9}$$

$$B_\varepsilon[F]\, g = \exp\left(\frac{F \circ \Sigma_\varepsilon^n - F}{\varepsilon}\right) B_\varepsilon g, \qquad B_0[F]\, g = \exp\left(\frac{dF}{dv_n}\right) B_0 g, \tag{10}$$

where

$$v_n(w, z) = \sum_{i=0}^{n-1} v_0 \circ \Sigma_0^i(w, z), \qquad v_0(w, z) = v(w, z, 0). \tag{11}$$

Equalities (9) and (10) imply the $C^\infty(M)$-linearity of the operator $B_0[F]$, i.e., the equality $B_0[F]fg = fB_0[F]\, g$ for all $f \in C^\infty(M)$ and $g \in \mathcal{F}^{pq}$. It follows from [5, Theorem 3.1] that, actually, this identity is valid for $F \in C^{p+q+1}(M)$ and $f \in C^{p+q-3}(M)$. The operator $B_0[F]$ with this property will be called *fibered*. Theorem 1.1 and (10) imply that the family $B_\varepsilon[F]$ analytically depends on $F$ and smoothly (in a certain weak sense) depends on $\varepsilon$. A perturbation $B_\varepsilon$ of the operator $B_0$ such that $B_\varepsilon[F]$ analytically depends on $F$ and smoothly depends on $\varepsilon$ is said to be *superregular*.

In this section, we prove that, for a superregular perturbation $B_\varepsilon$ of a fibered ergodic operator $B_0$, the family of operators $B_\varepsilon^{t/\varepsilon} e^{\xi F/\varepsilon}$ admits an asymptotic expansion in powers of the small parameters $\xi$ and $\varepsilon$. This expansion is similar to that obtained in [4] for a system with slow and Markov fast motions.

Theorem 3.1 from [5] and (10) imply that the operator $B_0[F]\colon \mathcal{F}^{pq} \to \mathcal{F}^{pq}$ analytically depends on the function $F \in C^{p+q+1}(M)$.

**Proposition 2.1.** *An arbitrary sufficiently small function* $F \in C^i(M)$, *where* $6 \le i \le N$, *uniquely determines the following objects that analytically depend on* $F$: *a function* $\lambda_F \in C^{i-4}(M)$, *a foliated function* $h_F \in \mathcal{F}^{1,i-2}$, *and a* $C^\infty(M)$-*linear functional* $\nu_F\colon \mathcal{F}^{i-5,4} \to C^2(M)$ *continuously mapping each space* $\mathcal{F}^{pq}$ *to* $C^{q-2}(M)$ (*for* $p \ge 1$ *and* $q \ge 4$ *with* $p + q \le i - 1$) *such that*

$$\lambda_0 = 0, \qquad h_0 = h, \qquad \nu_0 = \nu, \qquad \nu(h_F) = \nu_F(h) = 1,$$

$$B_0[F]\, h_F = e^{\lambda_F} h_F, \qquad \nu_F \circ B_0[F] = e^{\lambda_F} \nu_F.$$

*Here,* $h$ *and* $\nu$ *are the foliated function and the linear functional defined in Theorem* 1.2.

**Proof.** The space $\mathcal{F}^{1,i-2}$ decomposes into the direct sum $\ker \nu \oplus C^{i-4}(M)h$. Let us apply the implicit function theorem to the equation

$$B_0[F](h + \Delta h) - e^\lambda(h + \Delta h) = 0 \tag{12}$$

with respect to the function $\lambda \in C^{i-4}(M)$ and the foliated function $\Delta h \in \mathcal{F}^{1,i-2} \cap \ker \nu$. It follows from (7) that the restriction of the operator $\mathrm{I} - B_0$ to $\mathcal{F}^{1,i-2} \cap \ker \nu$ is invertible. Therefore, for zero $F$, $\lambda$, and $\Delta h$, the derivative of the left-hand side of equation (12) with respect to $\Delta h$ is an automorphism of $\mathcal{F}^{1,i-2} \cap \ker \nu$, while the derivative of the left-hand side of (12) with respect to $\lambda$ isomorphically maps $C^{i-4}(M)$ onto $C^{i-4}(M)h$. Hence, the implicit function theorem applies to this equation, and it uniquely determines $\lambda_F \in C^{i-4}(M)$ and $\Delta h_F \in \mathcal{F}^{1,i-2} \cap \ker \nu$, which analytically depend on $F$. We set $h_F = h + \Delta h_F$.

Consider the equation

$$\left.\Big((\nu + \Delta\nu) \circ B_0[F] - e^\lambda(\nu + \Delta\nu)\Big)\right|_{\ker \nu} = 0 \tag{13}$$

with respect to a $C^\infty(M)$-linear functional $\Delta\nu\colon \mathcal{F}^{i-5,4} \to C^2(M)$ such that $\Delta\nu(h) = 0$. The implicit function theorem applies to this equation as well because the operator $B_0 - \mathrm{I}$ is invertible on $\mathcal{F}^{i-5,4} \cap \ker\nu$. It uniquely determines a family $\Delta\nu_F$ that analytically depends on $F$.

The same argument applies to the cases when $\Delta\nu\colon \mathcal{F}^{pq} \to C^{q-2}(M)$ and $\Delta\nu\colon \mathcal{F}^{pq} \to C^2(M)$. Since the implicit function is unique, the restriction of the functional $\Delta\nu_F$ to $\mathcal{F}^{pq}$ must be the same in all the three cases. Therefore, $\Delta\nu_F$ continuously maps each $\mathcal{F}^{pq}$ to $C^{q-2}(M)$. To complete the proof, it remains to set $\nu_F = \nu + \Delta\nu_F$.

**Proposition 2.2.** *For small $F \in C^6(M)$, the function $\lambda_F$ is convex with respect to $F$.*

**Proof.** Since the operator $B_0$ is ergodic, we have

$$\lambda_F = \lim_{m\to\infty} \frac{\lambda_m(F)}{m}, \qquad \text{where} \quad \lambda_m(F) = \ln\nu\big((B_0[F])^m h\big).$$

Thus, it is sufficient to show that the functions $\lambda_m(F)$ are convex. By virtue of (9)–(11),

$$(B_0[F])^m h = \exp\left(\frac{dF}{dv_n} + \frac{dF}{dv_n}\circ\Sigma_0^n + \ldots + \frac{dF}{dv_n}\circ\Sigma_0^{n(m-1)}\right) B_0^m h = \exp\left(\frac{dF}{dv_{nm}}\right) h.$$

Therefore, $\lambda_m(F) = \ln\nu\big(e^{dF/dv_{nm}}h\big) = \ln\mu\big(e^{dF/dv_{nm}}\big)$. We know that the functional $\mu(f) = \nu(fh)$ is positive and normalized. To prove the convexity of $\lambda_m(F)$, it suffices to show that the function $\varphi(t) = \ln\mu(e^{u+tv})$ has a nonnegative second derivative for any $u, v \in C^5(W \times M)$. Let us calculate this derivative:

$$\varphi'(t) = \frac{\mu(e^{u+tv}v)}{\mu(e^{u+tv})}, \qquad \varphi''(t) = \frac{\mu(e^{u+tv}v^2)\mu(e^{u+tv}) - \mu(e^{u+tv}v)^2}{\mu(e^{u+tv})^2}.$$

The numerator of the second fraction is nonnegative because

$$\mu(e^{u+tv}v)^2 = \mu\big(e^{(u+tv)/2}v \cdot e^{(u+tv)/2}\big)^2 \le \mu(e^{u+tv}v^2)\mu(e^{u+tv})$$

by the Cauchy–Schwarz–Bunyakovsky inequality.

**Proposition 2.3.** *If the first differentials of two functions $F, F' \in C^N(M)$ coincide at a point $z_0 \in M$, then $\lambda_F(z_0) = \lambda_{F'}(z_0)$.*

**Proof.** Suppose that a point $z \in M$ has coordinates $z_k$. By the Hadamard lemma, all partial derivatives $\partial(F - F')/\partial z_k$ can be represented in the form

$$\frac{\partial(F - F')}{\partial z_k}(z) = \sum_l (z_l - z_{0l})f_{kl}(z), \qquad f_{kl} \in C^{N-2}(M).$$

Then, according to (10), the difference $B_0[F] - B_0[F']$ can be represented as

$$B_0[F] - B_0[F'] = \sum_l (z_l - z_{0l})\varphi_l B_0, \tag{14}$$

where $\varphi_l$ are some functions from $C^{N-2}(W \times M)$. By Proposition 2.1, there exist a vector $h_F \in \mathcal{F}^{1,N-2}$ and a $C^\infty(M)$-linear functional $\nu_{F'}\colon \mathcal{F}^{N-5,4} \to C^2(M)$ such that

$$B_0[F]h_F = e^{\lambda_F}h_F, \qquad \nu_{F'}\circ B_0[F'] = e^{\lambda_{F'}}\nu_{F'}. \tag{15}$$

Let us normalize the functional $\nu_{F'}$ by $\nu_{F'}(h_F) \equiv 1$. Then, relations (15) and (14) and the $C^\infty$-linearity of $\nu_{F'}$ imply the equalities

$$e^{\lambda_F} - e^{\lambda_{F'}} = \nu_{F'}(B_0[F]h_F) - \nu_{F'}(B_0[F']h_F) = \sum_l (z_l - z_{0l})\nu_{F'}(\varphi_l B_0 h_F).$$

Therefore, $\lambda_F(z_0) = \lambda_{F'}(z_0)$.

**Proposition 2.4.** *The function* $\lambda_F = \lambda_F(z)$ *depends only on* $z$ *and on the first partial derivatives of the function* $F$ *at the point* $z$; *i.e.,* $\lambda_F(z) = \lambda(z, p)$, *where* $p = dF(z)/dz$. *It is* $N - 4$ *times continuously differentiable with respect to* $z$ *and analytically depends on* $p$.

**Proof.** Proposition 2.3 implies that $\lambda_F(z) = \lambda(z, dF/dz)$. If $F(z)$ equals the inner product $(p, z)$, then, by Proposition 2.1, the function $\lambda_F(z) = \lambda(z, p)$ has smoothness of order $N - 4$ with respect to $z$ and analytically depends on $p$.

Let us define a vector field $\overline{v}_n$ on $M$ by

$$\overline{v}_n = \nu(v_n h). \tag{16}$$

Its order of smoothness is at least $N - 4$. The definition (11) of the field $v_n$ and the definition of the family of invariant measures $\mu_z$ show that $\overline{v}_n = n\overline{v}$, where $\overline{v}(z)$ is the mean value of the field $v(w, z, 0)$ with respect to the measure $\mu_z$

**Proposition 2.5.** *For any function* $f \in C^6(M)$,

$$\frac{df}{d\overline{v}_n} = \frac{d}{d\xi}\bigg|_{\xi=0} \nu(B_0[\xi f]h) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \nu(B_\varepsilon f h - f B_\varepsilon h) = \frac{d}{d\xi}\bigg|_{\xi=0} \lambda_{\xi f}. \tag{17}$$

**Proof.** The first equality follows from (10). By (9), $B_\varepsilon f - f B_\varepsilon = (f \circ \Sigma_\varepsilon^n - f)B_\varepsilon$. Obviously, $f \circ \Sigma_0^n - f = 0$ and

$$\frac{d(f \circ \Sigma_\varepsilon^n - f)}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{df}{dv_n}.$$

This yields the second equality in (17). The third equality follows from

$$\frac{d}{d\xi}\bigg|_{\xi=0} \lambda_{\xi f} = \frac{d}{d\xi}\bigg|_{\xi=0} \frac{\nu_{\xi f}(B_0[\xi f]h_{\xi f})}{\nu_{\xi f}(h_{\xi f})} = \frac{d}{d\xi}\bigg|_{\xi=0} \nu(B_0[\xi f]h).$$

We study the normalized weighted shift operator $B_\varepsilon$ obtained from $A_{\varepsilon,n}$ by replacing the weight $J_\varepsilon$ with the normalized weight $\widehat{J}_\varepsilon = J_\varepsilon e^{-\lambda_n/n}$. Let us define two more operators $\overline{B}$ and $\widetilde{B}$ by the formulas $\overline{B}g = \nu_n(g)h_n$ and $\widetilde{B} = \mathrm{I} - \overline{B}$. Obviously, these are mutually complementary projectors. We set

$$p_j = 1 + 2j, \qquad q_{ij} = N - 7 - 6i - 10j,$$

$$\mathfrak{D}' = \big\{(i, j) \in \mathbb{Z}_+^2 \ \big| \ 6i + 10j \leq N - 21\big\}.$$

**Definition 2.1.** A *regular Newton diagram* is an arbitrary bounded set $\mathfrak{D} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ that contains, together with its every point $(i, j)$, the entire rectangle $\Pi_{ij} = \big\{(i', j') \ \big| \ 0 \leq i' \leq i, \ 0 \leq j' \leq j\big\}$. A point $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ is called a *growth point* of a regular Newton diagram $\mathfrak{D}$ if it does not belong to $\mathfrak{D}$ while all the other points of the rectangle $\Pi_{ij}$ belong to $\mathfrak{D}$.

**Theorem 2.6.** *Suppose that* $N \geq 27$ *and a family of functions* $F_t(\xi) \in C^{N-5}(M)$ *satisfies the differential equation* $\dot{F}_t(\xi) = \lambda_{F_t(\xi)}$ *with the initial condition* $F_0(\xi) = \xi F$ *for all* $t \in [0, t_0]$ *and all sufficiently small real* $\xi$. *Then, for any foliated function* $g_0 \in \mathcal{F}^{1,N-2}$ *and any regular Newton diagram* $\mathfrak{D} \subset \mathfrak{D}'$ *with a set* $\mathfrak{G}$ *of growth points, there exist families of foliated functions*

$$U_t(\xi, \varepsilon) = \sum_{(i,j)\in\mathfrak{D}} \xi^i \varepsilon^j \big(\overline{U}_{ijt} h_{F_t(\xi)} + \widetilde{U}_{ijt}\big), \qquad V_{t\tau}(\xi, \varepsilon) = \sum_{(i,j)\in\mathfrak{D}} \xi^i \varepsilon^j \big(\overline{V}_{ijt\tau} + \widetilde{V}_{ijt\tau}\big) \tag{18}$$

*such that, for all $t \in [0, t_0]$ and $\tau \in \mathbb{Z}_+$, the identities*

$$g_0 = U_0(\xi, \varepsilon) + V_{00}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j), \tag{19}$$

$$B_\varepsilon e^{F_t(\xi)/\varepsilon} \big( U_t(\xi, \varepsilon) + V_{t\tau}(\xi, \varepsilon) \big) = e^{F_{t+\varepsilon}(\xi)/\varepsilon} \left( U_{t+\varepsilon}(\xi, \varepsilon) + V_{t+\varepsilon, \tau+1}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j) \right) \tag{20}$$

*hold, where the foliated functions $\xi^{-i} \varepsilon^{-j} O(\xi^i \varepsilon^j) \in \mathcal{F}^{[2N/5],4}$ are uniformly bounded. The coefficients of families (18) are such that the derivatives*

$$\frac{d^n \overline{U}_{ijt}}{dt^n} \in C^{q_{ij}-7-n}(M), \qquad n \le q_{ij} - 9,$$

$$\frac{d^n \overline{V}_{ijt\tau}}{dt^n} \in \overline{B} \mathcal{F}^{p_j, q_{ij}-n}, \qquad \frac{d^n \widetilde{U}_{ijt}}{dt^n}, \frac{d^n \widetilde{V}_{ijt\tau}}{dt^n} \in \widetilde{B} \mathcal{F}^{p_j, q_{ij}-n}, \qquad n \le q_{ij} - 4,$$

*are defined for all $t \in [0, t_0]$ and $\tau \in \mathbb{Z}_+$. All of them are uniformly bounded in the spaces specified and linearly and continuously depend on $g_0$, and the norms of the derivatives $d^n \overline{V}_{ijt\tau}/dt^n$ and $d^n \widetilde{V}_{ijt\tau}/dt^n$ exponentially decrease in $\tau$.*

Discarding the remainder terms in identities (19) and (20), we obtain the asymptotic expansion

$$B_\varepsilon^{t/\varepsilon} e^{\xi F/\varepsilon} g_0 \sim e^{F_t(\xi)/\varepsilon} \big( U_t(\xi, \varepsilon) + V_{t, t/\varepsilon}(\xi, \varepsilon) \big), \qquad t \in \varepsilon \mathbb{Z} \cap [0, t_0], \quad \varepsilon > 0. \tag{21}$$

**Addition 1.** The initial coefficients of expansions (18) are expressed as $\widetilde{U}_{00t} = \overline{V}_{00t\tau} = 0$ and $\widetilde{V}_{00t\tau} = B_0^\tau \widetilde{B} g_0$, and $\overline{U}_{00t}$ satisfies the differential equation

$$\frac{d\overline{U}_{00t}}{dt} = \frac{d\overline{U}_{00t}}{d\overline{v}} + \alpha \overline{U}_{00t}, \quad \overline{U}_{000} = \nu(g_0), \qquad \text{where} \quad \alpha = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \nu(B_\varepsilon h) \in C^{N-9}(M).$$

If the function $\overline{U}_{000}$ is bounded from below by a positive constant on $M$, then the family $\overline{U}_{00t}$ is bounded from below by a positive constant on the interval $[0, t_0]$.

We denote the Taylor polynomials of degree $k$ in $\xi$ for the function $F_t(\xi)$ and the families of operators $U_t(\xi, \varepsilon)$ by $J_\xi^k F_t(\xi)$ and $J_\xi^k U_t(\xi, \varepsilon)$, respectively.

**Addition 2.** In Theorem 2.6, $F_t(\xi)$ can be replaced by $J_\xi^k F_t(\xi)$ and $U_t(\xi, \varepsilon)$, by $J_\xi^k U_t(\xi, \varepsilon)$; moreover, $k$ can be taken in the range from $(N-26)/6$ to $3N/5 - 11$. In particular, $k = [N/5]$ is fit.

**Addition 3.** If a foliated function $g_0$ is strongly positive, then, for any standard trace $(\Gamma, \alpha, \Phi)$ with flowing density $\Phi$ (in the sense of Definition 5.1 from [5]) and any positive integer $k$ between $(N-26)/6$ and $3N/5 - 8$, the equality

$$\frac{e^{J_\xi^k F_{i\varepsilon}(\xi)/\varepsilon} \big( J_\xi^k U_{i\varepsilon}(\xi, \varepsilon) + V_{t, t/\varepsilon}(\xi, \varepsilon) \big)(\Gamma, \alpha, \Phi)}{e^{J_\xi^k F_{i\varepsilon}(\xi)/\varepsilon} \big( J_\xi^k U_{i\varepsilon}(\xi, \varepsilon) + V_{t, t/\varepsilon}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j) \big)(\Gamma, \alpha, \Phi)} = \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j)$$

holds uniformly with respect to $(\Gamma, \alpha, \Phi)$ and $t \in [0, t_0]$.

## 3. PROOF OF THE ASYMPTOTIC THEOREM 2.6

Recall that, for some $C_0$ and $\Lambda_0 \in (0, 1)$, the normalized averaged weighted shift operator $B_0 = e^{-\lambda_n} A_{0,n}$ satisfies estimate (7), i.e.,

$$\|B_0^m g\|_{pq} \le C_0 \Lambda_0^m \|g\|_{pq}, \qquad g \in \mathcal{F}^{pq} \cap \ker \overline{B}, \tag{22}$$

where $p \geq 1$, $q \geq 4$, and $p + q \leq N - 1$. Relations (10) imply the equality

$$e^{-F_{t+\varepsilon}(\xi)/\varepsilon} B_\varepsilon e^{F_t(\xi)/\varepsilon} g = e^{\Phi_t(\xi,\varepsilon)} B_\varepsilon g, \tag{23}$$

where

$$\Phi_t(\xi, \varepsilon) = \Phi_t(\xi, \varepsilon, w, z) = \frac{F_t(\xi, \Sigma_\varepsilon^n(w, z)) - F_{t+\varepsilon}(\xi, z)}{\varepsilon}.$$

By the conditions of Theorem 2.6, the family of functions $F_t(\xi, z)$ satisfies the differential equation $\dot{F} = \lambda_F$ with the initial condition $F_0(\xi, z) = \xi F(z)$ and has smoothness $N - 5$. The equality $\lambda_0 = 0$ implies $F_t(0, z) \equiv 0$ and, accordingly, $\Phi_t(0, \varepsilon) \equiv 0$.

**Lemma 3.1.** *Under the conditions of Theorem 2.6, the function $\Phi_t(\xi, \varepsilon, w, z)$ is $N - 6$ times continuously differentiable with respect to all variables and the family of foliated functions $h_{F_t(\xi)}$ has the property*

$$\frac{\partial^{i+n}}{\partial \xi^i \partial t^n} h_{F_t(\xi)} \in \mathcal{F}^{1, N-7-i-n}, \qquad i + n \leq N - 11.$$

**Proof.** The first assertion is obvious. The second assertion holds because, by Proposition 2.1, the function $h_{F_t(\xi)} \in \mathcal{F}^{1, N-7-i-n}$ analytically depends on $F_t(\xi) \in C^{N-5-i-n}(M)$, and this family is $i + n$ times differentiable with respect to $t$ and $\xi$ in the space specified.

**Lemma 3.2.** *Let $p + q \leq N - 6$. Suppose that a family of functions $\overline{U}_t \in C^{q-2}(M)$ and families of foliated functions $\widetilde{U}_t, V_{t\tau} \in \mathcal{F}^{pq}$ satisfy the estimates*

$$\left\| \frac{d^n \overline{U}_t}{dt^n} \right\|_{q-2-n} \leq M, \qquad \left\| \frac{d^n \widetilde{U}_t}{dt^n} \right\|_{p,q-n} \leq M, \qquad \left\| \frac{d^n V_{t\tau}}{dt^n} \right\|_{p,q-n} \leq M\Lambda^\tau, \qquad n \leq q - 4,$$

*where $M > 0$ and $\Lambda \in (\Lambda_0, 1)$ are constants independent of $t \in [0, t_0]$ and $\tau \in \mathbb{Z}_+$. Then, the foliated functions*

$$W_t^1(\xi, \varepsilon) = e^{\Phi_t(\xi,\varepsilon)} B_\varepsilon \big( \overline{U}_t h_{F_t(\xi)} \big) - \overline{U}_{t+\varepsilon} h_{F_{t+\varepsilon}(\xi)}, \qquad W_t^2(\xi, \varepsilon) = e^{\Phi_t(\xi,\varepsilon)} B_\varepsilon \widetilde{U}_t - \widetilde{U}_{t+\varepsilon},$$

$$W_{t\tau}^3(\xi, \varepsilon) = e^{\Phi_t(\xi,\varepsilon)} B_\varepsilon V_{t\tau} - V_{t+\varepsilon, \tau+1}, \qquad\qquad W^4(\xi) = \overline{U}_0 h_{\xi F} + \widetilde{U}_0 + V_{00}$$

*can be expanded in Taylor polynomials in $\xi$ and $\varepsilon$ up to an arbitrary degree $\nu \leq (q-4)/5$:*

$$W_t^1(\xi, \varepsilon) = \sum_{\substack{l>0 \\ k+l<\nu}} \xi^k \varepsilon^l W_{klt}^1 + \sum_{\substack{l>0 \\ k+l=\nu}} \xi^k \varepsilon^l W_{klt}^1(\xi, \varepsilon),$$

$$W_t^2(\xi, \varepsilon) = \sum_{k+l<\nu} \xi^k \varepsilon^l W_{klt}^2 + \sum_{k+l=\nu} \xi^k \varepsilon^l W_{klt}^2(\xi, \varepsilon),$$

$$W_{t\tau}^3(\xi, \varepsilon) = \sum_{k+l<\nu} \xi^k \varepsilon^l W_{klt\tau}^3 + \sum_{k+l=\nu} \xi^k \varepsilon^l W_{klt\tau}^3(\xi, \varepsilon),$$

$$W^4(\xi) = \sum_{k \geq 0} \xi^k W_k^4.$$

*There exist large constants $C_{kl}$ independent of $M$ and $\Lambda$ such that the coefficients of these expansions obey the estimates*

$$\big\| W_{klt}^1(\xi, \varepsilon) \big\|_{1+2\nu, q-5\nu} \leq C_{kl} M, \qquad \big\| W_{klt}^2(\xi, \varepsilon) \big\|_{p+2\nu, q-5\nu} \leq C_{kl} M,$$

$$\big\| W_{klt\tau}^3(\xi, \varepsilon) \big\|_{p+2\nu, q-5\nu} \leq C_{kl} M \Lambda^\tau$$

*and*

$$\left\|\frac{d^n W_{klt}^1}{dt^n}\right\|_{1+2l,q-k-5l-n} \le C_{kl}M, \qquad \left\|\frac{d^n W_{klt}^2}{dt^n}\right\|_{p+2l,q-k-5l-n} \le C_{kl}M,$$

$$\left\|\frac{d^n W_{klt\tau}^3}{dt^n}\right\|_{p+2l,q-k-5l-n} \le C_{kl}M\Lambda^\tau, \qquad \|W_k^4\|_{p,q} \le C_{k0}M$$

*for $n \le q - k - 5l - 4$.*

In the formulation of Lemma 3.2, an estimate for a norm $\|g\|_{\alpha,\beta}$ implies that the corresponding function $g$ is defined and belongs to $\mathcal{F}^{\alpha\beta}$.

**Proof.** Proposition 2.1 and Theorem 3.1 from [5] yield $\overline{U}_t h_{F_t(\xi)} \in \mathcal{F}^{1,q}$. According to Theorem 1.1 and Lemma 3.1, for $l \ge 1$, we have

$$\frac{\partial^l W_t^1(\xi,\varepsilon)}{\partial\varepsilon^l} \in \mathcal{F}^{1+2l,q-5l}, \qquad \frac{\partial^l W_t^2(\xi,\varepsilon)}{\partial\varepsilon^l} \in \mathcal{F}^{p+2l,q-5l}, \qquad \frac{\partial^l W_{t\tau}^3(\xi,\varepsilon)}{\partial\varepsilon^l} \in \mathcal{F}^{p+2l,q-5l}.$$

Further differentiation with respect to $\xi$ and $t$ with regard to Lemma 3.1 and Theorem 3.1 from [5] shows that

$$\frac{\partial^{k+l+n} W_t^1(\xi,\varepsilon)}{\partial\xi^k\partial\varepsilon^l\partial t^n} \in \mathcal{F}^{1+2l,q-k-5l-n}, \qquad \frac{\partial^{k+l+n} W_t^2(\xi,\varepsilon)}{\partial\xi^k\partial\varepsilon^l\partial t^n} \in \mathcal{F}^{p+2l,q-k-5l-n},$$

$$\frac{\partial^{k+l+n} W_{t\tau}^3(\xi,\varepsilon)}{\partial\xi^k\partial\varepsilon^l\partial t^n} \in \mathcal{F}^{p+2l,q-k-5l-n}.$$

For $\varepsilon = 0$, the operator $B_0$ continuously maps the space $\mathcal{F}^{pq}$ to itself. Therefore, these inclusions are also valid for zero $l$ and $\varepsilon$. They give the Taylor expansions for $W_t^i(\xi,\varepsilon)$ with $i = 1, 2, 3$ and the estimates for the norms of the coefficients in these expansions. The existence of an expansion and the estimates for $W^4(\xi)$ follow simply from the analytic dependence of the function $h_{\xi F} \in \mathcal{F}^{1,N-7}$ on $\xi$.

Equality (23) and Proposition 2.1 give the identities

$$e^{\Phi_t(\xi,0)} B_0 = e^{-\dot{F}_t(\xi)} B_0[F_t(\xi)], \qquad B_0[F_t(\xi)] h_{F_t(\xi)} = e^{\dot{F}_t(\xi)} h_{F_t(\xi)}.$$

These identities and the assumption that $B_0[F_t(\xi)]$ is fibered imply that $W_t^1(\xi,0) \equiv 0$. Therefore, $W_{k0t}^1 = 0$.

Under the conditions of Theorem 2.6, consider the discrepancies

$$P_t(\xi,\varepsilon) = e^{-F_{t+\varepsilon}(\xi)/\varepsilon} B_\varepsilon e^{F_t(\xi)/\varepsilon} U_t(\xi,\varepsilon) - U_{t+\varepsilon}(\xi,\varepsilon) = e^{\Phi_t(\xi,\varepsilon)} B_\varepsilon U_t(\xi,\varepsilon) - U_{t+\varepsilon}(\xi,\varepsilon), \tag{24}$$

$$Q_{t\tau}(\xi,\varepsilon) = e^{-F_{t+\varepsilon}(\xi)/\varepsilon} B_\varepsilon e^{F_t(\xi)/\varepsilon} V_{t\tau}(\xi,\varepsilon) - V_{t+\varepsilon,\tau+1}(\xi,\varepsilon) = e^{\Phi_t(\xi,\varepsilon)} B_\varepsilon V_{t\tau}(\xi,\varepsilon) - V_{t+\varepsilon,\tau+1}(\xi,\varepsilon), \tag{25}$$

$$R(\xi,\varepsilon) = g_0 - U_0(\xi,\varepsilon) - V_{00}(\xi,\varepsilon). \tag{26}$$

Obviously, the sum of "$O$" terms in (19) equals $R(\xi,\varepsilon)$, and the sum of such terms in (20) equals $P_t(\xi,\varepsilon) + Q_{t\tau}(\xi,\varepsilon)$. Suppose that

$$p_j = 1 + 2j, \qquad\qquad q_{ij} = N - 7 - 6i - 10j,$$

$$p'_{ij} = \max\{p_{j'} \mid (i',j') \in \mathfrak{D} \cap \Pi_{ij}\}, \qquad q'_{ij} = \min\{q_{i'j'} - 5 \mid (i',j') \in \mathfrak{D} \cap \Pi_{ij}\}.$$

We will prove Theorem 2.6 by extending the diagram $\mathfrak{D}$ by induction.

*Induction hypothesis.* Suppose that a regular Newton diagram $\mathfrak{D}$ is contained in the set $\mathfrak{D}' = \{(i,j) \in \mathbb{Z}_+^2 \mid 6i + 10j \leq N - 21\}$. We denote the set of growth points of $\mathfrak{D}$ by $\mathfrak{G}$ and the set of growth points of $\mathfrak{D}'$ by $\mathfrak{G}'$. Under the conditions of Theorem 2.6, suppose the following:

(a) Families of operators

$$\overline{U}_t(\xi,\varepsilon) = \sum_{(i,j)\in\mathfrak{D}} \xi^i\varepsilon^j\overline{U}_{ijt}, \qquad \widetilde{U}_t(\xi,\varepsilon) = \sum_{(i,j)\in\mathfrak{D}} \xi^i\varepsilon^j\widetilde{U}_{ijt}, \qquad U_t(\xi,\varepsilon) = \overline{U}_t(\xi,\varepsilon)h_{F_t(\xi)} + \widetilde{U}_t(\xi,\varepsilon);$$

$$\overline{V}_{t\tau}(\xi,\varepsilon) = \sum_{(i,j)\in\mathfrak{D}} \xi^i\varepsilon^j\overline{V}_{ijt\tau}, \qquad \widetilde{V}_\tau(\xi,\varepsilon) = \sum_{(i,j)\in\mathfrak{D}} \xi^i\varepsilon^j\widetilde{V}_{ijt\tau}, \qquad V_{t\tau}(\xi,\varepsilon) = \overline{V}_{t\tau}(\xi,\varepsilon) + \widetilde{V}_{t\tau}(\xi,\varepsilon)$$

are constructed.

(b) For all $t \in [0,t_0]$ and $\tau \in \mathbb{Z}_+$, the derivatives

$$\frac{d^n\overline{U}_{ijt}}{dt^n} \in C^{q_{ij}-7-n}(M), \qquad n \leq q_{ij} - 9,$$

$$\frac{d^n\overline{V}_{ijt\tau}}{dt^n} \in \overline{B}\mathcal{F}^{p_j,q_{ij}-n}, \qquad \frac{d^n\widetilde{U}_{ijt}}{dt^n}, \frac{d^n\widetilde{V}_{ijt\tau}}{dt^n} \in \widetilde{B}\mathcal{F}^{p_j,q_{ij}-n}, \qquad n \leq q_{ij} - 4,$$

are defined; these derivatives are uniformly bounded in the spaces specified and linearly and continuously depend on $g_0$; moreover, the norms of the derivatives $d^n\overline{V}_{ijt\tau}/dt^n$ and $d^n\widetilde{V}_{ijt\tau}/dt^n$ exponentially decrease in $\tau$.

(c) The initial condition

$$R(\xi,\varepsilon) = g_0 - U_0(\xi,\varepsilon) - V_{00}(\xi,\varepsilon) = \sum_{(i,j)\in\mathbb{Z}_+^2\setminus\mathfrak{D}} \xi^i\varepsilon^j R_{ij}, \qquad R_{ij} \in \mathcal{F}^{p'_{ij},q'_{ij}},$$

holds.

(d) The following equalities hold:

$$P_t(\xi,\varepsilon) = \sum_{(i,j)\in\mathfrak{D}'\setminus\mathfrak{D}} \xi^i\varepsilon^j P_{ijt} + \sum_{(i,j)\in\mathfrak{G}'} \xi^i\varepsilon^j P_{ijt}(\xi,\varepsilon), \tag{27}$$

$$\overline{B}P_{ijt} = 0 \qquad \text{for} \quad (i,j-1) \in \mathfrak{D} \quad \text{and} \quad (i,j) \in \mathfrak{D}' \setminus \mathfrak{D}, \tag{28}$$

$$Q_{t\tau}(\xi,\varepsilon) = \sum_{(i,j)\in\mathfrak{D}'\setminus\mathfrak{D}} \xi^i\varepsilon^j Q_{ijt\tau} + \sum_{(i,j)\in\mathfrak{G}'} \xi^i\varepsilon^j Q_{ijt\tau}(\xi,\varepsilon). \tag{29}$$

(e) There exist constants $C_{ij}$ and $\Lambda \in (\Lambda_0, 1)$ independent of $g_0$ such that, for all $t \in [0,t_0]$ and $\tau \in \mathbb{Z}_+$, the following estimates are valid:

$$\|P_{ijt}(\xi,\varepsilon)\|_{[2N/5],4} \leq C_{ij}\|g_0\|_{1,N-2}, \qquad \|Q_{ijt\tau}(\xi,\varepsilon)\|_{[2N/5],4} \leq C_{ij}\|g_0\|_{1,N-2},$$

$$\left\|\frac{d^n P_{ijt}}{dt^n}\right\|_{p_j,q_{ij}-n} \leq C_{ij}\|g_0\|_{1,N-2}, \qquad \left\|\frac{d^n Q_{ijt\tau}}{dt^n}\right\|_{p_j,q_{ij}-n} \leq C_{ij}\Lambda^\tau\|g_0\|_{1,N-2}, \tag{30}$$

where $n \leq q_{ij} - 4$, and

$$\left\|\frac{d^n P_{ijt}}{dt^n}\right\|_{p_j,q_{ij}+5-n} \leq C_{ij}\|g_0\|_{1,N-2}, \qquad n \leq q_{ij} + 1, \tag{31}$$

if $(i,j-1) \notin \mathfrak{D}$ (it is assumed that all the derivatives specified above exist and belong to the corresponding spaces).

(f) For $\varepsilon = 0$, the identity $\widetilde{U}_t(\xi,0) \equiv 0$ holds.

Obviously, if we prove this hypothesis for any regular diagram $\mathfrak{D} \subset \mathfrak{D}'$, we will thereby prove Theorem 2.6.

*Basis of induction.* The induction hypothesis holds if the diagram $\mathfrak{D}$ is empty, the functions $U_t(\xi, \varepsilon)$ and $V_{t\tau}(\xi, \varepsilon)$ and the discrepancies $P_t(\xi, \varepsilon)$ and $Q_{t\tau}(\xi, \varepsilon)$ are identically zero, and $R(\xi, \varepsilon) = R_{00} = g_0$. Formally, for $\mathfrak{D} = \varnothing$, the numbers $p'_{ij}$ and $q'_{ij}$ are not defined, but we can take $p'_{ij} = 1$ and $q'_{ij} = N - 2$.

*Induction step.* Consider an arbitrary growth point $(i, j) \in \mathfrak{G}$ of the diagram $\mathfrak{D}$ belonging to $\mathfrak{D}'$. We set

$$\overline{P}_{ijt} = \overline{B}P_{ijt}, \qquad \widetilde{P}_{ijt} = \widetilde{B}P_{ijt}, \qquad \overline{Q}_{ijt\tau} = \overline{B}Q_{ijt\tau}, \qquad \widetilde{Q}_{ijt\tau} = \widetilde{B}Q_{ijt\tau}.$$

A. *Calculation of* $\widetilde{U}_{ijt}$. First, note that $\overline{P}_{ijt} \equiv 0$. For $j > 0$, this follows from (28), and for $j = 0$, from the equality

$$P_t(\xi, 0) = e^{-\dot{F}_t(\xi)} B_0[F_t(\xi)]\big(\overline{U}_t(\xi, 0)h_{F_t(\xi)}\big) - \overline{U}_t(\xi, 0)h_{F_t(\xi)} \equiv 0, \tag{32}$$

which results from condition (f) of the induction hypothesis and from the fact that $B_0[F_t(\xi)]$ is fibered. Therefore, to "kill" the term $\xi^i \varepsilon^j P_{ijt}$ in (27), it is sufficient to add a monomial $\xi^i \varepsilon^j \widetilde{U}_{ijt}$ to the foliated function $U_t(\xi, \varepsilon)$ in (24) such that

$$B_0 \widetilde{U}_{ijt} - \widetilde{U}_{ijt} = -\widetilde{P}_{ijt}.$$

This equation has a unique solution

$$\widetilde{U}_{ijt} = (I - B_0)^{-1} \widetilde{P}_{ijt}. \tag{33}$$

By virtue of (22) and (30), we have $d^n \widetilde{U}_{ijt}/dt^n \in \mathcal{F}^{p_j, q_{ij} - n}$ for $n \le q_{ij} - 4$. Note that, for $j = 0$, identity (32) implies $P_{i0t} \equiv 0$, whence $\widetilde{U}_{i0t} \equiv 0$. Therefore, the addition of $\xi^i \varepsilon^j \widetilde{U}_{ijt}$ to $U_t(\xi, \varepsilon)$ does not violate condition (f) of the induction hypothesis.

B. *Calculation of* $\widetilde{V}_{ijt\tau}$. To "kill" the term $\xi^i \varepsilon^j \widetilde{Q}_{ijt\tau}$ in (29), we must add a monomial $\xi^i \varepsilon^j \widetilde{V}_{ijt\tau}$ to the function $V_{t\tau}(\xi, \varepsilon)$ in (25) for which

$$B_0 \widetilde{V}_{ijt\tau} - \widetilde{V}_{ijt,\tau+1} = -\widetilde{Q}_{ijt\tau}.$$

This equality implies

$$\widetilde{V}_{ijt\tau} = \widetilde{Q}_{ijt,\tau-1} + B_0 \widetilde{V}_{ijt,\tau-1} = \widetilde{Q}_{ijt,\tau-1} + B_0 \widetilde{Q}_{ijt,\tau-2} + B_0^2 \widetilde{V}_{ijt,\tau-2} = \ldots$$

$$= \widetilde{Q}_{ijt,\tau-1} + B_0 \widetilde{Q}_{ijt,\tau-2} + \ldots + B_0^{\tau-1} \widetilde{Q}_{ijt0} + B_0^{\tau} \widetilde{V}_{ijt0}. \tag{34}$$

The function $\widetilde{V}_{ijt0}$ should be chosen so that, after the completion of the induction step, the initial condition (c) is satisfied by the new diagram $\mathfrak{D}$ to which the point $(i, j) \in \mathfrak{G} \cap \mathfrak{D}'$ under consideration is added. This can be done only if $\widetilde{V}_{ij00} = \widetilde{R}_{ij} - \widetilde{U}_{ij0}$. We still have to choose $\widetilde{V}_{ijt0}$ for $t > 0$. The simplest choice is $\widetilde{V}_{ijt0} = \widetilde{V}_{ij00}$. Obviously, $p'_{ij} \le p_j$ and, if $(i, j) \notin \mathfrak{D}$, $q'_{ij} \ge q_{ij}$. By the induction hypothesis, $R_{ij} \in \mathcal{F}^{p'_{ij}, q'_{ij}} \subset \mathcal{F}^{p_j, q_{ij}}$ and $\widetilde{Q}_{ijt\tau} \in \mathcal{F}^{p_j, q_{ij}}$; by construction, $\widetilde{U}_{ij0} \in \mathcal{F}^{p_j, q_{ij}}$. Therefore, $\widetilde{V}_{ijt\tau} \in \mathcal{F}^{p_j, q_{ij}}$.

C. *Calculation of* $\overline{V}_{ijt\tau}$. To "kill" the term $\xi^i \varepsilon^j \overline{Q}_{ijt\tau}$ in (29), it is necessary and sufficient to add a monomial $\xi^i \varepsilon^j \overline{V}_{ijt\tau}$ to the function $V_{t\tau}(\xi, \varepsilon)$ in (25) such that

$$\overline{V}_{ijt\tau} - \overline{V}_{ijt,\tau+1} = -\overline{Q}_{ijt\tau}.$$

We should find a solution $\overline{V}_{ijt\tau}$ to this equation that tends to zero as $\tau \to \infty$. The equation gives

$$\overline{V}_{ijt\tau} = -\overline{Q}_{ijt\tau} + \overline{V}_{ijt,\tau+1} = -\overline{Q}_{ijt\tau} - \overline{Q}_{ijt,\tau+1} + \overline{V}_{ijt,\tau+2} = \dots$$
$$= -\overline{Q}_{ijt\tau} - \overline{Q}_{ijt,\tau+1} - \overline{Q}_{ijt,\tau+2} - \dots, \tag{35}$$

and (30) implies

$$\left\| \frac{d^n \overline{V}_{ijt\tau}}{dt^n} \right\|_{p_j, q_{ij}-n} \le C_{ij} \frac{\Lambda^\tau}{1 - \Lambda} \|g_0\|_{1,N-2}.$$

D. *Calculation of* $\overline{U}_{ijt}$. Let us replace the foliated function $U_t(\xi, \varepsilon)$ by $U_t(\xi, \varepsilon) + \xi^i \varepsilon^j \widetilde{U}_{ijt}$ and calculate the corresponding discrepancy (24) again. It takes the form

$$P'_t(\xi, \varepsilon) = P_t(\xi, \varepsilon) + \xi^i \varepsilon^j W_t^2(\xi, \varepsilon), \qquad \text{where} \quad W_t^2(\xi, \varepsilon) = e^{\Phi_t(\xi, \varepsilon)} B_\varepsilon \widetilde{U}_{ijt} - \widetilde{U}_{ij,t+\varepsilon}.$$

Applying Lemma 3.2, we construct the Taylor expansion of $W_t^2(\xi, \varepsilon)$ with respect to $\xi$ and $\varepsilon$ up to the degree $\nu = [q_{ij}/5] - 1$. As a result, we obtain the Taylor expansion for $P'_t(\xi, \varepsilon)$. In this new expansion, the coefficient $P_{i,j+1,t}$ (of $\xi^i \varepsilon^{j+1}$) is different from that in the old expansion; it equals

$$P'_{i,j+1,t} = P_{i,j+1,t} + \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left( B_\varepsilon \widetilde{U}_{ijt} - \widetilde{U}_{ij,t+\varepsilon} \right) = P_{i,j+1,t} + W^2_{01t}. \tag{36}$$

Lemma 3.2 and (31) yield

$$\frac{d^n W^2_{01t}}{dt^n} \in \mathcal{F}^{p_j+2, q_{ij}-5-n}, \qquad \frac{d^n P_{i,j+1,t}}{dt^n} \in \mathcal{F}^{p_{j+1}, q_{i,j+1}+5-n},$$

whence

$$\frac{d^n P'_{i,j+1,t}}{dt^n} \in \mathcal{F}^{p_j+2,\, q_{ij}-5-n}, \qquad n \le q_{ij} - 9. \tag{37}$$

It remains to "kill" the term $\xi^i \varepsilon^{j+1} \overline{B} P'_{i,j+1,t}$ in (27), or, equivalently, to find a monomial $\xi^i \varepsilon^j \overline{U}_{ijt} e_{F_t(\xi)}$ such that, after adding it to the function $U_t(\xi, \varepsilon)$ in (24), equality (28) is satisfied. To do this, it is necessary and sufficient to solve the equation

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \overline{B}\big( B_\varepsilon(\overline{U}_{ijt} h) - \overline{U}_{ij,t+\varepsilon} h \big) = -\overline{B} P'_{i,j+1,t}.$$

It is equivalent to

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \nu\Big( B_\varepsilon(\overline{U}_{ijt} h) - \overline{U}_{ijt} B_\varepsilon h + \overline{U}_{ijt} B_\varepsilon h - \overline{U}_{ijt} B_0 h + \overline{U}_{ijt} h - \overline{U}_{ij,t+\varepsilon} h \Big) = -\nu(P'_{i,j+1,t}).$$

By Proposition 2.5, this last equation is equivalent to

$$\frac{d\overline{U}_{ijt}}{d\overline{v}_n} + \alpha \overline{U}_{ijt} - \frac{d\overline{U}_{ijt}}{dt} = -\nu(P'_{i,j+1,t}), \qquad \text{where} \quad \alpha = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \nu(B_\varepsilon h) \in C^{N-9}(M). \tag{38}$$

This is an inhomogeneous linear partial differential equation of the first order. It has a unique solution for any initial condition $\overline{U}_{ij0}$. For condition (c) in the induction hypothesis to be satisfied, we should take $\overline{U}_{ij0} = \nu(R_{ij} - \overline{V}_{ij00})$. We have $R_{ij} \in \mathcal{F}^{p'_{ij}, q'_{ij}} \subset \mathcal{F}^{p_j, q_{ij}}$ and $\overline{V}_{ij00} \in \mathcal{F}^{p_j, q_{ij}}$. Therefore, $\overline{U}_{ij0} \in C^{q_{ij}-2}(M)$. On the other hand, (37) implies $d^n \nu(P'_{i,j+1,t})/dt^n \in C^{q_{ij}-7-n}(M)$. Hence, the solution $\overline{U}_{ijt}$ to equation (38) satisfies $d^n \overline{U}_{ijt}/dt^n \in C^{q_{ij}-7-n}(M)$.

E. *Construction of new discrepancies.* First, let us prove the estimates

$$\left\|\frac{d^n\overline{U}_{ijt}}{dt^n}\right\|_{q_{ij}-7-n} \le C\|g_0\|_{1,N-2}, \qquad \left\|\frac{d^n\widetilde{U}_{ijt}}{dt^n}\right\|_{p_j,q_{ij}-n} \le C\|g_0\|_{1,N-2},$$

$$\left\|\frac{d^n\overline{V}_{ijt\tau}}{dt^n}\right\|_{p_j,q_{ij}-n} \le C\Lambda^\tau\|g_0\|_{1,N-2}, \qquad \left\|\frac{d^n\widetilde{V}_{ijt\tau}}{dt^n}\right\|_{p_j,q_{ij}-n} \le C\Lambda^\tau\|g_0\|_{1,N-2},$$

where $C$ is a constant independent of $g_0$. The first three estimates follow from the definitions of the functions $\overline{U}_{ijt}$, $\widetilde{U}_{ijt}$, and $\overline{V}_{ijt\tau}$ and from the corresponding estimates (e) in the induction hypothesis. Estimate (22), the definition of $\widetilde{V}_{ijt\tau}$, and the estimates for $d^nQ_{ijt\tau}/dt^n$ imply

$$\left\|\frac{d^n\widetilde{V}_{ijt\tau}}{dt^n}\right\|_{p_j,q_{ij}-n} \le \|\widetilde{B}\|C_{ij}\left(\Lambda^{\tau-1}+C_0\Lambda_0\Lambda^{\tau-2}+\ldots+C_0\Lambda_0^{\tau-1}\right)\|g_0\|_{1,N-2}+C_0\Lambda_0^\tau\left\|\frac{d^n\widetilde{V}_{ijt0}}{dt^n}\right\|_{p_j,q_{ij}-n}$$

$$\le \|\widetilde{B}\|C_{ij}C_0\frac{\Lambda^\tau}{\Lambda-\Lambda_0}\|g_0\|_{1,N-2}+C_0\Lambda^\tau\left\|\frac{d^n\widetilde{V}_{ijt0}}{dt^n}\right\|_{p_j,q_{ij}-n}.$$

It is easy to see that the right-hand side of this inequality is of the order no higher than $\Lambda^\tau\|g_0\|_{1,N-2}$.

Let us replace the functions $U_t(\xi,\varepsilon)$ and $V_{t\tau}(\xi,\varepsilon)$ in the definitions of discrepancies (24)–(26) by

$$U_t'(\xi,\varepsilon) = U_t(\xi,\varepsilon) + \xi^i\varepsilon^j\left(\overline{U}_{ijt}h_{F_t(\xi)} + \widetilde{U}_{ijt}\right), \qquad V_{t\tau}'(\xi,\varepsilon) = V_{t\tau}(\xi,\varepsilon) + \xi^i\varepsilon^j\left(\overline{V}_{ijt\tau} + \widetilde{V}_{ijt\tau}\right),$$

respectively. Then, the new discrepancies are rewritten as

$$P_t'(\xi,\varepsilon) = P_t(\xi,\varepsilon) + \xi^i\varepsilon^j\left(W_t^1(\xi,\varepsilon) + W_t^2(\xi,\varepsilon)\right), \tag{39}$$

$$Q_{t\tau}'(\xi,\varepsilon) = Q_{t\tau}(\xi,\varepsilon) + \xi^i\varepsilon^jW_{t\tau}^3(\xi,\varepsilon), \tag{40}$$

$$R'(\xi,\varepsilon) = R(\xi,\varepsilon) + \xi^i\varepsilon^jW^4(\xi), \tag{41}$$

where

$$W_t^1(\xi,\varepsilon) = e^{\Phi_t(\xi,\varepsilon)}B_\varepsilon\left(\overline{U}_{ijt}h_{F_t(\xi)}\right) - \overline{U}_{ij,t+\varepsilon}h_{F_{t+\varepsilon}(\xi)},$$

$$W_t^2(\xi,\varepsilon) = e^{\Phi_t(\xi,\varepsilon)}B_\varepsilon\widetilde{U}_{ijt} - \widetilde{U}_{ij,t+\varepsilon},$$

$$W_{t\tau}^3(\xi,\varepsilon) = e^{\Phi_t(\xi,\varepsilon)}B_\varepsilon\left(\overline{V}_{ijt\tau} + \widetilde{V}_{ijt\tau}\right) - \left(\overline{V}_{ij,t+\varepsilon,\tau+1} + \widetilde{V}_{ij,t+\varepsilon,\tau+1}\right),$$

$$W^4(\xi) = \overline{U}_{ij0}h_{\xi F} + \widetilde{U}_{ij0} + \overline{V}_{ij00} + \widetilde{V}_{ij00}.$$

Applying Lemma 3.2, we expand the additional functions $W_t^1(\xi,\varepsilon)$, $W_t^2(\xi,\varepsilon)$, and $W_{t\tau}^3(\xi,\varepsilon)$ in Taylor polynomials with respect to $\xi$ and $\varepsilon$ up to the degree $\nu = [q_{ij}/5] - 2$ and $W^4(\xi)$ in a power series about $\xi$. It remains to verify that what we obtain is expansions for discrepancies (39)–(41) satisfying requirements (c) and (e) of the induction hypothesis for the diagram $\mathfrak{D} \cup \{(i,j)\}$.

F. *Verification of conditions* (c) *and* (e). Conditions (c) and (e) are verified similarly for all the three discrepancies. For this reason, we consider only discrepancy (39), for which the verification is most complicated. In the expansion of $P_t'(\xi,\varepsilon)$ in powers of $\xi$ and $\varepsilon$, we denote the coefficient of $\xi^{i'}\varepsilon^{j'}$ by $P_{i'j't}'$. If $i' < i$ or $j' < j$, then $P_{i'j't}' = P_{i'j't}$, and estimates (30) and (31) hold for $P_{i'j't}'$. Suppose that $i' \ge i$ and $j' \ge j$. Then $i' = i + k$ and $j' = j + l$.

First, consider the case $k + l < \nu$. We have $P'_{i'j't} = P_{i'j't} + W^1_{klt} + W^2_{klt}$ (the coefficients $W^1_{klt}$ and $W^2_{klt}$ are taken from Lemma 3.2). By Lemma 3.2, we have

$$\frac{d^n W^1_{klt}}{dt^n} \in \mathcal{F}^{1+2l, q_{ij}-5-k-5l-n}, \qquad \frac{d^n W^2_{klt}}{dt^n} \in \mathcal{F}^{p_j+2l, q_{ij}-k-5l-n}, \tag{42}$$

and $W^1_{k0t} = 0$. It is easy to see that $1 + 2l \le p_j + 2l = p_{j'}$ and $q_{ij} - k - 5l \ge q_{i'j'} + 5(k+l)$. Therefore, derivatives (42) belong to the space $\mathcal{F}^{p_{j'}, q_{i'j'}-n}$ and, if $(k,l) \ne (0,1)$, to the space $\mathcal{F}^{p_{j'}, q_{i'j'}+5-n}$, to which they should belong according to (e).

Consider the case $k + l = \nu$. We have $\nu = [q_{ij}/5] - 2$. Therefore,

$$6i' + 10j' > 6i + 10j + 5\nu \ge 6i + 10j + q_{ij} - 14 = N - 21,$$

and the point $(i', j')$ lies outside $\mathfrak{D}'$. Hence, it suffices to verify that $W^1_{klt}(\xi, \varepsilon), W^2_{klt}(\xi, \varepsilon) \in \mathcal{F}^{[2N/5],4}$. This is so because, by Lemma 3.2, the functions $W^1_{klt}(\xi, \varepsilon)$ and $W^2_{klt}(\xi, \varepsilon)$ belong to $\mathcal{F}^{p_j+2\nu, q_{ij}-5-5\nu}$, where

$$p_j + 2\nu \le 1 + 2j + 2q_{ij}/5 - 4 = -3 + 2j + 2(N - 7 - 6i - 10j)/5 < 2N/5,$$

$$q_{ij} - 5 - 5\nu \ge q_{ij} - 5 - q_{ij} + 10 = 5.$$

Lemma 3.2 also implies that the norms of the derivatives $d^n W^1_{klt}/dt^n$ and $d^n W^2_{klt}/dt^n$ in the corresponding spaces are estimated in terms of $\|g_0\|_{1,N-2}$. Therefore, $d^n P'_{i'j't}/dt^n$ are also estimated in terms of $\|g_0\|_{1,N-2}$. This completes the proof of Theorem 2.6.

To prove Addition 1, it is sufficient to perform only one induction step at $i = j = 0$. Since $P_{00t} = Q_{00t\tau} = P_{01t} = 0$ and $R_{00} = g_0$, formulas (33)–(35) give

$$\widetilde{U}_{00t} = 0, \qquad \widetilde{V}_{00t\tau} = B_0^\tau \widetilde{V}_{00t0} = B_0^\tau \widetilde{R}_{00} = B_0^\tau \widetilde{B} g_0, \qquad \overline{V}_{00t\tau} = 0.$$

Thus, (36) yields $P'_{01t} = 0$, and equation (38) for finding $\overline{U}_{00t}$ takes the form

$$\frac{d\overline{U}_{00t}}{dt} = \frac{d\overline{U}_{00t}}{d\overline{v}_n} + \alpha \overline{U}_{00t}, \qquad \text{where} \quad \overline{U}_{000} = \nu(R_{00}) = \nu(g_0).$$

Let $g^t$ be the phase flow of the vector field $-\overline{v}_n$ on $M$. Then, the function $u_t = \overline{U}_{00t} \circ g^t$ satisfies the linear ordinary differential equation $\dot{u}_t = \alpha \circ g^t u_t$. It is seen from the definition (38) of the function $\alpha$ that this function is bounded on $M$. If $\inf_M u_0 = m > 0$, then $u_t \ge m e^{t \inf \alpha}$.

To prove Addition 2, it is sufficient to verify that equalities (19) and (20) remain valid when $F_t(\xi)$ is replaced by $F'_t(\xi) = J^k_\xi F_t(\xi)$ and $U_t(\xi, \varepsilon)$ by $U'_t(\xi, \varepsilon) = J^k_\xi U_t(\xi, \varepsilon)$ in these equalities. This imposes natural constraints on $k$. On the one hand, $k$ must be so large that the additional discrepancies arising after such a replacement in (19), (20) have the form $\sum_{(i,j)\in\mathfrak{G}} O(\xi^i \varepsilon^j)$. Actually, they have the form $O(\xi^{k+1})$, whereby this requirement is satisfied if the point $(k+1, 0)$ lies outside $\mathfrak{D}'$, or, equivalently, if $k \ge (N - 26)/6$. On the other hand, $k$ must not be too large in order that the additional discrepancies have the necessary smoothness and belong to the space $\mathcal{F}^{[2N/5],4}$. This requirement is met if the four functions

$$\left(J^k_\xi U_0(\xi, \varepsilon) - U_0(\xi, \varepsilon)\right)/\xi^{k+1}, \qquad \left(J^k_\xi U_{t+\varepsilon}(\xi, \varepsilon) - U_{t+\varepsilon}(\xi, \varepsilon)\right)/\xi^{k+1},$$

$$\left(e^{J^k_\xi \Phi_t(\xi, \varepsilon)} B_\varepsilon J^k_\xi U_t(\xi, \varepsilon) - e^{\Phi_t(\xi, \varepsilon)} B_\varepsilon U_t(\xi, \varepsilon)\right)/\xi^{k+1}, \tag{43}$$

$$\left(e^{J^k_\xi \Phi_t(\xi, \varepsilon)} B_\varepsilon V_{t\tau}(\xi, \varepsilon) - e^{\Phi_t(\xi, \varepsilon)} B_\varepsilon V_{t\tau}(\xi, \varepsilon)\right)/\xi^{k+1}$$

belong to $\mathcal{F}^{[2N/5],4}$. By Lemma 3.1, the smoothness of the function $(J^k_\xi \Phi_t(\xi,\varepsilon) - \Phi_t(\xi,\varepsilon))/\xi^{k+1}$ is no lower than $N - 7 - k$ with respect to all variables and the foliated function $(J^k_\xi h_{F_t(\xi)} - h_{F_t(\xi)})/\xi^{k+1}$ belongs to the space $\mathcal{F}^{1,N-8-k}$. It is easy to derive from the definition of $U_t(\xi,\varepsilon)$ and $V_{t\tau}(\xi,\varepsilon)$ that the inclusion of foliated functions (43) in the space $\mathcal{F}^{[2N/5],4}$ can be ensured if this space is a module over $C^{N-7-k}(W \times M)$. This is so if $N - 7 - k \geq [2N/5] + 4$; thus, it suffices to take $k \leq 3N/5 - 11$. We omit the elementary proof of this assertion.

Let us prove Addition 3. First, note that the definitions of a flowing density and of the norm of a foliated function (Definitions 5.1 and 3.3 in [5]) imply the existence of a large $C$ such that any standard trace $(\Gamma, \alpha, \Phi)$ with flowing density and any function $g \in \mathcal{F}^{pq}$ satisfy the estimate

$$|g(\Gamma, \Phi)| \leq C\|g\|_{pq} \inf \Phi. \tag{44}$$

By assumption, the functions $g_0$ and $h_0$ are strongly positive, which means that they are positive and there exists $c > 0$ such that $g_0(\Gamma, \alpha, 1) \geq c$ and $h_0(\Gamma, \alpha, 1) \geq c$ for all standard leaves $(\Gamma, \alpha)$. We call a foliated function *weakly positive* if it takes nonnegative values on standard traces with flowing densities. Since the image of a trace with flowing density also has flowing density (see Corollary 5.1.1 in [5]), the operator $B_0$ preserves both strong and weak positivity.

Take a standard trace $(\Gamma, \alpha, \Phi)$ with flowing density. The strong positivity of $g_0$ and estimate (44) for the function $h_0$ imply that, for some $\delta > 0$, the function $g_0 - \delta h_0$ is weakly positive. The function $B^k_0(g_0 - \delta h_0) = B^k_0 g_0 - \delta h_0$ is weakly positive as well. Therefore,

$$B^k_0 g_0(\Gamma, \Phi) \geq \delta h_0(\Gamma, \Phi) \geq \delta c \inf \Phi.$$

Identities (19) and (20) with $\xi = \varepsilon = 0$ give $U_0(0,0) + V_{0,k}(0,0) = B^k_0 g_0$. By continuity, for any positive integer $m$, there exists a small domain of variation of the parameters $\xi$ and $\varepsilon$ in which

$$\left[U_{k\varepsilon}(\xi,\varepsilon) + V_{k\varepsilon,k}(\xi,\varepsilon)\right](\Gamma, \alpha, \Phi) \geq \frac{\delta c}{2} \inf \Phi, \qquad k = 0, \ldots, m. \tag{45}$$

It is easy to derive from Addition 1 that there exists $c_0 > 0$ such that

$$U_t(\xi,\varepsilon)(\Gamma, \alpha, \Phi) \geq c_0 \inf \Phi \tag{46}$$

for all sufficiently small $\xi$ and $\varepsilon$ and for any $t \in [0, t_0]$. Since $V_{t\tau}$ exponentially decreases in $\tau$, (45) and (46) guarantee the existence of $c_1 > 0$ such that

$$\left[U_t(\xi,\varepsilon) + V_{t,t/\varepsilon}(\xi,\varepsilon)\right](\Gamma, \alpha, \Phi) \geq c_1 \inf \Phi.$$

On the other hand, by (44),

$$\left[\sum_{(i,j)\in\mathfrak{G}} O(\xi^i \varepsilon^j)\right](\Gamma, \alpha, \Phi) = \sum_{(i,j)\in\mathfrak{G}} O(\xi^i \varepsilon^j) \cdot \inf \Phi$$

uniformly with respect to $(\Gamma, \alpha, \Phi)$ and $t \in [0, t_0]$. These two estimates prove Addition 3.

## 4. CRAMÉR ASYMPTOTICS

Consider an arbitrary probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We say that another probability space $(\Omega', \mathcal{A}', \mathbf{P}')$ is its *extension* if there is a measurable projection $\pi\colon (\Omega', \mathcal{A}', \mathbf{P}') \to (\Omega, \mathcal{A}, \mathbf{P})$ that is measure-preserving (i.e., $\mathbf{P}(M) = \mathbf{P}'(\pi^{-1}(M))$ for all $M \in \mathcal{A}$). Any random variable $\chi$ on $\Omega$ induces a random variable $\chi' = \chi \circ \pi$ on $\Omega'$. The distribution functions of $\chi$ and $\chi'$ coincide. Below, we pass from a probability space $\Omega$ to its extension $\Omega'$ to calculate the distribution of a random variable $\chi$.

Recall that the averaged weighted shift operator $A_{\varepsilon,n}$ is defined by

$$A_{\varepsilon,n}g(\Gamma, \alpha, \Phi) = \int_{\Gamma'_\varepsilon} \xi_\Gamma(\alpha, \beta)g(\Gamma'_\varepsilon, \beta', \Phi'_\varepsilon)\, d\mu(\beta'). \tag{47}$$

In this formula, $(\Gamma, \alpha, \Phi)$ is an arbitrary standard trace centered at $\alpha$, the point $\beta$ belongs to $\Gamma$, $\beta' = \Sigma_\varepsilon^n(\beta)$, the trace $(\Gamma'_\varepsilon, \Phi'_\varepsilon)$ coincides with $(\Sigma_\varepsilon J_\varepsilon)^n(\Gamma, \Phi)$, and $\mu$ is the Riemannian volume on $\Gamma'_\varepsilon$. A standard nonnegative function $\xi_\Gamma(\alpha, \beta)$ on $\Gamma \times \Gamma$ is defined in such a way that it is strictly positive if the distance between $\alpha$ and $\beta$ is smaller than some sufficiently small fixed number $r_0 > 0$, vanishes if the distance between $\alpha$ and $\beta$ is greater than $2r_0$, and satisfies the identity

$$\int_\Gamma \xi_\Gamma(\alpha, \beta)\, d\mu(\alpha) \equiv 1. \tag{48}$$

As above, we assume that the weight function $J_\varepsilon$ is bounded from below by a positive constant. Then, the *normalized* averaged weighted shift operator $B_\varepsilon$ is obtained from $A_{\varepsilon,n}$ by replacing the weight $J_\varepsilon$ by the normalized weight $\widehat{J}_\varepsilon = e^{-\lambda_n/n}J_\varepsilon$ in the definition of $A_{\varepsilon,n}$ ($e^{\lambda_n}$ is the maximum eigenvalue of $A_{0,n}$). If the initial weight is normalized in advance, then $A_{\varepsilon,n}$ coincides with $B_\varepsilon$.

Let $(G_0, \Phi_0)$ be a trace with sufficiently large linear dimensions and positive density $\Phi_0$. Suppose that the set $\Gamma_0$ is a compact subset of $G_0$ and, for any point $\alpha \in \Gamma_0$, there exists a standard leaf $(\Gamma, \alpha)$ centered at $\alpha$ and lying entirely in $G_0$. Take a foliated function $g_0$ defined so that its value on any standard trace $(\Gamma, \alpha, \Phi)$ equals $\Phi(\alpha)$. Consider the linear functionals $\nu_k \colon C^\infty(G_0) \to \mathbb{R}$ defined by

$$\nu_k(f) = \int_{\Gamma_0} A_{\varepsilon,n}^k g_0(\Gamma, \alpha, f\Phi_0)\, d\mu(\alpha). \tag{49}$$

Using definition (47), we can represent these functionals as multiple integrals. For this purpose, we construct sequences of traces $(G_i, \Phi_i) = (\Sigma_\varepsilon J_\varepsilon)^{ni}(G_0, \Phi_0)$ and $(G_i, f_i) = \Sigma_\varepsilon^{ni}(G_0, f)$. Obviously,

$$\Phi_i\big(\Sigma_\varepsilon^{ni}(\alpha)\big) = \big(\Phi \times J_\varepsilon \times J_\varepsilon \circ \Sigma_\varepsilon \times \ldots \times J_\varepsilon \circ \Sigma_\varepsilon^{ni-1}\big)(\alpha),$$

$$f_i\big(\Sigma_\varepsilon^{ni}(\alpha)\big) = f(\alpha), \qquad (G_i, f_i\Phi_i) = (\Sigma_\varepsilon J_\varepsilon)^{ni}(G_0, f\Phi_0).$$

Consider sequences of pairs of points $(\alpha_i, \beta_i) \in G_i$, where $i = 0, \ldots, k$, such that $\alpha_{i+1} = \Sigma_\varepsilon^n(\beta_i)$ for all $i = 0, \ldots, k-1$. Formulas (47) and (49) yield

$$\nu_k(f) = \int_{G_k} f_k(\alpha_k)\Phi_k(\alpha_k)\, d\mu(\alpha_k) \int_{G_{k-1}} \xi_{G_{k-1}}(\alpha_{k-1}, \beta_{k-1})\, d\mu(\alpha_{k-1}) \ldots$$

$$\ldots \int_{G_1} \xi_{G_1}(\alpha_1, \beta_1)\, d\mu(\alpha_1) \int_{\Gamma_0} \xi_{G_0}(\alpha_0, \beta_0)\, d\mu(\alpha_0) \tag{50}$$

$$= \int_{\Gamma_0} d\mu(\alpha_0) \int_{G_1} \xi_{G_0}(\alpha_0, \beta_0)\, d\mu(\alpha_1) \ldots \int_{G_k} \xi_{G_{k-1}}(\alpha_{k-1}, \beta_{k-1}) f_k(\alpha_k)\Phi_k(\alpha_k)\, d\mu(\alpha_k). \tag{51}$$

For an arbitrary $R > 0$, let $\Gamma_0^{+R}$ denote the set of points in the leaf $G_0$ that lie at distances smaller than $R$ from $\Gamma_0$, and let $\Gamma_0^{-R}$ be the set of points in $\Gamma_0$ that lie at distances greater than $R$ from $G_0 \setminus \Gamma_0$. Without loss of generality, we can assume that the mapping $\Sigma_\varepsilon$ expands all leaves with the expansion coefficient no smaller than $\Lambda^{-1}$, where the number $\Lambda \in (0, 1)$ does not depend on the leaf.

**Proposition 4.1.** *For any $\varepsilon$ and $k$, there exists a nonnegative smooth function $\zeta \colon G_0 \to [0,1]$ that takes value 1 on $\Gamma_0^{-R}$, vanishes on $G_0 \setminus \Gamma_0^{+R}$, where $R = 2r_0(1-\Lambda)^{-1}$, and satisfies*

$$\nu_k(f) = \int\limits_{G_0} \zeta f \Phi_0 \times J_\varepsilon \times J_\varepsilon \circ \Sigma_\varepsilon \times \ldots \times J_\varepsilon \circ \Sigma_\varepsilon^{nk-1} \, d\mu \circ \Sigma_\varepsilon^{nk} \tag{52}$$

*(where $\mu$ is the Riemannian volume on $\Sigma_\varepsilon^{nk}(G_0)$). In particular, for the canonical weight $J_\varepsilon$,*

$$\nu_k(\Phi) = \int\limits_{G_0} \zeta \Phi \, d\mu,$$

*where $\mu$ is the Riemannian volume on $G_0$.*

**Proof.** Let

$$\zeta(\alpha) = \int\limits_{G_{k-1}} \xi_{G_{k-1}}(\alpha_{k-1}, \beta_{k-1}) \, d\mu(\alpha_{k-1}) \ldots \int\limits_{\Gamma_0} \xi_{G_0}(\alpha_0, \beta_0) \, d\mu(\alpha_0), \tag{53}$$

where $\beta_{k-1} = \Sigma_\varepsilon^{n(k-1)}(\alpha)$. Then, (52) follows from (50). We set

$$\Gamma_0^- = \Gamma_0^{-2r_0}, \quad \Gamma_1^- = \left(\Sigma_\varepsilon^n(\Gamma_0^-)\right)^{-2r_0}, \quad \ldots, \quad \Gamma_k^- = \left(\Sigma_\varepsilon^n(\Gamma_{k-1}^-)\right)^{-2r_0}.$$

Since the mapping $\Sigma_\varepsilon$ expands all leaves with the expansion coefficient no smaller than $\Lambda^{-1}$, we have $\Sigma_\varepsilon^{ni}(\Gamma_0^{-R}) \subset \Gamma_i^-$ for all $i$. Successively calculating the integrals in (53) from right to left and taking into account (48), we obtain

$$\int\limits_{G_i} \xi_{G_i}(\alpha_i, \beta_i) \, d\mu(\alpha_i) \ldots \int\limits_{G_1} \xi_{G_1}(\alpha_1, \beta_1) \, d\mu(\alpha_1) \int\limits_{\Gamma_0} \xi_G(\alpha_0, \beta_0) \, d\mu(\alpha_0) = 1$$

for $\beta_i \in \Gamma_i^-$. This equality (with $i = k-1$) and the inclusion $\Sigma_\varepsilon^{n(k-1)}(\Gamma_0^{-R}) \subset \Gamma_{k-1}^-$ imply $\zeta(\alpha) = 1$ for $\alpha \in \Gamma_0^{-R}$. Similarly, $\zeta(\alpha) = 0$ for $\alpha \in G_0 \setminus \Gamma_0^{+R}$. Finally, it is seen from (48) and (53) that $\zeta(\alpha) \in [0,1]$ for $\alpha \in G_0$. This completes the proof of Proposition 4.1.

We study a trace $(G_0, \Phi_0)$ with positive density $\Phi_0$ and a leaf $\Gamma_0 \subset G_0$ such that, for each point $\alpha \in \Gamma_0$, there exists a standard leaf $(\Gamma, \alpha)$ lying entirely in $G_0$. Let us define linear functionals $\mathbf{P}_k$ on $C^\infty(G_0)$ by the formula

$$\mathbf{P}_k(f) = \frac{\nu_k(f)}{\nu_k(1)}. \tag{54}$$

By (52), the functional $\mathbf{P}_k$ is positive (i.e., it takes nonnegative values on nonnegative functions). In addition, it is normalized (i.e., $\mathbf{P}_k(1) = 1$). According to the Riesz theorem, $\mathbf{P}_k$ determines a Borel probability measure on $G_0$.

Consider the probability space $(G_0, \mathcal{B}, \mathbf{P}_k)$, where $\mathcal{B}$ is the Borel $\sigma$-algebra on $G_0$. We define a probability space $\Omega_k$ as the set of sequences of pairs of points $(\alpha_i, \beta_i) \in G_i$, $i = 0, \ldots, k-1$, where the distance between $\alpha_i$ and $\beta_i$ does not exceed $2r_0$ and $\alpha_{i+1} = \Sigma_\varepsilon^n(\beta_i)$ for all $i$. Obviously, in every such sequence, the points $\beta_i$ are uniquely determined by $\alpha_i$. Therefore, any function $f$ on $\Omega_k$ can be represented as $f = f(\alpha_0, \ldots, \alpha_k)$. We define a probability measure $\mathbf{P}_k'$ on $\Omega_k$ as the linear functional $\mathbf{P}_k'(f) = \nu_k'(f)/\nu_k'(1)$, where

$$\nu_k'(f) = \int\limits_{\Gamma_0} d\mu(\alpha_0) \int\limits_{G_1} \xi_{G_0}(\alpha_0, \beta_0) \, d\mu(\alpha_1) \ldots \int\limits_{G_k} \xi_{G_{k-1}}(\alpha_{k-1}, \beta_{k-1}) f(\alpha_0, \ldots, \alpha_k) \Phi_k(\alpha_k) \, d\mu(\alpha_k). \tag{55}$$

If $f(\alpha_0, \ldots, \alpha_k) = f_k(\alpha_k)$, then (55) coincides with (51). Therefore, $(\Omega_k, \mathcal{B}, \mathbf{P}'_k)$ is an extension of $(G_0, \mathcal{B}, \mathbf{P}_k)$. The corresponding projection $\pi \colon \Omega_k \to G_0$ maps a sequence $(\alpha_0, \ldots, \alpha_k)$ to a point $\alpha \in G_0$ such that $\Sigma_\varepsilon^{nk}(\alpha) = \alpha_k$. Consider the random variable $f(\alpha) = F \circ \Sigma_\varepsilon^{nk}(\alpha)$ on $G_0$. It induces a random variable $f \circ \pi(\alpha_0, \ldots, \alpha_k) = F(\alpha_k)$ on $\Omega_k$. Therefore, the distribution of $F \circ \Sigma_\varepsilon^{nk}$ with respect to the measure $\mathbf{P}_k$ coincides with the distribution of $F(\alpha_k)$ with respect to the measure $\mathbf{P}'_k$.

**Theorem 4.2.** *Suppose that $J_\varepsilon$ is a normalized weight, $\Phi_0$ is a flowing density on the leaf $G_0$, and $g_0$ is a foliated function taking the value $\Phi(\alpha)$ on each standard trace $(\Gamma, \alpha, \Phi)$. Then, under the conditions of Theorem 2.6, there exists a family of continuous functions $G_k(\xi, \varepsilon) \colon \Omega_k \to \mathbb{C}$ that analytically depends on $\xi$, is real for $\xi \in \mathbb{R}$, and satisfies the equalities*

$$\mathbf{P}'_{t/\varepsilon}\big(\exp(\xi F(\alpha_{t/\varepsilon})/\varepsilon + G_{t/\varepsilon}(\xi, \varepsilon))\big) = \frac{e^{F_t^*(\xi)/\varepsilon} \int_{\Gamma_0} \big(U_t^*(\xi, \varepsilon) + V_{t,t/\varepsilon}(\xi, \varepsilon)\big)(\Gamma, \alpha, \Phi_0)\, d\mu(\alpha)}{\int_{\Gamma_0} \big(U_t^*(0, \varepsilon) + V_{t,t/\varepsilon}(0, \varepsilon)\big)(\Gamma, \alpha, \Phi_0)\, d\mu(\alpha)} \tag{56}$$

*for all sufficiently small $\xi$, $\varepsilon$ and $t \in \varepsilon\mathbb{Z} \cap [0, t_0]$; here, $F_t^*(\xi) = J_\xi^{[N/5]} F_t(\xi)$ and $U_t^*(\xi, \varepsilon) = J_\xi^{[N/5]} \overline{U}_t(\xi, \varepsilon) + \widetilde{U}_t(\xi, \varepsilon)$ are Taylor polynomials of degree $[N/5]$ with respect to the variable $\xi$ and*

$$G_{t/\varepsilon}(\xi, \varepsilon) = \varepsilon^{-1} \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j). \tag{57}$$

**Proof.** For convenience, we set $k = t/\varepsilon$ and rewrite identities (19) and (20) from Theorem 2.6 with regard to its Addition 2, with $k = [N/5]$:

$$g_0 = U_0^*(\xi, \varepsilon) + V_{00}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j), \tag{58}$$

$$B_\varepsilon e^{F_t^*(\xi)/\varepsilon}\big(U_t^*(\xi, \varepsilon) + V_{t\tau}(\xi, \varepsilon)\big) = e^{F_{t+\varepsilon}^*(\xi)/\varepsilon}\Bigg(U_{t+\varepsilon}^*(\xi, \varepsilon) + V_{t+\varepsilon, \tau+1}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j)\Bigg). \tag{59}$$

Let $(\Gamma_i, \alpha_i)$ be a standard leaf in $G_i$. Consider the functions

$$\psi_{k-i}(\xi, \varepsilon)(\alpha_{k-i}) = \frac{e^{F_{i\varepsilon}^*(\xi)/\varepsilon}\big(U_{i\varepsilon}^*(\xi, \varepsilon) + V_{t,t/\varepsilon}(\xi, \varepsilon)\big)(\Gamma_{k-i}, \alpha_{k-i}, \Phi_{k-i})}{e^{F_{i\varepsilon}^*(\xi)/\varepsilon}\big(U_{i\varepsilon}^*(\xi, \varepsilon) + V_{t,t/\varepsilon}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j)\big)(\Gamma_{k-i}, \alpha_{k-i}, \Phi_{k-i})},$$

$$\psi_k(\xi, \varepsilon)(\alpha_k) = \frac{g_0(\Gamma_k, \alpha_k, \Phi_k)}{\big(U_0^*(\xi, \varepsilon) + V_{00}(\xi, \varepsilon) + \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j)\big)(\Gamma_k, \alpha_k, \Phi_k)},$$

where the numerators and denominators are taken from (58) and (59), and

$$\Psi_k(\xi, \varepsilon)(\alpha_0, \ldots, \alpha_k) = \prod_{i=0}^k \psi_{k-i}(\xi, \varepsilon)(\alpha_{k-i}).$$

Take the function $f = \Psi_k(\xi, \varepsilon) e^{\xi F(\alpha_k)/\varepsilon}$ in (55). Let us successively calculate all integrals in (55) by using the definition of the operator $B_\varepsilon$ and equalities (58) and (59) at each step. We obtain

$$\nu'_k\big(\Psi_k(\xi, \varepsilon) e^{\xi F(\alpha_k)/\varepsilon}\big) = \int_{\Gamma_0} e^{F_t^*(\xi)/\varepsilon}\big(U_t^*(\xi, \varepsilon) + V_{t,t/\varepsilon}(\xi, \varepsilon)\big)(\Gamma, \alpha, \Phi_0)\, d\mu(\alpha), \tag{60}$$

$$\nu'_k\big(\Psi_k(0, \varepsilon)\big) = \int_{\Gamma_0} \big(U_t^*(0, \varepsilon) + V_{t,t/\varepsilon}(0, \varepsilon)\big)(\Gamma, \alpha, \Phi_0)\, d\mu(\alpha). \tag{61}$$

Let

$$G_k(\xi, \varepsilon) = \sum_{i=0}^{k} \ln \psi_i(\xi, \varepsilon)(\alpha_i) - \ln \frac{\nu_k'(\Psi_k(0, \varepsilon))}{\nu_k'(1)}.$$

The division of (60) by (61) yields (56). Addition 3 implies that $\psi_i(\xi, \varepsilon)(\alpha_i) - 1 = \sum_{(i,j) \in \mathfrak{G}} O(\xi^i \varepsilon^j)$ uniformly with respect to $i = 1, \ldots, k$. This gives estimates (57) and proves the theorem.

Below, we state a theorem that allows one to derive from (56) exact asymptotics of large deviation probabilities for the random variable $F \circ \Sigma_\varepsilon^{t/\varepsilon}$ with respect to the probability measure $\mathbf{P}_{t/\varepsilon}$. Suppose that some real random variable $S$ has a distribution function $\Psi$ and $\Phi_d$ is the Gaussian distribution with mathematical expectation zero and variance $d$.

**Theorem 4.3** [3]. *Suppose that* $\mathbf{E}\, e^{\xi S + G(\xi)} = e^{\varphi(\xi)}$ *for* $|\xi| \leq R$, *where* $G(\xi)$ *is a random variable that analytically depends on a complex parameter* $\xi$ *and is real for real* $\xi$, *and* $\varphi$ *is an analytic function satisfying the conditions* $\varphi(0) = \varphi'(0) = 0$ *and* $\varphi''(0) = d > 0$. *If a positive constant* $\Gamma$ *and a function* $\gamma(r)$ *nondecreasing for* $r > 0$ *are such that* $\sup_{|\xi| < R} |\varphi(\xi)| \leq \Gamma$ *and* $\sup_{|\xi| < r} \operatorname{ess\,sup} |G(\xi)| \leq \gamma(r)$, *then*

$$\frac{1 - \Psi(\varphi'(\alpha))}{1 - \Phi_d(\varphi'(\alpha))} = e^{\varphi(\alpha) - \alpha\varphi'(\alpha) + \varphi'(\alpha)^2/2d} \left[ 1 + O\left( \inf_{0 < r < dR^3/\Gamma} \left( \frac{1}{r\sqrt{d}} + \gamma(r) \right) (\alpha\sqrt{d} + 1) \right) \right] \qquad (62)$$

*for all* $\alpha > 0$.

In this formula, the symbol $O(\zeta)$ denotes a function such that $|O(\zeta)| \leq \Delta|\zeta|$ for all $|\zeta| \leq \delta$, where $\Delta$ and $\delta$ are universal positive constants.

Let us apply this theorem to the random variable $F \circ \Sigma_\varepsilon^{t/\varepsilon}$. Under the conditions of Theorems 2.6 and 4.2, introduce the notation

$$f_t(\xi, \varepsilon) = \frac{F_t^*(\xi)}{\varepsilon} + \ln \frac{\int_{\Gamma_0} (U_t^*(\xi, \varepsilon) + V_{t,t/\varepsilon}(\xi, \varepsilon))(\Gamma, \alpha, \Phi_0) \, d\mu(\alpha)}{\int_{\Gamma_0} (U_t^*(0, \varepsilon) + V_{t,t/\varepsilon}(0, \varepsilon))(\Gamma, \alpha, \Phi_0) \, d\mu(\alpha)}, \qquad (63)$$

$$a = \left. \frac{\partial f_t(\xi, \varepsilon)}{\partial \xi} \right|_{\xi=0}, \qquad S = \frac{F \circ \Sigma_\varepsilon^{t/\varepsilon}}{\varepsilon} - a, \qquad \varphi(\xi) = f_t(\xi, \varepsilon) - a\xi. \qquad (64)$$

Let $\Psi$ be the distribution function of the random variable $S$ with respect to the measure $\mathbf{P}_{t/\varepsilon}$, and let $\Phi_d$ be the Gaussian distribution with mathematical expectation zero and variance $d = \varphi''(0)$.

**Theorem 4.4.** *If the conditions of Theorem 4.2 hold, the derivative* $\partial^2 F_t(0)/\partial\xi^2$ *is positive, a diagram* $\mathfrak{D}$ *coincides with the set* $\mathfrak{D}' = \{(i, j) \in \mathbb{Z}_+^2 \mid 6i + 10j \leq N - 21\}$, *and* $N > 32$, *then*

$$\frac{1 - \Psi(\varphi'(\alpha))}{1 - \Phi_d(\varphi'(\alpha))} = e^{\varphi'(\alpha) - \alpha\varphi'(\alpha) + \varphi'(\alpha)^2/2d} \left[ 1 + O\left( \varepsilon^{-9/(N-14)}(\alpha + \sqrt{\varepsilon}) \right) \right] \qquad (65)$$

*for any* $\delta > 9/(N - 14)$ *and all* $\alpha \in (0, \varepsilon^\delta)$, *where the notation* (63) *and* (64) *is used.*

This is a Cramér asymptotics for the random variable $S$ and, thereby, for the random variable $F \circ \Sigma_\varepsilon^{t/\varepsilon}/\varepsilon$ because the latter differs from $S$ only by the nonrandom term $a$. It holds in the domain where $\alpha < \varepsilon^\delta$. In this domain, the order of $\varphi'(\alpha)$ does not exceed $\varepsilon^{\delta-1}$ and, accordingly, the order of the difference $F \circ \Sigma_\varepsilon^{t/\varepsilon} - \varepsilon a$ does not exceed $\varepsilon^\delta$.

**Proof of Theorem 4.4.** Theorem 4.2 implies the identity $\mathbf{E}\, e^{\xi S + G_t(\xi, \varepsilon)} = e^{\varphi(\xi)}$. Therefore, it is sufficient to refer to Theorem 4.3. We only need to estimate the discrepancy in equality (62). In the situation under consideration, the variance $d = \varphi''(0) = \varepsilon^{-1} F_t''(0) + O(1)$ is on the order of $\varepsilon^{-1}$. It is easy to see that we can take $R$ of order 1 and $\Gamma$ of the order $\varepsilon^{-1}$ in (62). Choose a

function $\gamma(r,\varepsilon)$ such that $\sup_{|\xi|<r} \operatorname{ess\,sup} |G_t(\xi,\varepsilon)| \le \gamma(r,\varepsilon)$. Let us apply the well-known inequality $a^p b^q \le a + b$, where $p + q = 1$. If $(i,j) \in \mathfrak{G}$, then $6i + 10j \ge N - 20$ and

$$|\xi^i \varepsilon^j| \le |\xi|^{(6i+10j)/6} + \varepsilon^{(6i+10j)/10} \le |\xi|^{(N-20)/6} + \varepsilon^{(N-20)/10}.$$

Therefore, for small $\xi$ and $\varepsilon$ and a sufficiently large $C$, we have

$$|G_t(\xi,\varepsilon)| = \varepsilon^{-1} \sum_{(i,j)\in\mathfrak{G}} O(|\xi|^i \varepsilon^j) \le C\varepsilon^{-1}\big(|\xi|^{(N-20)/6} + \varepsilon^{(N-20)/10}\big).$$

Hence, we can take $\gamma(r,\varepsilon) = C\varepsilon^{-1}\big(r^{(N-20)/6} + \varepsilon^{(N-20)/10}\big)$. An elementary calculation shows that, in this situation, we have

$$\inf_{0<r<dR^3/\Gamma} \left(\frac{1}{r\sqrt{d}} + \gamma(r,\varepsilon)\right) \sim \varepsilon^{(N-32)/(2N-28)}$$

(the infimum is attained at $r \sim \varepsilon^{9/(N-14)}$). This estimate and (62) imply (65).

To conclude, we add a few words about the mathematical expectation and variance of the random variable $F \circ \Sigma_\varepsilon^{t/\varepsilon}$. Let us expand the function $F_t(\xi)$ in powers of $\xi$: $F_t(\xi) = \xi F_{1t} + \xi^2 F_{2t} + \dots$. It follows from (56) that the mathematical expectation of $F \circ \Sigma_\varepsilon^{t/\varepsilon}$ is equal to $F_{1t} + O(\varepsilon)$, and the variance is equal to $2\varepsilon F_{2t} + O(\varepsilon^2)$. By construction, the function $F_t(\xi)$ is a solution to the differential equation $\dot{F} = \lambda_F$ with the initial condition $F_0(\xi) = \xi F$. By Proposition 2.5, $d\lambda_{\xi f}/d\xi\big|_{\xi=0} = df/d\overline{v}_n$, where $\overline{v}_n = n\overline{v}$. Therefore, the coefficients $F_{1t}$ and $F_{2t}$ satisfy the differential equations

$$\frac{dF_{1t}}{dt} = n\frac{dF_{1t}}{d\overline{v}}, \qquad \frac{dF_{2t}}{dt} = n\frac{dF_{2t}}{d\overline{v}} + \frac{1}{2}\left.\frac{d^2\lambda_{\xi F_{1t}}}{d\xi^2}\right|_{\xi=0}$$

with the initial conditions $F_{10} = F$ and $F_{20} = 0$. Let $g^t$ be the phase flow of the vector field $\overline{v}$ on $M$. Then, $F_{1t}(z) = F \circ g^{nt}(z)$. By Proposition 2.2, the second derivative $d^2\lambda_{\xi f}/d\xi^2$ is always nonnegative. Therefore, the function $F_{2t}$ is also nonnegative (in the case of general position, it is strictly positive). These facts should be interpreted as follows. If $\{(w_i, z_i)\}_{i=0}^\infty$ is a trajectory of initial cascade (1) with the initial condition $(w_0, z_0) \in G_0$, then the point $z_{t/\varepsilon}$ has asymptotically normal distribution whose mathematical expectation in the first approximation equals $g^t(z_0)$ and variance is of order no higher than $\varepsilon$. Formula (65) allows one to calculate asymptotic expansions in powers of $\varepsilon$ for the mathematical expectation, variance, and higher-order semi-invariants of the random variable $F(z_{t/\varepsilon})$; in addition, it gives sharp asymptotics for the probabilities of large deviations of $F(z_{t/\varepsilon})$ from $F \circ g^t(z_0)$ by distances of order no higher than $\varepsilon^\delta$.

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## REFERENCES

1. V. I. Bakhtin, "A Direct Method for Constructing an Invariant Measure on a Hyperbolic Attractor," Izv. Ross. Akad. Nauk, Ser. Mat. **56** (5), 934–957 (1992) [Russ. Acad. Sci., Izv. Math. **41** (2), 207–227 (1993)].
2. V. I. Bakhtin, "Random Processes Generated by a Hyperbolic Sequence of Mappings. I, II," Izv. Ross. Akad. Nauk, Ser. Mat. **58** (2), 40–72; (3), 184–195 (1994) [Russ. Acad. Sci., Izv. Math. **44** (2), 247–279; (3), 617–627 (1995)].
3. V. I. Bakhtin, "Large Deviations of a Random Variable with a Finite Number of Semi-Invariants Whose Values Are Calculated Approximately," Teor. Veroyatn. Ee Primen. **42** (3), 603–608 (1997) [Theory Probab. Appl. **42** (3), 513–517 (1997)].

4. V. I. Bakhtin, "Cramér Asymptotics in a System with Slow and Fast Markovian Motions," Teor. Veroyatn. Primen. **44** (1), 14–33 (1999) [Theory Probab. Appl. **44** (1), 1–17 (1999)].

5. V. I. Bakhtin, "Foliated Functions and an Averaged Weighted Shift Operator for Perturbations of Hyperbolic Mappings," Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk **244**, 35–64 (2004) [Proc. Steklov Inst. Math. **244**, 29–57 (2004)].

6. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, Berlin, 1975), Lect. Notes Math. **470**; "Markov Partitions and Minimal Sets for Axiom A Diffeomorphisms," Am. J. Math. **92**, 907–917 (1970); "Symbolic Dynamics for Hyperbolic Flows," Am. J. Math. **95**, 429–459 (1973); R. Bowen and D. Ruelle, "The Ergodic Theory of Axiom A Flows," Invent. Math. **29** (3), 181–202 (1975); R. Bowen, "A Horseshoe with Positive Measure," Invent. Math. **29** (3), 203–204 (1975); "Topological Entropy for Noncompact Sets," Trans. Am. Math. Soc. **184**, 125–136 (1973); *Methods of Symbolic Dynamics* (Mir, Moscow, 1979) [in Russian].

7. V. B. Minasyan, "An Averaging Principle and a Theorem on Large Deviations for a Family of Extensions of a Y-Flow," Usp. Mat. Nauk **35** (2), 215–216 (1980) [Russ. Math. Surv. **35** (2), 251–252 (1980)].

8. Ya. B. Pesin, "A General Theory of Smooth Hyperbolic Dynamical Systems," in *Itogi Nauki Tekh., Ser. Sovr. Probl. Mat., Fundam. Napravleniya*, Vol. 2 (VINITI, Moscow, 1985), pp. 123–173.

9. Yu. Kifer, "Averaging in Dynamical Systems and Large Deviations," Invent. Math. **110** (2), 337–370 (1992).

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