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Foliated Functions and an Averaged Weighted Shift Operator for Perturbations of Hyperbolic Mappings

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Abstract—In order to study the perturbations of a family of mappings with a hyperbolic mixing attractor, an apparatus of foliated functions is developed. Foliated functions are analogues of distributions based on smooth measures on leaves (traces), which are embedded manifolds in a neighborhood of the attractor. The dimension of such manifolds must coincide with the dimension of the expanding foliation, and the values of a foliated function on a trace must vary smoothly under smooth transverse deformations of the trace (which include deformations of the measure itself).

In [1–3], the notion of regular functions was introduced to study the stochastic properties of hyperbolic sequences of mappings. It was applied to prove that invariant measures on a hyperbolic attractor depend smoothly on a parameter and to obtain limit theorems for random processes on an attractor. In this paper, a similar apparatus is developed for a dynamical system that represents a perturbation of a family of mappings with a hyperbolic mixing attractor. Here, we use the term "foliated function" instead of "regular function" for two reasons: first, "foliated" better matches the nature of the object, and second, "regular" has a too broad meaning.

Consider a dynamical system with discrete time generated by the mapping

$$\begin{cases} w' = S(w, z, \varepsilon), \\ z' = z + \varepsilon v(w, z, \varepsilon). \end{cases}$$

We use the abbreviated notation $(w', z') = \Sigma_\varepsilon(w, z)$ for this mapping. Suppose that the mapping $w \mapsto S(w, z, 0)$ has a hyperbolic mixing attractor for every z. In this paper, we introduce a notion of foliated functions for such systems, which is a remote analogue of distributions. Distributions are defined as linear functionals on some space of base functions. The base for foliated functions are the so-called *traces* rather than usual functions. A trace is a smooth measure on a leaf, and a leaf is a submanifold in a neighborhood of a hyperbolic attractor with a distinguished point, which is called a *center*. This submanifold must have the same dimension as the expanding foliation. We consider only leaves whose directions little differ from the direction of expanding fibers. It is required that the value of a foliated function on a trace should vary smoothly under smooth transverse deformations of the trace (which include deformations of the measure itself).

If Γ is a leaf centered at α and Φ is a measure on this leaf, then the product of the trace (Γ, Φ) and a function $J = J(w, z)$ is the trace $(\Gamma, J\Phi)$. The image of the trace (Γ, Φ) is the trace $\Sigma_\varepsilon(\Gamma, \Phi) = (\Sigma_\varepsilon(\Gamma), \Phi \circ \Sigma_\varepsilon^{-1})$. On the set of traces, an operator $\Sigma_\varepsilon J$ is defined by the formula $\Sigma_\varepsilon J(\Gamma, \Phi) = \Sigma_\varepsilon(\Gamma, J\Phi)$. Consider an arbitrary foliated function $g(\Gamma, \alpha, \Phi)$. For this function, we define a new foliated function $A_{\varepsilon,n} g(\Gamma, \alpha, \Phi)$ as follows. We set $(\Gamma', \Phi') = (\Sigma_\varepsilon J)^n(\Gamma, \Phi)$, $\beta' = \Sigma_\varepsilon^n(\beta)$,

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and

$$A_{\varepsilon,n}g(\Gamma,\alpha,\Phi)=\int\limits_{\Gamma'}\xi_\Gamma(\alpha,\beta)g(\Gamma',\beta',\Phi')\,d\mu(\beta').$$

In this formula, μ is the Riemannian volume on Γ' and $\xi_\Gamma(\alpha,\beta)$ is a standard nonnegative function on $\Gamma\times\Gamma$. We have thus defined an *averaged weighted shift operator* $A_{\varepsilon,n}$ (with weight function J) in the space of foliated functions. In this paper, we study its simplest properties.

The main results are as follows. When endowed with a suitable norm, the space of foliated functions becomes a Banach module over the algebra of finitely smooth functions depending on the variables w and z. If the weight function J is positive and bounded away from zero, then, for the operator $A_{0,n}$ (with $\varepsilon=0$) and large n, there exist a smooth function $\lambda_n=\lambda_n(z)$ and a foliated function h_n such that $A_{0,n}h_n=e^{\lambda_n}h_n$. Moreover, the sequence of operators $(e^{-\lambda_n}A_{0,n})^m$ converges as $m\to\infty$ to some projector that maps the entire module of foliated functions to a submodule generated by h_n (over the algebra of functions depending only on z). For nonzero ε, the dependence of the family $A_{\varepsilon,n}$ on ε is smooth in a certain sense (to be more precise, it is *quasidifferentiable* in the terminology of [2]).

The paper is organized as follows. In Section 1, the system under examination is described in detail, and a special hyperbolic atlas and the notion of a leaf are defined for this system. In Section 2, a Σ_ε-invariant class of leaves is constructed. In Section 3, spaces of foliated functions and averaged weighted shift operators are rigorously defined, and the main results (Theorems 3.1–3.3) are stated. The following six sections contain detailed proofs. Almost all of them are computational. They are based on carefully differentiating implicit functions and estimating their derivatives. Unfortunately, the geometrically evident ideas related to foliated functions have a rather cumbersome analytic formalization. For this reason, the proofs are lengthy and contain many formulas but very few nontrivial tricks.

1. A HYPERBOLIC ATLAS AND LEAVES

Let M be a convex domain in a standard Euclidean space, and let W be a Riemannian manifold. Consider N times continuously differentiable family of self-mappings of the direct product $W\times M$ of the form

$$\begin{cases} w'=S(w,z,\varepsilon),\\ z'=z+\varepsilon v(w,z,\varepsilon), \end{cases} \tag{1}$$

where ε is a small positive parameter, $w\in W$, and $z\in M$. For such mappings, we use the abbreviated notation $(w',z')=\Sigma_\varepsilon(w,z)$. Suppose that $\Sigma_\varepsilon(W\times M)\subset W\times M$ for all sufficiently small values of ε. Then, Σ_ε generates a dynamical system with discrete time (a cascade) on $W\times M$ with fast motions on W and slow motions on M (at velocities of order ε). Suppose also that, for every $z\in M$, the mapping $S_z(w)=S(w,z,0)$ has a mixing hyperbolic attractor that continuously depends on z (see the definitions in [4, 5]). Below, we introduce a notion of a uniformly hyperbolic atlas for such systems. It generalizes the notion of hyperbolic atlas given in [1] for an attractor that is independent of a parameter.

Consider two Euclidean spaces $\mathbb{R}^u$ and $\mathbb{R}^s$, where $u+s=\dim W$. We denote an arbitrary point in $\mathbb{R}^u$ by x and an arbitrary point in $\mathbb{R}^s$ by y. Let $\mathbb{B}^u_r$ and $\mathbb{B}^s_r$ denote open balls of radius r centered at zero in $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively. Let $\mathfrak{A}$ be a finite set of charts on W that have the form $w=\chi(x,y)$, where $\chi\colon\mathbb{B}^u_5\times\mathbb{B}^s_5\to W$. We do not require these charts to cover the entire W. The part of W covered by the charts from $\mathfrak{A}$ is denoted by $W_{\mathfrak{A}}$. As a representation of Σ_ε in the charts

$\chi, \chi' \in \mathfrak{A}$, we consider the mapping

$$\begin{cases} x' = X(x, y, z, \varepsilon), \\ y' = Y(x, y, z, \varepsilon), \\ z' = Z(x, y, z, \varepsilon) = z + \varepsilon Z'(x, y, z, \varepsilon), \end{cases} \tag{2}$$

where

$$(x', y') = \big(X(x, y, z, \varepsilon),\, Y(x, y, z, \varepsilon)\big) = (\chi')^{-1} \circ S\big(\chi(x, y), z, \varepsilon\big), \tag{3}$$

$$z' = Z(x, y, z, \varepsilon) = z + \varepsilon v\big(\chi(x, y), z, \varepsilon\big), \quad Z'(x, y, z, \varepsilon) = v\big(\chi(x, y), z, \varepsilon\big). \tag{4}$$

Definition 1.1. We say that a finite set $\mathfrak{A}$ of charts of the form $\chi\colon \mathbb{B}_5^u \times \mathbb{B}_5^s \to W$ is a *uniformly hyperbolic atlas* for system (1) if there exist positive numbers a, b, and θ such that

(a) $a + \theta^{-1} b < 1$ and θ is sufficiently small (namely, $\theta < 1/10$);

(b) $\Sigma_\varepsilon(W_{\mathfrak{A}} \times M) \subset \bigcup_{\chi \in \mathfrak{A}} \chi(\mathbb{B}_1^u \times \mathbb{B}_1^s) \times M$ for small ε;

(c) for any charts $\chi, \chi' \in \mathfrak{A}$ and arbitrary x, y, z, and ε from the domain of mapping (3),

$$\left\| \left(\frac{\partial X}{\partial x}\right)^{-1} \right\| \le a, \qquad \left\| \frac{\partial Y}{\partial y} \right\| \le a, \qquad \left\| \frac{\partial X}{\partial y} \right\| \le b, \qquad \left\| \frac{\partial Y}{\partial x} \right\| \le b; \tag{5}$$

(d) all partial derivatives of X, Y, Z, and Z' up to the order N are bounded.

Definition 1.2. We say that an atlas $\mathfrak{A}$ is *mixing* if there exists a set of charts $\mathfrak{B} \subset \mathfrak{A}$ and a positive integer n_0 such that, for all $n \ge n_0$ and $z \in M$, the following conditions hold:

(a) for any charts $\chi \in \mathfrak{A}$ and $\chi' \in \mathfrak{B}$, the intersection $S_z^n\big(\chi(\mathbb{B}_1^u \times \mathbb{B}_1^s)\big) \cap \chi'(\mathbb{B}_1^u \times \mathbb{B}_1^s)$ is nonempty;

(b) $S_z^n(W_{\mathfrak{A}}) \subset \bigcup_{\chi \in \mathfrak{B}} \chi(\mathbb{B}_1^u \times \mathbb{B}_1^s)$.

In [1], it is shown that, for a fixed $z_0 \in M$, the mapping $S_{z_0}(w) = S(w, z_0, 0)$ has a hyperbolic mixing attractor if and only if it admits a hyperbolic mixing atlas. If we somewhat increase the numbers a and b, then the same atlas will become hyperbolic and mixing for the mappings S_z with any z close to z_0. We make an even stronger assumption; namely, we assume that system (1) has a uniformly hyperbolic mixing atlas $\mathfrak{A}$ that serves all $z \in M$. We fix this atlas for the rest of the paper. Thus, for each $z \in M$, the mapping S_z has a hyperbolic attractor $H_z = \bigcap_n S_z^n(W_{\mathfrak{A}})$.

Suppose that numbers d and η and the parameter ε satisfy the conditions

$$\sup \left\| \frac{\partial(X, Y, Z, Z')}{\partial(x, y, z, \varepsilon)} \right\| < d, \qquad \eta = \frac{b}{d}, \qquad |\varepsilon| < \theta \eta^2. \tag{6}$$

Here, the supremum is taken over all charts $\chi, \chi' \in \mathfrak{A}$ and all admissible values of the variables x, y, z, and ε.

Definition 1.3. A *skew leaf* (or simply leaf) is an arbitrary smooth submanifold $\Gamma \subset W_{\mathfrak{A}} \times M$ of dimension u that is represented in some chart $\chi \in \mathfrak{A}$ by the graph $\{(x, y(x), z(x)) \mid x \in U\} \subset \mathbb{B}_5^u \times \mathbb{B}_5^s \times M$ of a pair of functions $y(x)$ and $z(x)$ of class $C^N(U)$ that satisfy the inequalities $\|dy(x)/dx\| \le \theta$ and $\|dz(x)/dx\| \le \theta\eta$. We identify the triple $(y = y(x), z = z(x), \chi)$ with the leaf Γ and write $\Gamma \sim (y(x), z(x), \chi)$. We say that a leaf is *straight* if it is entirely contained in a layer $z = \text{const}$.

It can be proved that the mapping Σ_ε expands the skew leaves in the x-direction and does not violate the conditions on the derivatives of $y(x)$ and $z(x)$. Below, we give appropriate statements.

Proposition 1.1. *If $V = (V_x, V_y, V_z)$ is a vector in the (x, y, z)-space, the vector $V' = (V'_x, V'_y, V'_z)$ is its image under mapping* (2), *and conditions* (5) *and* (6) *hold, then the inequalities $\|V_y\| \le \theta\|V_x\|$ and $\|V_z\| \le \theta\eta\|V_x\|$ imply*

$$\|V'_y\| \le \theta\|V'_x\|, \qquad \|V'_z\| \le \theta\eta\|V'_x\|, \qquad \|V'_x\| \ge (a+b)^{-1}\|V_x\|.$$

Proof. By virtue of (5) and (6), we have

$$\frac{\|V'_y\|}{\|V'_x\|} \le \frac{b\|V_x\| + a\|V_y\| + d\|V_z\|}{a^{-1}\|V_x\| - b\|V_y\| - d\|V_z\|} \le \frac{b + a\theta + d\theta\eta}{a^{-1} - b\theta - d\theta\eta} \le \frac{\theta(a + \theta^{-1}b + b)}{a^{-1} - 2\theta b} < \theta,$$

$$\frac{\|V'_z\|}{\|V'_x\|} \le \frac{\varepsilon d\|V_x\| + \varepsilon d\|V_y\| + (1+\varepsilon d)\|V_z\|}{a^{-1}\|V_x\| - b\|V_y\| - d\|V_z\|} \le \frac{\varepsilon d(1+\theta) + (1+\varepsilon d)\theta\eta}{a^{-1} - 2\theta b} < \theta\eta,$$

$$\frac{\|V'_x\|}{\|V_x\|} \ge \frac{a^{-1}\|V_x\| - b\|V_y\| - d\|V_z\|}{\|V_x\|} \ge a^{-1} - 2\theta b > (a+b)^{-1}.$$

Let $B(x, r)$ denote an open ball of radius r centered at x.

Proposition 1.2. *Suppose that $\Gamma \sim (y(x), z(x), \chi)$ is a leaf, x varies in the domain $U = B(x_0, r)$, and the point $\Sigma^n_\varepsilon\big(\chi(x_0, y(x_0)), z(x_0)\big)$ is represented in a chart $\chi' \in \mathfrak{A}$ as $(x'_0, y'_0, z'_0) \in \mathbb{B}^u_4 \times \mathbb{B}^s_4 \times M$. Then, there is a unique leaf $\Gamma' \sim (y'(x'), z'(x'), \chi')$ that is a subset of $\Sigma^n_\varepsilon(\Gamma)$ such that $y'(x'_0) = y'_0$, $z'(x'_0) = z'_0$, and x' varies in the domain $U' = B(x'_0, r/(a+b)^n) \cap \mathbb{B}^u_5$.*

Proof. Consider the case $n = 1$. By the preceding proposition, the mapping $x'(x) = X(x, y(x), z(x), \varepsilon)$ is locally diffeomorphic and expands all distances at least by a factor of $(a+b)^{-1}$. For every point $x' \in U'$, consider the path $x'_t = x'_0 + t(x' - x'_0)$, $t \in [0, 1]$. For small t, this path has a unique preimage under the mapping $x'(x)$, which is a smooth path x_t starting at x_0. Obviously, $\|x_t - x_0\| \le (a+b)t\|x' - x'_0\|$. Therefore, x_t can be reconstructed for all $t \in [0, 1]$. Let us define functions $y'(x')$ and $z'(x')$ by

$$y'(x') = Y\big(x_1, y(x_1), z(x_1), \varepsilon\big), \qquad z'(x') = Z\big(x_1, y(x_1), z(x_1), \varepsilon\big).$$

By virtue of the preceding proposition, $\|dy'(x')/dx'\| \le \theta$ and $\|dz'(x')/dx'\| \le \theta\eta$. This proves the required assertion for $n = 1$. For other values of n, the proof is similar.

In addition to separate leaves, we will consider their deformations $\Gamma_t \sim (y_t(x), z_t(x), \chi)$ depending on a finite-dimensional parameter $t = (t_1, \dots, t_n)$ of arbitrary dimension n. We use the notation i for an arbitrary nonnegative integer, $j = (j_1, \dots, j_n)$ for an integer multiindex, $v = (v_1, \dots, v_n)$ for an n-vector with nonnegative components, and σ for a number in the interval $[0, \eta]$. To each pair of nonnegative integers (i, k), we assign a positive number α_{ik}. In what follows, we use the abbreviated notation $v^j = v_1^{j_1} \dots v_n^{j_n}$.

Definition 1.4. We say that a vector v *majorizes* a deformation $y_t(x)$ of a function (and write $y_t \prec v$) if, for all x and t,

$$\left\| \frac{\partial^{i+|j|} y_t(x)}{\partial x^i \partial t^j} \right\| \le \alpha_{i|j|} v^j, \qquad 1 \le i + |j| \le N. \tag{7}$$

We say that a pair (v, σ) *majorizes* a deformation $(y_t(x), z_t(x))$ of a pair of functions (and write $(y_t, z_t) \prec (v, \sigma)$) if, for all x and t, in addition to (7), the following inequalities hold:

$$\left\| \frac{\partial^{i+|j|} z_t(x)}{\partial x^i \partial t^j} \right\| \le \alpha_{i|j|} \sigma v^j, \qquad i > 0, \quad i + |j| \le N; \tag{8}$$

$$\left\| \frac{\partial^{|j|} z_t(x)}{\partial t^j} \right\| \le \alpha_{0|j|} \eta v^j, \qquad 1 \le |j| \le N. \tag{9}$$

Consider an arbitrary mapping of the form (2). Suppose that all partial derivatives up to the order N of the functions X, Y, Z, and Z' are bounded, estimates (5) and (6) hold, $a + \theta^{-1}b < 1$, and $\theta < 1/10$. Take a deformation (y_t, z_t) of a pair of functions and construct a new deformation $(y'_{t\varepsilon}, z'_{t\varepsilon})$ in such a way that, for any t and ε, the graph $\{(x', y'_{t\varepsilon}(x'), z'_{t\varepsilon}(x'))\}$ is the image of the graph $\{(x, y_t(x), z_t(x))\}$ under mapping (2).

Theorem 1.3. *Under conditions* (5) *and* (6), *there exists a set of constants* $\alpha_{ik} \geq \theta$ *independent of* ε $(\alpha_{10} = \theta)$ *and a large number* C *such that*

(a) *if* $\varepsilon = 0$, $z_t(x) = \mathrm{const}$, *and* $y_t \prec v$, *then* $y'_{t0} \prec (1-b)^{1/N} v$;

(b) *if* $\sigma \in [0, \eta]$ *and* $(y_t, z_t) \prec (v, \sigma)$, *then* $(y'_{t\varepsilon}, z'_{t\varepsilon}) \prec (v', \sigma')$, *where* $v' = v + C(\sigma + \varepsilon)v$ *and* $\sigma' = (1-b)(\sigma + C\varepsilon)$.

The proof of Theorem 1.3 is given in the next section. Hereafter, we assume that the constants α_{ik} are the same throughout the paper. If $\Gamma_t \sim (y_t(x), z_t(x), \chi)$ and $(y_t, z_t) \prec (v, \sigma)$, we write simply $\Gamma_t \prec (v, \sigma)$. If $z_t(x) = \mathrm{const}$ and $y_t \prec v$, then we write $\Gamma_t \prec v$. Certainly, Theorem 1.3 can be directly applied to any deformation Γ_t of a leaf. We will write $\Gamma' \subset \Sigma^n_\varepsilon(\Gamma)$ only if there exists an open domain $G \subset \Gamma$ such that Σ^n_ε maps it homeomorphically onto Γ'. Repeatedly applying assertion (b) of Theorem 1.3 to the leaf deformation Γ_t, we obtain the following corollary.

Corollary 1.3.1. *If* $\Gamma'_t \subset \Sigma^n_0(\Gamma_t)$ *and* $\Gamma_t \prec (v, \sigma)$, *then* $\Gamma'_t \prec (e^{c\sigma} v,\ (1-b)^n \sigma)$, *where* $c = C/b$.

Take a number $\sigma_0 \in (0, \eta]$.

Definition 1.5. A skew leaf $\Gamma \sim (y(x), z(x), \chi)$ is said to be *flowing* if

$$\left\| \frac{d^i y(x)}{dx^i} \right\| \leq \alpha_{i0}, \qquad \left\| \frac{d^i z(x)}{dx^i} \right\| \leq \alpha_{i0}\sigma_0, \qquad i = 1, \dots, N. \tag{10}$$

Since $\alpha_{10} = \theta$, these conditions include the constraints $\|dy/dx\| \leq \theta$ and $\|dz/dx\| \leq \theta\eta$ on the derivatives, which hold for any leaf by definition.

Corollary 1.3.2. *If a leaf* Γ *is flowing and* ε *is small, then any leaf* $\Gamma' \subset \Sigma_\varepsilon(\Gamma)$ *is flowing.*

Proof. Let us identify Γ and Γ' with trivial deformations that are absolutely independent of t. Then, the leaf Γ is flowing if and only if $\Gamma \prec (0, \sigma_0)$. According to assertion (b) of Theorem 1.3, we have $\Gamma' \prec (0, \sigma')$, where $\sigma' = (1-b)(\sigma_0 + C\varepsilon) < \sigma_0$, which implies that Γ' is flowing.

Corollary 1.3.3. *If* $\Gamma' \subset \Sigma_\varepsilon(\Gamma)$ *and* $\Gamma \prec (0, C\varepsilon/b)$, *then* $\Gamma' \prec (0, C\varepsilon/b)$ *as well.*

This corollary is proved by setting $v = 0$ and $\sigma = C\varepsilon/b$ in assertion (b) of Theorem 1.3.

Hereafter, we use the term "leaf" only for *flowing leaves.*

Theorem 1.4. *There exist large constants* C_n *such that if* $\Gamma_t \prec (v, \eta)$ *and* $\Gamma'_{t\varepsilon} \subset \Sigma^n_\varepsilon(\Gamma_t)$, *where* $\Gamma'_{t\varepsilon} \sim \big(y'_{t\varepsilon}(x'), z'_{t\varepsilon}(x'), \chi'\big)$, *then*

$$\left\| \frac{\partial^{i+|j|+l} y'_{t\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^l} \right\| \leq C_n v^j, \qquad \left\| \frac{\partial^{i+|j|+l} z'_{t\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^l} \right\| \leq C_n v^j, \qquad 1 \leq i + |j| + l \leq N. \tag{11}$$

This theorem is also proved in the next section.

2. PROOFS OF THE THEOREMS ABOUT IMAGES OF LEAVES

Obviously, the functions $y'_{t\varepsilon}$ and $z'_{t\varepsilon}$ from Theorem 1.3 are specified by the parametric equations

$$x' = X\big(x, y_t(x), z_t(x), \varepsilon\big), \tag{12}$$

$$y'_{t\varepsilon}(x') = Y\big(x, y_t(x), z_t(x), \varepsilon\big), \tag{13}$$

$$z'_{t\varepsilon}(x') = Z\big(x, y_t(x), z_t(x), \varepsilon\big) = z_t(x) + \varepsilon Z'\big(x, y_t(x), z_t(x), \varepsilon\big). \tag{14}$$

Proposition 1.1 implies that the derivative $\partial x'/\partial x$ expands vectors from $\mathbb{R}^u$ at least by a factor of $(a+b)^{-1}$. Therefore, equation (12) determines an implicit function $x = x(x', t, \varepsilon)$, and

$$\left\| \frac{\partial x}{\partial x'} \right\| \le a + b. \tag{15}$$

In what follows, we always assume that

$$X = X(x,y,z,\varepsilon), \qquad Y = Y(x,y,z,\varepsilon), \qquad Z = z + \varepsilon Z'(x,y,z,\varepsilon),$$
$$y = y_t(x), \qquad z = z_t(x), \qquad x = x(x',t,\varepsilon),$$

but, for the sake of brevity, we do not explicitly indicate the arguments of these functions. We also use the notation

$$\frac{dX}{dx} = \frac{\partial X}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial x}, \qquad \frac{dY}{dx} = \frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial Y}{\partial z}\frac{\partial z}{\partial x},$$
$$\frac{dZ}{dx} = \frac{\partial z}{\partial x} + \varepsilon\frac{dZ'}{dx} = \frac{\partial z}{\partial x} + \varepsilon\left(\frac{\partial Z'}{\partial x} + \frac{\partial Z'}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial Z'}{\partial z}\frac{\partial z}{\partial x}\right).$$

Let us calculate the partial derivatives on the basis of (12)–(14):

$$\frac{\partial x}{\partial x'} = \left(\frac{dX}{dx}\right)^{-1} = \left(\frac{\partial X(x,y,z,\varepsilon)}{\partial x} + \frac{\partial X(x,y,z,\varepsilon)}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial X(x,y,z,\varepsilon)}{\partial z}\frac{\partial z}{\partial x}\right)^{-1}, \tag{16}$$

$$\frac{\partial x}{\partial t} = -\left(\frac{dX}{dx}\right)^{-1}\left(\frac{\partial X}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial t}\right) = -\frac{\partial x}{\partial x'}\left(\frac{\partial X}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial t}\right), \tag{17}$$

$$\frac{\partial x}{\partial \varepsilon} = -\left(\frac{dX}{dx}\right)^{-1}\frac{\partial X}{\partial \varepsilon} = -\frac{\partial x}{\partial x'}\frac{\partial X}{\partial \varepsilon}, \tag{18}$$

$$\frac{\partial y'_{t\varepsilon}(x')}{\partial x'} = \frac{dY}{dx}\frac{\partial x}{\partial x'} = \left(\frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial Y}{\partial z}\frac{\partial z}{\partial x}\right)\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial x}\right)^{-1}, \tag{19}$$

$$\begin{aligned}\frac{\partial y'_{t\varepsilon}(x')}{\partial t} &= \frac{dY}{dx}\frac{\partial x}{\partial t} + \frac{\partial Y}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial Y}{\partial z}\frac{\partial z}{\partial t} \\ &= \left(\frac{\partial Y}{\partial y} - \frac{dY}{dx}\frac{\partial x}{\partial x'}\frac{\partial X}{\partial y}\right)\frac{\partial y}{\partial t} + \left(\frac{\partial Y}{\partial z} - \frac{dY}{dx}\frac{\partial x}{\partial x'}\frac{\partial X}{\partial z}\right)\frac{\partial z}{\partial t},\end{aligned} \tag{20}$$

$$\frac{\partial z'_{t\varepsilon}(x')}{\partial x'} = \frac{dZ}{dx}\frac{\partial x}{\partial x'} = \left(\frac{\partial z}{\partial x} + \varepsilon\frac{dZ'}{dx}\right)\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial x}\right)^{-1}, \tag{21}$$

$$\begin{aligned}\frac{\partial z'_{t\varepsilon}(x')}{\partial t} &= \frac{\partial Z}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial y}{\partial t} + \frac{dZ}{dx}\frac{\partial x}{\partial t} \\ &= \left(\mathrm{I} + \varepsilon\frac{\partial Z'}{\partial z} - \left(\frac{\partial z}{\partial x} + \varepsilon\frac{dZ'}{dx}\right)\frac{\partial x}{\partial x'}\frac{\partial X}{\partial z}\right)\frac{\partial z}{\partial t} + \left(\varepsilon\frac{\partial Z'}{\partial y} - \left(\frac{\partial z}{\partial x} + \varepsilon\frac{dZ'}{dx}\right)\frac{\partial x}{\partial x'}\frac{\partial X}{\partial y}\right)\frac{\partial y}{\partial t}.\end{aligned} \tag{22}$$

Equality (16) defines $\partial x/\partial x'$ as a function of six variables x, y, z, ε, $\partial y/\partial x$, and $\partial z/\partial x$. Under the conditions $\|\partial y/\partial x\| \le \theta$ and $\|\partial z/\partial x\| \le \theta\eta$, this function is $N-1$ times continuously differentiable with respect to the set of variables, and all of its partial derivatives are bounded. The same is true for the derivatives on the left-hand sides of equalities (17)–(22).

We say that an index pair (i,j), where $j=(j_1,\dots,j_n)$, *precedes* a pair (i',j') if $i+|j|\le i'+|j'|$, $|j|\le|j'|$, and $(i,|j|)\ne(i',|j'|)$. This partial ordering determines a priority of partial derivatives of the form $\partial^{i+|j|}y/\partial x^i\partial t^j$ and $\partial^{i+|j|}z/\partial x^i\partial t^j$.

Lemma 2.1. *If $i\ge 2$ or $|j|\ge 1$, then, for any vector $V'\in\mathbb{R}^u$,*

$$\frac{\partial^{i+|j|}y'_{t\varepsilon}(x')}{(\partial x')^i\partial t^j}(V')^i=\left(\frac{\partial Y}{\partial y}-\frac{dY}{dx}\frac{\partial x}{\partial x'}\frac{\partial X}{\partial y}\right)\frac{\partial^{i+|j|}y_t(x)}{\partial x^i\partial t^j}\left(\frac{\partial x}{\partial x'}V'\right)^i$$
$$+\left(\frac{\partial Y}{\partial z}-\frac{dY}{dx}\frac{\partial x}{\partial x'}\frac{\partial X}{\partial z}\right)\frac{\partial^{i+|j|}z_t(x)}{\partial x^i\partial t^j}\left(\frac{\partial x}{\partial x'}V'\right)^i+P_{ij},$$

$$\frac{\partial^{i+|j|}z'_{t\varepsilon}(x')}{(\partial x')^i\partial t^j}(V')^i=\left(\mathrm{I}+\varepsilon\frac{\partial Z'}{\partial z}-\left(\frac{\partial z}{\partial x}+\varepsilon\frac{dZ'}{dx}\right)\frac{\partial x}{\partial x'}\frac{\partial X}{\partial z}\right)\frac{\partial^{i+|j|}z_t(x)}{\partial x^i\partial t^j}\left(\frac{\partial x}{\partial x'}V'\right)^i$$
$$+\left(\varepsilon\frac{\partial Z'}{\partial y}-\left(\frac{\partial z}{\partial x}+\varepsilon\frac{dZ'}{dx}\right)\frac{\partial x}{\partial x'}\frac{\partial X}{\partial y}\right)\frac{\partial^{i+|j|}y_t(x)}{\partial x^i\partial t^j}\left(\frac{\partial x}{\partial x'}V'\right)^i+Q_{ij}.$$

Each coordinate of the vector P_{ij} is a finite sum of products of the form $p_1p_2p_3$, and each coordinate of the vector Q_{ij} is a finite sum of products of the form $q_1q_2q_3$, where

(a) *p_1 and q_1 are smooth functions of variables x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, and ε all of whose partial derivatives are bounded;*

(b) *p_2 and q_2 are products of some coordinates of the vector V' with i multipliers;*

(c) *p_3 and q_3 are finite products of some partial derivatives of y and z that precede (i,j), and, for each of these products, the sum of all multiindices corresponding to differentiation with respect to t equals j;*

(d) *each product $q_1q_2q_3$ contains a factor that either coincides with ε or is a component of some derivative of the form $\partial^{i'+|j'|}z/\partial x^{i'}\partial t^{j'}$, where $i'>0$.*

Proof. For $i=0$ and $|j|=1$, the assertion of the lemma immediately follows from equalities (20) and (22) with zero P_{ij} and Q_{ij}. For $i=2$ and $j=0$, the required equalities are obtained by a direct differentiation of (19) and (21). For other pairs of indices, the assertion is proved by induction based on the ordering introduced above. One should merely successively differentiate the equalities and, at each step, represent the current derivatives of $y'_{t\varepsilon}(x')$ and $z'_{t\varepsilon}(x')$ as functions of x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, ε, V', and lower derivatives of y and z by using substitutions (16) and (17). We leave the details to the reader.

Let us return to the proof of Theorem 1.3. Suppose that, for some $\sigma\le\eta$, we have $\|\partial y/\partial x\|\le\theta$ and $\|\partial z/\partial x\|\le\theta\sigma$. Then, by Proposition 1.1, $\|\partial y'_{t\varepsilon}/\partial x'\|\le\theta$ and, by (15) and (21),

$$\left\|\frac{\partial z'_{t\varepsilon}}{\partial x'}\right\|\le(\theta\sigma+\varepsilon d)(a+b)\le\theta\left(\sigma+\frac{\varepsilon d}{\theta}\right)(1-b)=\theta\sigma',$$

where $\sigma'=(1-b)(\sigma+\varepsilon d/\theta)$. Therefore, we can take $\alpha_{10}=\theta$ in (7) and (8).

The remaining α_{ik} are determined by induction based on the ordering of indices introduced above. Let us describe an induction step. Take a pair (I,J). Suppose that, for all pairs of indices (i,j) that precede (I,J), some set of constants $\alpha_{i|j|}$ has already been determined. Let us show how to determine $\alpha_{I|J|}$. Suppose that inequalities (7)–(9) hold. Let us estimate the two derivatives $\partial^{I+|J|}y'_{t\varepsilon}(x')/(\partial x')^I\partial t^J$ and $\partial^{I+|J|}z'_{t\varepsilon}(x')/(\partial x')^I\partial t^J$ with the use of Lemma 2.1. By Lemma 2.1, there exists a large constant C' (depending on $\alpha_{i|j|}$ that have already been chosen but not on $\alpha_{I|J|}$) such that the vectors P_{IJ} and Q_{IJ} from this lemma obey the estimates

$$\|P_{IJ}\|\le C'\|V'\|^I v^J,\qquad \|Q_{IJ}\|\le C'\|V'\|^I(\sigma+\varepsilon)v^J. \tag{23}$$

By virtue of (19),

$$\left\|\frac{dY}{dx}\frac{\partial x}{\partial x'}\right\| = \left\|\frac{\partial y'_{t\varepsilon}}{\partial x'}\right\| \le \theta. \tag{24}$$

Substituting inequalities (5)–(9), (15), (23), and (24) into the equalities of Lemma 2.1, we obtain the estimates

$$\begin{aligned}\left\|\frac{\partial^{I+|J|}y'_{t\varepsilon}(x')}{(\partial x')^I\partial t^J}\right\| &\le (a+\theta b)\alpha_{I|J|}v^J(a+b)^I + 2d\alpha_{I|J|}\eta v^J(a+b)^I + C'v^J\\ &\le (a+3b)\alpha_{I|J|}v^J + C'v^J \le \alpha_{I|J|}(a+3b+C'/\alpha_{I|J|})v^J;\\ \left\|\frac{\partial^{|J|}z'_{t\varepsilon}(x')}{\partial t^J}\right\| &\le (1+\theta\sigma d+2\varepsilon d^2)\alpha_{0|J|}\eta v^J + (\theta\sigma b+2\varepsilon d)\alpha_{0|J|}v^J + C'(\sigma+\varepsilon)v^J\\ &\le \alpha_{0|J|}\eta\Big(1+\theta\sigma d+2\varepsilon d^2+\theta\sigma b/\eta+2\varepsilon d/\eta + C'(\sigma+\varepsilon)/(\alpha_{0|J|}\eta)\Big)v^J\end{aligned}$$

for $J\ne 0$; and

$$\begin{aligned}\left\|\frac{\partial^{I+|J|}z'_{t\varepsilon}(x')}{(\partial x')^I\partial t^J}\right\| &\le (1+\theta\sigma d+2\varepsilon d^2)\alpha_{I|J|}\sigma v^J(a+b)^I + (\theta\sigma b+2\varepsilon d)\alpha_{I|J|}v^J(a+b)^I + C'(\sigma+\varepsilon)v^J\\ &\le \alpha_{I|J|}\Big[(1+\theta\sigma d+\theta b)(a+b)\sigma + 2\varepsilon d^2\sigma + 2\varepsilon d + C'(\sigma+\varepsilon)/\alpha_{I|J|}\Big]v^J\\ &\le \alpha_{I|J|}\Big[(1-2b+C'/\alpha_{I|J|})\sigma + (4d+C'/\alpha_{I|J|})\varepsilon\Big]v^J\end{aligned}$$

for $I>0$.

If the numbers $\alpha_{I|J|}$ and C are large in comparison with C', then these inequalities give the estimates

$$\left\|\frac{\partial^{I+|J|}y'_{t\varepsilon}(x')}{(\partial x')^I\partial t^J}\right\| \le \alpha_{I|J|}(1-b)v^J,$$

$$\left\|\frac{\partial^{I+|J|}z'_{t\varepsilon}(x')}{(\partial x')^I\partial t^J}\right\| \le \alpha_{I|J|}(1-b)(\sigma+C\varepsilon)v^J \qquad \text{for}\quad I>0,$$

$$\left\|\frac{\partial^{|J|}z'_{t\varepsilon}(x')}{\partial t^J}\right\| \le \alpha_{0|J|}\eta(1+C(\sigma+\varepsilon))v^J \qquad \text{for}\quad J\ne 0.$$

These estimates prove Theorem 1.3.

Suppose that the deformations of the functions $y_t(x)$ and $z_t(x)$ in (12)–(14) depend not only on the parameter t but also on ε. Let us denote them by $y_{t\varepsilon}(x)$ and $z_{t\varepsilon}(x)$, respectively.

Lemma 2.2. *If conditions* (12)–(14) *hold and* $1\le i+|j|+l\le N$, *then*

$$\frac{\partial^{i+|j|+l}y'_{t\varepsilon}(x')}{(\partial x')^i\partial t^j\partial\varepsilon^l}(V')^i = P_{ijl}, \qquad \frac{\partial^{i+|j|+l}z'_{t\varepsilon}(x')}{(\partial x')^i\partial t^j\partial\varepsilon^l}(V')^i = Q_{ijl};$$

every coordinate of the vector P_{ijl} *is a finite sum of products of the form* $p_1p_2p_3$, *and every coordinate of the vector* Q_{ijl} *is a finite sum of products of the form* $q_1q_2q_3$, *where*

(a) p_1 *and* q_1 *are smooth functions of the variables* x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, *and* ε *all of whose partial derivatives are bounded*;

(b) *p_2 and q_2 are products of i coordinates of the vector V';*

(c) *p_3 and q_3 are finite products of some partial derivatives of $y_{t\varepsilon}(x)$ and $z_{t\varepsilon}(x)$ with respect to x, t, and ε of orders not exceeding $i+|j|+l$; for each of these products, the sum of all multiindices corresponding to differentiation with respect to t equals j.*

Proof. For $i \geq 2$ or $|j| \geq 1$, this assertion is proved by a straightforward differentiation with respect to ε of the equalities from Lemma 2.1; for $j = 0$ and $i = 1$, it is proved by differentiating equalities (19) and (21); finally, for $j = 0$ and $i = 0$, it is proved by differentiating equalities (13) and (14). All differentiations should be performed with the use of substitution (18).

Theorem 1.4 is proved with the use of Lemma 2.2 by induction on n. At every induction step, estimates (11) obtained at the preceding step should be substituted into the equalities of Lemma 2.2. For $n = 1$, (7)–(9) should be taken as such estimates.

3. TRACES AND FOLIATED FUNCTIONS

We consider a dynamical system of the form (1) for which a uniformly hyperbolic mixing atlas $\mathfrak{A}$ is fixed and the notions of flowing skew leaves and majorized deformations of leaves are defined.

Definition 3.1. A *skew trace* (or simply trace) is a pair (Γ, Φ) consisting of a flowing skew leaf Γ and a smooth real-valued function Φ on it (the latter is called a *density*). We say that a trace is *straight* if the corresponding leaf Γ is straight. A trace (Γ', Φ') is called a *restriction* of a trace (Γ, Φ) if $\Gamma' \subset \Gamma$ and the density Φ' coincides with Φ on Γ'. The *image* of a trace (Γ, Φ) is the trace $(\Gamma', \Phi') = \Sigma_\varepsilon(\Gamma, \Phi)$ such that $\Gamma' = \Sigma_\varepsilon(\Gamma)$ and $\Phi = \Phi' \circ \Sigma_\varepsilon$.

Definition 3.2. A *weight function* (weight) is an arbitrary smooth function J defined on the manifold of u-dimensional subspaces tangent to $W_{\mathfrak{A}} \times M$.

In the representation in hyperbolic charts, a weight J is a function of the variables x, y, z, $p = dy/dx$, and $q = dz/dx$. Let us point out at once that we are interested in the behavior of this function only in the domain where $\|p\| \leq \theta$ and $\|q\| \leq \theta\eta$.

A *canonical weight function* is the reciprocal of the expansion coefficient of u-dimensional volumes under the mapping Σ_ε. Obviously, it depends on ε; thus, it is more appropriate to say that this is a family of weight functions J_ε.

For a weight J_ε and a skew leaf Γ, the restriction of J_ε to Γ is defined in a natural way; we denote it by the same symbol. A self-mapping $\Sigma_\varepsilon J_\varepsilon$ of the set of skew traces is defined by $\Sigma_\varepsilon J_\varepsilon(\Gamma, \Phi) = \Sigma_\varepsilon(\Gamma, J_\varepsilon \Phi)$. If the weight J_ε is canonical and Φ is the density of some measure with respect to the Riemannian volume on Γ, then the density of the trace $\Sigma_\varepsilon J_\varepsilon(\Gamma, \Phi)$ is automatically the density of the corresponding measure on $\Sigma_\varepsilon(\Gamma)$ induced by the mapping Σ_ε.

A leaf $\Gamma \sim (y(x), z(x), \chi)$ is said to be *standard* if the domains of the functions $y(x)$ and $z(x)$ coincide with an open unit ball $B(x_0, 1) \subset \mathbb{B}^u_5$. For standard leaves, we use the notation $\Gamma \sim (y(x), z(x), \chi, x_0)$; we assume that $x \in B(x_0, 1)$. The point $\alpha = \big(\chi(x_0, y(x_0)), z(x_0)\big)$ is called the *center* of the standard leaf. When it is necessary to emphasize that a standard leaf Γ is centered at α, we denote this leaf by $\Gamma(\alpha)$. We say that a trace (Γ, Φ) is standard if the corresponding leaf Γ is standard. For any standard trace (Γ, Φ), we can assume that, in a chart representation, the domain of the density $\Phi = \Phi(x)$ is the same ball $B(x_0, 1)$. We say that two standard traces (Γ_1, Φ_1) and (Γ_2, Φ_2) are *equivalent* if their centers coincide, the intersection $\Gamma_1 \cap \Gamma_2$ is open in Γ_1 and in Γ_2, and the densities Φ_1 and Φ_2 coincide on this intersection. A *standard deformation* of a leaf is a deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ such that all functions $y_t(x)$ and $z_t(x)$ have the same domain $B(x_0, 1) \subset \mathbb{B}^u_5$. A trace deformation (Γ_t, Φ_t) is standard if the corresponding deformation of the leaf Γ_t is standard.

We call a deformation (Γ_t, Φ_t) of standard traces *sliding* if all traces in this deformation are restrictions of the same trace (Γ, Φ). Under a sliding deformation, the leaf Γ_t and its center move along Γ.

Definition 3.3. A *foliated function* (on skew traces) of type (p, q), where $p \geq 1$, $q \geq 4$, and $p+q \leq N-1$, is an arbitrary real-valued function $g(\Gamma, \Phi)$ defined on the set of standard skew traces (with flowing leaves) and possessing the following properties:

(a) it linearly depends of Φ;

(b) its values at equivalent traces coincide;

(c) there exists a number $c \geq 0$ such that, for any standard deformation $\Gamma_{tz} \sim (y_t(x), z, \chi, x_0)$ of flowing straight leaves and any density $\Phi = \Phi(x)$, the relation $y_t \prec v$ implies the estimates

$$\left| \frac{\partial^{|j|+|k|} g(\Gamma_{tz}, \Phi)}{\partial t^j \partial z^k} \right| \leq c v^j \|\Phi\|_{p+|j|+|k|}, \qquad |j| + |k| \leq q, \quad |k| \leq q-2; \tag{25}$$

(d) there exists a $c \geq 0$ such that if a standard deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ of flowing skew leaves is majorized by a pair (v, σ), where $\sigma \in [0, \sigma_0]$, and a family of straight leaves $\overline{\Gamma}_t$ has the form $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$, then, for any density $\Phi = \Phi(x)$, the following estimates hold:

$$\left| \frac{\partial^{|j|} g(\Gamma_t, \Phi)}{\partial t^j} - \frac{\partial^{|j|} g(\overline{\Gamma}_t, \Phi)}{\partial t^j} \right| \leq c\sigma v^j \|\Phi\|_{p+1+|j|}, \qquad |j| \leq q-4; \tag{26}$$

(e) all the derivatives in (25) and (26) are continuous in t and z and vary continuously under sliding deformations of traces.

The exact value of the small positive number $\sigma_0 \in (0, \eta]$ in the definition of flowing leaves is chosen below, in Theorems 3.2 and 3.3. We denote the minimum c for which inequalities (25) and (26) hold by $\|g\|_{pq}$. Obviously, $\|g\|_{pq}$ is a norm. The set of all foliated functions of type (p, q) endowed with this norm forms a Banach space; we denote it by $\mathcal{F}^{pq}$. Estimates (25) and (26) imply that $\mathcal{F}^{pq} \subset \mathcal{F}^{p'q'}$ and $\|g\|_{pq} \geq \|g\|_{p'q'}$ for $p' \geq p$ and $q' \leq q$. Foliated functions can be multiplied by ordinary functions according to the following rule: if $f \in C^\infty(W \times M)$, then $fg(\Gamma, \Phi) = g(\Gamma, f\Phi)$. Formally, a foliated function $g \in \mathcal{F}^{pq}$ is defined on the set of standard traces with infinitely differentiable densities. However, it can be extended by continuity to the traces with densities of smoothness $p+q$.

Consider simple examples of foliated functions. For any standard trace $(\Gamma(\alpha), \Phi)$ centered at α, we set $g_0(\Gamma(\alpha), \Phi) = \Phi(\alpha)$. Let $\xi_0(t)$ be an infinitely differentiable function on the real axis with support in a sufficiently small neighborhood of zero. We set

$$g_1(\Gamma(\alpha), \Phi) = \int_\Gamma \xi_0\bigl(\rho(\alpha, \beta)\bigr) \Phi(\beta)\, d\mu(\beta),$$

where $\rho(\alpha, \beta)$ is the distance between α and a point $\beta \in \Gamma$ and μ is the Riemannian volume on Γ. It is easy to see that g_0 and g_1 are foliated functions of type $(0, N-1)$.

Theorem 3.1. *The space $\mathcal{F}^{pq}$ is a Banach module over the algebra of weight functions of smoothness $p+q$, over $C^{p+q}(W \times M)$, and over $C^{p+q-3}(M)$.*

This theorem is proved in the next section.

Without loss of generality, we can assume that the Riemannian metric on W is equivalent to the Euclidean metric on hyperbolic charts. This means that the length of a tangent vector to W

and the length of its image in a chart may differ by at most a finite factor that is independent of the vector and the chart.

Take a small positive number r_0 such that the restriction $\Gamma^* \sim (y(x), z(x), \chi)$ of any standard leaf $\Gamma \sim (y(x), z(x), \chi, x_0)$, where $x \in B(x_0, 1-a-b)$, contains the intersection of Γ with the $2r_0$-neighborhood of its center $\alpha = (\chi(x_0, y(x_0)), z(x_0))$. Take an infinitely differentiable nonnegative function $\xi_0(t)$ on the real axis that takes value 1 for $|t| \le r_0$ and vanishes for $|t| \ge 2r_0$. On each leaf Γ contained in the Riemannian manifold $W \times M$, the Riemannian volume μ is induced. Let us define a function of two variables on Γ by

$$\xi_\Gamma(\alpha, \beta) = \frac{\xi_0(\rho(\alpha, \beta))}{\int_\Gamma \xi_0(\rho(\alpha, \beta))\, d\mu(\alpha)},$$

where $\rho(\alpha, \beta)$ is the distance between α and β. This function vanishes if $\rho(\alpha, \beta) \ge 2r_0$ and is strictly positive if $\rho(\alpha, \beta) \le r_0$, and

$$\int\limits_\Gamma \xi_\Gamma(\alpha, \beta)\, d\mu(\alpha) \equiv 1. \tag{27}$$

To obviate the necessity of handling the boundary of Γ, it is convenient to assume that this function is defined on the $2r_0$-interior of Γ (i.e., on the set of pairs of points in Γ lying at a distance greater than $2r_0$ from the boundary of Γ).

Take a family of weight functions J_ε. Let (Γ, Φ) be an arbitrary standard trace centered at α, and let $(\Gamma'_\varepsilon, \Phi'_\varepsilon) = (\Sigma_\varepsilon J_\varepsilon)^n (\Gamma, \Phi)$. By Proposition 1.2, for any point $\beta \in \Gamma$ in the $2r_0$-neighborhood of α, there exists a standard trace $(\Gamma'_\varepsilon(\beta'), \Phi'_\varepsilon) \subset (\Gamma'_\varepsilon, \Phi'_\varepsilon)$ centered at $\beta' = \Sigma_\varepsilon^n(\beta)$.

Definition 3.4. We define the *averaged weighted shift operator* $A_{\varepsilon,n}$ on the space $\mathcal{F}^{pq}$ of foliated functions by

$$\bigl(A_{\varepsilon,n} g\bigr)(\Gamma(\alpha), \Phi) = \int\limits_{\Gamma'_\varepsilon} \xi_\Gamma(\alpha, \beta) g(\Gamma'_\varepsilon(\beta'), \Phi'_\varepsilon)\, d\mu(\beta'). \tag{28}$$

Obviously, for all $g \in \mathcal{F}^{pq}$ and $f \in C^{p+q}(W \times M)$, the following *homological identity* holds:

$$A_{\varepsilon,n}(fg) = f \circ \Sigma_\varepsilon^n \cdot A_{\varepsilon,n} g. \tag{29}$$

Theorem 3.2. *For every n, there exist small positive numbers $\sigma_0 \in (0, \eta]$ and $\varepsilon_0 = \varepsilon_0(\sigma_0)$ such that, for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $l \le q-8$, the operator $d^l A_{\varepsilon,n}/d\varepsilon^l$ maps continuously the space $\mathcal{F}^{pq}$ to $\mathcal{F}^{p+1+l,q-4-l}$ (and, hence, to $\mathcal{F}^{p+2l,q-5l}$ provided that $l \ne 0$).*

This theorem is proved in Section 6.

We say that a foliated function is *positive* if it takes nonnegative values on traces with nonnegative densities. We call a positive foliated function g *strongly positive* if there exists $c > 0$ such that $g(\Gamma, 1) \ge c$ for any standard leaf Γ. A linear functional on the space of foliated functions is said to be positive if it takes nonnegative values on positive functions.

Now, suppose that a family J_ε of weight functions is positive and bounded away from zero in the domain $\|dy/dx\| \le \theta$, $\|dz/dx\| \le \theta\eta$, where x, y, z, dy/dx, and dz/dx are the coordinates corresponding to an arbitrary hyperbolic chart. Then, the operator $A_{\varepsilon,n}$ maps positive (strongly positive) foliated functions to positive (respectively, strongly positive) foliated functions. Under these conditions, the following theorem is valid.

Theorem 3.3. *For any sufficiently large $n \in \mathbb{N}$, there exist a small number $\sigma_0 \in (0, \eta]$, a function $\lambda = \lambda_n \in C^{N-4}(M)$, a strongly positive foliated function $h = h_n \in \mathcal{F}^{1,N-2}$, and a positive $C^\infty(M)$-linear functional $\nu = \nu_n \colon \mathcal{F}^{N-5,4} \to C^2(M)$ such that*

(a) *the functional* ν *maps continuously every space* $\mathcal{F}^{pq}$ *to the space* $C^{q-2}(M)$;

(b) *the operator* $A_{0,n}$ *maps continuously every space* $\mathcal{F}^{pq}$ *to itself*;

(c) $A_{0,n}h = e^{\lambda}h$, $\nu \circ A_{0,n} = e^{\lambda}\nu$, *and* $\nu(h) \equiv 1$;

(d) *the sequence of operators* $[e^{-\lambda}A_{0,n}]^m \colon \mathcal{F}^{pq} \to \mathcal{F}^{pq}$ *converges in the uniform operator norm to the projector* $\overline{A}_{0,n}g = \nu(g)h$ *as* $m \to \infty$.

This is the main theorem of the paper. It is proved in Section 9.

4. PROOF OF THEOREM 3.1

Consider a foliated function $g \in \mathcal{F}^{pq}$ and a weight function J of smoothness $p+q$. To prove that the product Jg also belongs to $\mathcal{F}^{pq}$ and continuously depends on the factors, one should directly estimate all the derivatives in the definition of foliated function. This involves no ideas and only requires that the derivatives should be carefully calculated and written out.

Consider a standard deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$. Take the corresponding deformation $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$ of straight leaves. We connect them by the homotopy

$$\Gamma_{t\tau} \sim \bigl(y_t(x),\, \tau z_t(x) + (1-\tau)z_t(x_0),\, \chi, x_0\bigr), \qquad \tau \in [0,1].$$

We can assume that $\Gamma_{t\tau}$ is a standard deformation of skew leaves that depends on the composite parameter (t,τ). It may be majorized by pairs of the form $((v,v_\tau),\sigma)$, where (v,v_τ) is a composite vector of the same structure as the composite parameter (t,τ). Definition 1.4 of majorizing pairs directly implies the following assertion.

Lemma 4.1. *If* $\Gamma_t \prec (v,\sigma)$, *where* $\sigma \in [0,\eta]$, *then* $\Gamma_{t\tau} \prec ((v, B_1\sigma),\eta)$, *where* $B_1 = \max_{i,k}\alpha_{ik}/\theta\eta$.

Lemma 4.2. *There exists a large* B_2 *such that, for any foliated function* $g \in \mathcal{F}^{pq}$ *and any standard deformation* (Γ_t,Φ) *of straight traces, the relation* $\Gamma_t \prec (v,0)$ *implies the estimates*

$$\left|\frac{\partial^{|j|} g(\Gamma_t,\Phi)}{\partial t^j}\right| \le B_2\|g\|_{pq} v^j \|\Phi\|_{p+|j|}, \qquad |j| \le q-2.$$

Proof. Let $\Gamma_t \sim (y_t(x), z_t, \chi, x_0)$. Consider $\Gamma_{tz} \sim (y_t(x), z, \chi, x_0)$. We have

$$\frac{\partial^{|j|} g(\Gamma_t,\Phi)}{\partial t^j} = \sum_{k\le j} C_j^k\, \frac{\partial^{|j|} g(\Gamma_{t,z_T},\Phi)}{\partial t^{j-k}\partial T^k}\bigg|_{T=t} \qquad \text{and} \qquad \left\|\frac{\partial^{|k|} z_T}{\partial T^k}\right\| \le \alpha_{0|k|}\eta v^k.$$

These formulas and the definition of $\|g\|_{pq}$ imply the required estimates.

Let J be a weight function. In the coordinates corresponding to an arbitrary hyperbolic chart χ, it is represented as a function $J = J(x,y,z,dy/dx,dz/dx)$ of five variables. Let $\|J\|_q^\chi$ denote its C^q-norm with respect to these variables, and let $\|J\|_q = \max_{\chi\in\mathfrak{A}}\|J\|_q^\chi$. Consider the family of straight leaves $\Gamma_{tz} \sim (y_t(x), z, \chi)$. The restriction of the weight J to the leaf Γ_{tz} is represented as a function $\Psi_{tz}(x) = J(x, y_t(x), z, dy_t(x)/dx, 0)$. Differentiating $\Psi_{tz}(x)$ as a composite function and substituting estimates (7), we obtain the following assertion.

Lemma 4.3. *There exists a constant* B_3 *such that the relation* $y_t \prec v$ *implies the estimates*

$$\left\|\frac{\partial^{i+|j|+|k|}\Psi_{tz}(x)}{\partial x^i \partial t^j \partial z^k}\right\| \le B_3 v^j \|J\|_{i+|j|+|k|}, \qquad i+|j|+|k| \le N-1.$$

Consider a deformation of skew leaves $\Gamma_t \sim (y_t, z_t, \chi)$. The restriction of the weight J to the leaf Γ_t is represented as a function

$$\Psi_t(x) = J\big(x, y_t(x), z_t(x), dy_t(x)/dx, dz_t(x)/dx\big).$$

Differentiating it and substituting estimates (7)–(9), we obtain the following lemma.

Lemma 4.4. *There exists a large B_4 such that the relation $\Gamma_t \prec (v, \eta)$ implies the estimates*

$$\left\| \frac{\partial^{i+|j|}\Psi_t(x)}{\partial x^i \partial t^j} \right\| \le B_4 v^j \|J\|_{i+|j|}, \qquad i + |j| \le N - 1.$$

Lemma 4.5. *There exists a large B_5 such that, for any standard deformation of straight leaves $\Gamma_{tz} \sim (y_t(x), z, \chi, x_0)$ and any family of densities $\Phi_{tz}(x)$, the relation $y_t \prec v$ and the estimates*

$$\left\| \frac{\partial^{i+|j'|+|k'|}\Phi_{tz}(x)}{\partial x^i \partial t^{j'} \partial z^{k'}} \right\| \le \beta v^{j'}, \qquad i + |j'| + |k'| \le p + |j| + |k|,$$

which hold for some $\beta \ge 0$ and fixed multiindices j and k, imply

$$\left| \frac{\partial^{|j|+|k|} g(\Gamma_{tz}, \Phi_{tz})}{\partial t^j \partial z^k} \right| \le B_5 \|g\|_{pq} \beta v^j, \qquad |j| + |k| \le q, \quad |k| \le q - 2.$$

Proof. Differentiation by the Leibniz formula yields

$$\frac{\partial^{|j|+|k|} g(\Gamma_{tz}, \Phi_{tz})}{\partial t^j \partial z^k} = \sum_{j' \le j} \sum_{k' \le k} C_j^{j'} C_k^{k'} \frac{\partial^{|j-j'|+|k-k'|}}{\partial t^{j-j'} \partial z^{k-k'}} g(\Gamma_{tz}, \Phi_{TZ}^{j'k'}) \bigg|_{\substack{T=t \\ Z=z}}, \tag{30}$$

where

$$\Phi_{tz}^{j'k'}(x) = \frac{\partial^{|j'|+|k'|}\Phi_{tz}(x)}{\partial t^{j'} \partial z^{k'}}.$$

If $\Phi_{tz}^{j'k'}$ is treated as a function of x, then the hypothesis of the lemma implies the estimates $\|\Phi_{tz}^{j'k'}\|_{p+|j-j'|+|k-k'|} \le \beta v^{j'}$. From (30) and the definition of $\|g\|_{pq}$, we obtain

$$\left| \frac{\partial^{|j|+|k|} g(\Gamma_{tz}, \Phi_{tz})}{\partial t^j \partial z^k} \right| \le \sum_{j' \le j} \sum_{k' \le k} C_j^{j'} C_k^{k'} \|g\|_{pq} v^{j-j'} \beta v^{j'}.$$

Lemma 4.6. *There exists a large B_6 such that, for any standard deformation of skew leaves $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$, any deformation of straight leaves $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$, and any family of densities $\Phi_t(x)$, the relation $\Gamma_t \prec (v, \sigma)$ and the estimates*

$$\left\| \frac{\partial^{i+|j'|}\Phi_t(x)}{\partial x^i \partial t^{j'}} \right\| \le \beta v^{j'}, \qquad i + |j'| \le p + 1 + |j|, \tag{31}$$

which hold for some $\beta \ge 0$ and a fixed multiindex j, imply the estimates

$$\left| \frac{\partial^{|j|} g(\Gamma_t, \Phi_t)}{\partial t^j} - \frac{\partial^{|j|} g(\overline{\Gamma}_t, \Phi_t)}{\partial t^j} \right| \le B_6 \|g\|_{pq} \beta \sigma v^j, \qquad |j| \le q - 4. \tag{32}$$

If estimates (31) *hold for all indices such that* $i+|j'|\le p+|j|$, *then*

$$\left|\frac{\partial^{|j|}g(\overline{\Gamma}_t,\Phi_t)}{\partial t^j}\right|\le B_6\|g\|_{pq}\beta v^j,\qquad |j|\le q-2. \tag{33}$$

Therefore, under conditions (31),

$$\left|\frac{\partial^{|j|}g(\Gamma_t,\Phi_t)}{\partial t^j}\right|\le B_6(1+\sigma)\|g\|_{pq}\beta v^j,\qquad |j|\le q-4. \tag{34}$$

Proof. By the Leibniz formula,

$$\frac{\partial^{|j|}g(\Gamma_t,\Phi_t)}{\partial t^j}-\frac{\partial^{|j|}g(\overline{\Gamma}_t,\Phi_t)}{\partial t^j}=\sum_{j'\le j}C_j^{j'}\left(\frac{\partial^{|j-j'|}g(\Gamma_t,\Phi_T^{j'})}{\partial t^{j-j'}}-\frac{\partial^{|j-j'|}g(\overline{\Gamma}_t,\Phi_T^{j'})}{\partial t^{j-j'}}\right)\Bigg|_{T=t}, \tag{35}$$

$$\frac{\partial^{|j|}g(\overline{\Gamma}_t,\Phi_t)}{\partial t^j}=\sum_{j'\le j}C_j^{j'}\left.\frac{\partial^{|j-j'|}g(\overline{\Gamma}_t,\Phi_T^{j'})}{\partial t^{j-j'}}\right|_{T=t}, \tag{36}$$

where

$$\Phi_t^{j'}(x)=\frac{\partial^{|j'|}\Phi_t(x)}{\partial t^{j'}}.$$

Let us consider $\Phi_t^{j'}$ as a function of x. Then, by the condition of the lemma, $\|\Phi_t^{j'}\|_{p+1+|j-j'|}\le\beta v^{j'}$. It is easy to see that the definition of $\|g\|_{pq}$ and (35) imply (32). The relation $\Gamma_t\prec(v,\sigma)$ implies $\overline{\Gamma}_t\prec(v,0)$. If estimates (31) hold for $i+|j'|\le p+|j|$, then $\|\Phi_t^{j'}\|_{p+|j-j'|}\le\beta v^{j'}$. Hence, Lemma 4.2 and (36) imply (33).

Now, we can prove Theorem 3.1.

Choose a constant B_0 such that, for any functions $\Phi,\Psi\in C^q(\mathbb{B}_5^u)$, where $q\le N$, we have $\|\Phi\Psi\|_q\le B_0\|\Phi\|_q\|\Psi\|_q$. Suppose that $g\in\mathcal{F}^{pq}$ and J is an infinitely differentiable weight function. Obviously, the product Jg satisfies conditions (a) and (b) of Definition 3.3.

Let us verify that the function Jg satisfies condition (c) of Definition 3.3. Consider a standard deformation of straight leaves $\Gamma_{tz}\sim(y_t(x),z,\chi,x_0)$, where $y\prec v$, and a density $\Phi(x)$. Let $\Psi_{tz}(x)$ be the restriction of J to Γ_{tz}. The function Ψ_{tz} satisfies the hypothesis of Lemma 4.3. Therefore, the product $\Psi_{tz}\Phi$ satisfies the hypothesis of Lemma 4.5, where we can take $\beta=B_0B_3\|J\|_{p+q}\|\Phi\|_{p+|j|+|k|}$. By this lemma, we have

$$\left|\frac{\partial^{|j|+|k|}Jg(\Gamma_{tz},\Phi)}{\partial t^j\partial z^k}\right|=\left|\frac{\partial^{|j|+|k|}g(\Gamma_{tz},\Psi_{tz}\Phi)}{\partial t^j\partial z^k}\right|\le B_5\|g\|_{pq}B_0B_3\|J\|_{p+q}\|\Phi\|_{p+|j|+|k|}.$$

This inequality is quite similar to estimate (25) in which the foliated function g is replaced by Jg and the number c, by $B_0B_3B_5\|g\|_{pq}\|J\|_{p+q}$.

Let us verify that Jg satisfies condition (d) of Definition 3.3. Consider a standard deformation of skew leaves $\Gamma_t\sim(y_t(x),z_t(x),\chi,x_0)$ majorized by a pair (v,σ), a deformation $\overline{\Gamma}_t\sim(y_t(x),z_t(x_0),\chi,x_0)$, and a rectilinear homotopy $\Gamma_{t\tau}\sim\big(y_t(x),\tau z_t(x)+(1-\tau)z_t(x_0),\chi,x_0\big)$ that connects them. Let $\Psi_{t\tau}(x)$ be the restriction of the weight J to $\Gamma_{t\tau}$. Then, $Jg(\Gamma_t,\Phi)=g(\Gamma_{t1},\Psi_{t1}\Phi)$ and $Jg(\overline{\Gamma}_t,\Phi)=g(\Gamma_{t0},\Psi_{t0}\Phi)$. Obviously,

$$Jg(\Gamma_t,\Phi)-Jg(\overline{\Gamma}_t,\Phi)=\big[g(\Gamma_{t1},\Psi_{t1}\Phi)-g(\Gamma_{t0},\Psi_{t1}\Phi)\big]+g(\Gamma_{t0},\Psi_{t1}\Phi-\Psi_{t0}\Phi). \tag{37}$$

By Lemma 4.1, $\Gamma_{t\tau} \prec ((v, B_1\sigma), \eta)$. Applying Lemma 4.4 to the deformation $\Gamma_{t\tau}$, we obtain

$$\left\| \frac{\partial^{i+|j|+l}\Psi_{t\tau}(x)}{\partial x^i \partial t^j \partial \tau^l} \right\| \le B_4 v^j (B_1\sigma)^l \|J\|_{i+|j|+l}, \qquad i+|j|+l \le N-1.$$

Therefore,

$$\left\| \frac{\partial^{i+|j'|}(\Psi_{t1}(x)\Phi(x))}{\partial x^i \partial t^{j'}} \right\| \le B_0 B_4 v^{j'} \|J\|_{p+1+|j|} \|\Phi\|_{p+1+|j|}, \qquad i+|j'| \le p+1+|j|,$$

and

$$\left\| \frac{\partial^{i+|j'|}}{\partial x^i \partial t^{j'}} \big(\Psi_{t1}(x)\Phi(x) - \Psi_{t0}(x)\Phi(x)\big) \right\| = \left\| \int_0^1 \frac{\partial^{i+|j'|+1}(\Psi_{t\tau}(x)\Phi(x))}{\partial x^i \partial t^{j'} \partial \tau}\, d\tau \right\|$$

$$\le B_0 B_4 v^{j'} B_1 \sigma \|J\|_{p+1+|j|} \|\Phi\|_{p+|j|}$$

for $i+|j'| \le p+|j|$. These inequalities coincide with conditions (31) of Lemma 4.6 for the functions $\Psi_{t1}\Phi$ and $\Psi_{t1}\Phi - \Psi_{t0}\Phi$. According to this lemma,

$$\left| \frac{\partial^{|j|} g(\Gamma_{t1}, \Psi_{t1}\Phi)}{\partial t^j} - \frac{\partial^{|j|} g(\overline{\Gamma}_{t1}, \Psi_{t1}\Phi)}{\partial t^j} \right| \le B_6 \|g\|_{pq} B_0 B_4 \|J\|_{p+1+|j|} \|\Phi\|_{p+1+|j|} \sigma v^j,$$

$$\left| \frac{\partial^{|j|}}{\partial t^j} g(\Gamma_{t0}, \Psi_{t1}\Phi - \Psi_{t0}\Phi) \right| \le B_6 \|g\|_{pq} B_0 B_1 B_4 \sigma \|J\|_{p+1+|j|} \|\Phi\|_{p+|j|} v^j.$$

Combining these inequalities with (37), we obtain estimates (26) for the function Jg with a constant c of order $\|g\|_{pq}\|J\|_{p+q-3}$.

If the weight function J is infinitely differentiable, then, obviously, Jg satisfies the last condition (e) in Definition 3.3 as well. Therefore, $Jg \in \mathcal{F}^{pq}$, and the norm $\|Jg\|_{pq}$ is bounded by a quantity of order $\|g\|_{pq}\|J\|_{p+q}$. These results can be transferred to the weight functions of smoothness $p+q$ by continuity. This proves the theorem for weight functions J of smoothness $p+q$ and for $J \in C^{p+q}(W \times M)$. In the case of $J \in C^{p+q-3}(M)$, all conditions of Definition 3.3 for Jg are verified in precisely the same way, with the only exception that, to verify (c), we can write at once

$$\left| \frac{\partial^{|j|+|k|} J(z) g(\Gamma_{tz}, \Phi)}{\partial t^j \partial z^k} \right| = \left| \sum_{k' \le k} C_k^{k'} \frac{\partial^{|k-k'|} J(z)}{\partial z^{k-k'}} \frac{\partial^{|j|+|k'|} g(\Gamma_{tz}, \Phi)}{\partial t^j \partial z^{k'}} \right| \le \sum_{k' \le k} C_k^{k'} \|J\|_{q-2} \|g\|_{pq} v^j \|\Phi\|_{p+|j|+|k|}.$$

Therefore, we can take a constant c of order $\|J\|_{q-2}\|g\|_{pq} \le \|J\|_{p+q-3}\|g\|_{pq}$ in estimate (25) for the function Jg. This completes the proof of Theorem 3.1.

5. PROPERTIES OF IMAGES OF TRACES

As above, we consider a dynamical system of the form (1) for which a uniformly hyperbolic mixing atlas $\mathfrak{A}$ is fixed and the notions of weight functions, flowing skew leaves, and majorized deformations of leaves are defined.

Suppose that a family of weight functions J_ε is strictly positive and bounded away from zero in the domain $\|dy/dx\| \le \theta$, $\|dz/dx\| \le \theta\eta$, where x, y, z, dy/dx, and dz/dx are the coordinates corresponding to an arbitrary hyperbolic chart. Denote by $\nu = (\nu_1, \dots, \nu_n)$ an arbitrary vector with nonnegative components and by β_{ik}, where $i, k \in \mathbb{Z}_+$, a set of positive constants.

Theorem 5.1. *Suppose that a family of weights* J_ε *is bounded away from zero. Then, there exists a set of constants* $\beta_{ik} \ge 2$ *such that if the trace deformations* $(\Gamma'_{t\varepsilon}, \Phi'_{t\varepsilon}) = \Sigma_\varepsilon J_\varepsilon(\Gamma_t, \Phi_t)$, *where* $\Gamma_t \sim (y_t(x), z_t(x), \chi)$ *and* $\Gamma'_{t\varepsilon} \sim (y'_{t\varepsilon}(x'), z'_{t\varepsilon}(x'), \chi')$, *satisfy the conditions*

$$\Gamma_t \prec (v,\eta), \qquad \left\| \frac{\partial^{i+|j|}\Phi_t(x)}{\partial x^i \partial t^j} \right\| \le \beta_{i|j|}\nu^j |\Phi_t(x)|, \qquad i+|j| \le q, \quad q \le N-1, \tag{38}$$

for some x *and* t, *then, for the corresponding* x' *and* t,

$$\left\| \frac{\partial^{i+|j|}\Phi'_{t\varepsilon}(x')}{(\partial x')^i \partial t^j} \right\| \le (1-b)\beta_{i|j|}(\nu+v)^j |\Phi'_{t\varepsilon}(x')|, \qquad 1 \le i \le q-|j|, \tag{39}$$

$$\left\| \frac{\partial^{|j|}\Phi'_{t\varepsilon}(x')}{\partial t^j} \right\| \le \beta_{0|j|}(\nu+v)^j |\Phi'_{t\varepsilon}(x')|, \qquad |j| \le q. \tag{40}$$

This theorem is proved later on in this section.

Definition 5.1. A density $\Phi = \Phi(x)$ of smoothness $q \le N-1$ is said to be *flowing* if it is nonnegative and satisfies the inequalities $\|d^i\Phi(x)/dx^i\| \le \beta_{i0}\Phi(x)$ for all x and $i = 1, \dots, q$.

Corollary 5.1.1. *If* $(\Gamma'_\varepsilon, \Phi'_\varepsilon) = \Sigma_\varepsilon J_\varepsilon(\Gamma, \Phi)$ *and the density* Φ *of smoothness* q *is flowing, then the density* Φ'_ε *is also flowing.*

Proof. Consider the trace (Γ, Φ) as a deformation that does not depend on the parameter t. For such a deformation, we can set $v = \nu = 0$ in (38). Therefore, by (39), we have $\|d^i\Phi'_\varepsilon(x')/(dx')^i\| \le (1-b)\beta_{i0}\Phi'_\varepsilon(x')$; thus, the density Φ'_ε is flowing.

Repeatedly applying Theorem 5.1 with $\varepsilon = 0$ and Corollary 1.3.1, we obtain the following assertion.

Corollary 5.1.2. *Suppose that* $C_n = ne^{c\eta}$, *and let* $(\Gamma'_t, \Phi'_t) \subset (\Sigma_0 J_0)^n(\Gamma_t, \Phi_t)$, *where* $\Gamma_t \sim (y_t(x), z_t(x), \chi)$ *and* $\Gamma'_t \sim (y'_t(x'), z'_t(x'), \chi')$. *Then, the conditions*

$$\Gamma_t \prec (v,\eta), \qquad \left\| \frac{\partial^{i+|j|}\Phi_t(x)}{\partial x^i \partial t^j} \right\| \le \beta_{i|j|}\nu^j |\Phi_t(x)|, \qquad i+|j| \le q,$$

imply the estimates

$$\left\| \frac{\partial^{i+|j|}\Phi'_t(x')}{(\partial x')^i \partial t^j} \right\| \le \beta_{i|j|}(\nu+C_n v)^j |\Phi'_t(x')|, \qquad i+|j| \le q.$$

Theorem 5.2. *For any family of weights* J_ε, *there exist large constants* C_n *such that if some leaf deformation* $\Gamma_t \sim (y_t(x), z_t(x), \chi)$ *is majorized by a pair* (v,η), *and a family of densities* $\Phi_t(x)$ *satisfies the estimates*

$$\left\| \frac{\partial^{i+|j|}\Phi_t(x)}{\partial x^i \partial t^j} \right\| \le \beta v^j, \qquad i+|j| \le q, \quad q \le N-1,$$

for some $\beta \ge 0$, *then any family of traces* $(\Gamma'_{t\varepsilon}, \Phi'_{t\varepsilon}) \subset (\Sigma_\varepsilon J_\varepsilon)^n(\Gamma_t, \Phi_t)$ *obeys the estimates*

$$\left\| \frac{\partial^{i+|j|+l}\Phi'_{t\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^l} \right\| \le C_n \beta v^j, \qquad i+|j|+l \le q. \tag{41}$$

To prove Theorems 5.1 and 5.2, we need two lemmas.

Lemma 5.3. *If* $(\Gamma'_{t\varepsilon}, \Phi'_{t\varepsilon}) = \Sigma_\varepsilon J_\varepsilon(\Gamma_t, \Phi_t)$, *then*

$$\frac{\partial^{i+|j|}\Phi'_{t\varepsilon}(x')}{(\partial x')^i \partial t^j}(V')^i = J_\varepsilon\left(x, y, z, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) \frac{\partial^{i+|j|}\Phi_t(x)}{\partial x^i \partial t^j}\left(\frac{\partial x}{\partial x'}V'\right)^i + P_{ij}.$$

Here, the function P_{ij} *is a finite sum of products of the form* $p_1p_2p_3p_4$, *where*

(a) p_1 *is a smooth function of the variables* x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, *and* ε *all of whose partial derivatives are bounded*;

(b) p_2 *is a product of* i *coordinates of the vector* V';

(c) p_3 *is a partial derivative of* $\Phi_t(x)$ *preceding the index pair* (i, j) *in the ordering introduced in Section* 2;

(d) p_4 *is a product of several partial derivatives of* $y_t(x)$ *and* $z_t(x)$ *of order at most* $i + 1 + |j|$.

The sum of all multiindices in the product p_3p_4 *that correspond to the differentiations with respect to* t *equals* j.

Proof. The lemma is proved by successively differentiating the identity $\Phi'_{t\varepsilon}(x') = J_\varepsilon(x, y, z, \partial y/\partial x, \partial z/\partial x)\Phi_t(x)$, where $x = x(x', t, \varepsilon)$, $y = y_t(x)$, and $z = z_t(x)$. After every differentiation, we have to represent the result as a function of x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, ε, V', and the partial derivatives of y, z, and Φ_t with the use of substitutions (16) and (17).

Lemma 5.4. *Suppose that* $(\Gamma'_{t\varepsilon}, \Phi'_{t\varepsilon}) \subset \Sigma_\varepsilon J_\varepsilon(\Gamma_{t\varepsilon}, \Phi_{t\varepsilon})$, *where* $\Gamma_{t\varepsilon} \sim (y_{t\varepsilon}(x), z_{t\varepsilon}(x), \chi)$ *and* $\Gamma'_{t\varepsilon} \sim (y'_{t\varepsilon}(x'), z'_{t\varepsilon}(x'), \chi')$. *Then,*

$$\frac{\partial^{i+|j|+l}\Phi'_{t\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^l}(V')^i = P_{ijl};$$

here, P_{ijl} *is a finite sum of products of the form* $p_1p_2p_3p_4$, *where*

(a) p_1 *is a smooth function of the variables* x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, *and* ε *all of whose partial derivatives are bounded*;

(b) p_2 *is a product of* i *coordinates of the vector* V';

(c) p_3 *is a partial derivative of* $\Phi_{t\varepsilon}(x)$ *of order no higher than* $i + |j| + l$;

(d) p_4 *is a product of several partial derivatives of* $y_{t\varepsilon}(x)$ *and* $z_{t\varepsilon}(x)$ *of order no higher than* $i + |j| + l + 1$.

The sum of all multiindices in the product p_3p_4 *that correspond to the differentiations with respect to* t *equals* j.

Proof. The lemma is proved by successively differentiating the identity $\Phi'_{t\varepsilon}(x') = J_\varepsilon(x, y, z, \partial y/\partial x, \partial z/\partial x)$, where $x = x(x', t, \varepsilon)$, $y = y_{t\varepsilon}(x)$, and $z = z_{t\varepsilon}(x)$. After every differentiation, we have to represent the result as a function of x, y, z, $\partial y/\partial x$, $\partial z/\partial x$, ε, V', and the partial derivatives of $y_{t\varepsilon}(x)$, $z_{t\varepsilon}(x)$, and $\Phi_{t\varepsilon}(x)$ with the use of substitutions (16)–(18).

Proof of Theorem 5.1. We will determine the constants β_{ik} by induction based on the ordering of indices introduced in Section 2. Suppose that, for a pair of indices (I, J), we have already determined a set of numbers $\beta_{i|j|}$ that does not contain $\beta_{I|J|}$ but contains all numbers corresponding to the pairs of indices (i, j) preceding (I, J). Let us show how to find $\beta_{I|J|}$. We denote by $\mathfrak{D}$ the set of pairs (i, j) preceding (I, J) and such that $j \le J$. By Lemma 5.3 and conditions (38), there exists a large constant C (independent of $\beta_{i|j|}$) such that

$$\left\|\frac{\partial^{I+|J|}\Phi'_{t\varepsilon}(x')}{(\partial x')^I \partial t^J}\right\| \le J_\varepsilon\left(x, y, z, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right)\left\|\frac{\partial^{I+|J|}\Phi_t(x)}{\partial x^I \partial t^J}\right\|(a+b)^I + C\sum_{(i,j)\in\mathfrak{D}} \beta_{i|j|}\nu^j|\Phi_t(x)|v^{J-j}.$$

This inequality and the identity $\Phi'_{t\varepsilon}(x') = J_\varepsilon \Phi_t(x)$ give the implication

$$\left\| \frac{\partial^{I+|J|}\Phi_t(x)}{\partial x^I \partial t^J} \right\| \le \beta_{I|J|}\nu^J |\Phi_t(x)| \qquad \Longrightarrow \qquad \left\| \frac{\partial^{I+|J|}\Phi'_{t\varepsilon}(x')}{(\partial x')^I \partial t^J} \right\| \le \Delta |\Phi'_{t\varepsilon}(x')|,$$

where

$$\Delta = \beta_{I|J|}\nu^J (a+b)^I + \frac{C}{J_\varepsilon} \sum_{(i,j)\in\mathfrak{D}} \beta_{i|j|}\nu^j v^{J-j}.$$

Obviously,

$$\sum_{(i,j)\in\mathfrak{D}} \nu^j v^{J-j} \le N \sum_{j\le J} \nu^j v^{J-j} \le N(\nu+v)^J.$$

If $I > 0$, then, choosing $\beta_{I|J|}$ large enough in comparison with C, N, and $\beta_{i|j|}$, we can ensure that the inequality $\Delta \le (1-b)\beta_{I|J|}(\nu+v)^J$ holds for all ν and v. If $I = 0$, then the definition of the ordering of indices implies that j cannot coincide with J. In this case, taking $\beta_{0|J|} \ge \sup C J_\varepsilon^{-1} \beta_{i|j|} N$, we obtain $\Delta \le \beta_{0|J|}(\nu+v)^J$. In any case, we have secured estimate (39) if $I > 0$ and (40) if $I = 0$. The fulfillment of these estimates for all pairs of indices (I, J) is precisely what Theorem 5.1 asserts. This completes the induction step.

Proof of Theorem 5.2. Theorem 5.2 is proved by induction on n with the use of Lemma 5.4 and Theorem 1.4. At each induction step, one should substitute the estimates from Theorem 1.4 and inequalities of the form (41) obtained at the preceding step into the equality of Lemma 5.4.

6. PROOF OF THEOREM 3.2

First, we represent the value of the foliated function $A_{\varepsilon,n}g$ on an arbitrary standard trace (Γ, Φ) in a form convenient for calculations. Recall that

$$(A_{\varepsilon,n}g)(\Gamma(\alpha), \Phi) = \int\limits_{\Gamma'_\varepsilon} \xi_\Gamma(\alpha,\beta) g(\Gamma'_\varepsilon(\beta'), \Phi'_\varepsilon)\, d\mu(\beta'). \tag{42}$$

Here, $\Gamma \sim (y(x), z(x), \chi, x_0)$ is a standard leaf; the point $\alpha = (\chi(x_0, y(x_0)), z(x_0))$ is its center; β is an arbitrary point of Γ; the standard trace $(\Gamma'_\varepsilon, \beta', \Phi'_\varepsilon)$ is determined by its center $\beta' = \Sigma^n_\varepsilon(\beta)$ and by the inclusion $(\Gamma'_\varepsilon, \Phi'_\varepsilon) \subset (\Sigma_\varepsilon J_\varepsilon)^n(\Gamma, \Phi)$; and μ is the Riemannian volume on the leaf $\Gamma'_\varepsilon = \Sigma^n_\varepsilon(\Gamma)$. The function $\xi_\Gamma(\alpha,\beta)$ is defined by

$$\xi_\Gamma(\alpha,\beta) = \frac{\xi_0(d(\alpha,\beta))}{\int_\Gamma \xi_0(d(\alpha,\beta))\, d\mu(\alpha)}, \tag{43}$$

where d is the distance in $W \times M$ and $\xi_0(t)$ is a fixed smooth nonnegative function taking value 1 for $|t| \le r_0$ and vanishing for $|t| \ge 2r_0$. Here, r_0 is a sufficiently small positive number.

Now, we will modify (42) so that it admits differentiation with respect to the parameter under a transverse deformation of the standard skew trace $\Gamma \sim (y(x), z(x), \chi, x_0)$. Obviously, the variable $x \in B(x_0, 1)$ is a coordinate on Γ. The function $\xi_\Gamma(\alpha,\beta)$ depends only on the point $\beta(x) = (\chi(x, y(x)), z(x))$ because the point α (the center of Γ) is fixed. We set $\xi_\Gamma(x) = \xi_\Gamma(\alpha, \beta(x))$ and write the Riemannian volume on Γ in the form $d\mu = \rho(x)\, dx$.

Let $\Gamma^* = \{\beta \in \Gamma \mid d(\alpha,\beta) \le 2r_0\}$. Then, $\operatorname{supp} \xi_\Gamma \subset \Gamma^*$. By the definition of a hyperbolic atlas, for any point $\beta \in \Gamma^*$, there exists a chart $\chi' \in \mathfrak{A}$ such that $\Sigma^n_\varepsilon(\beta) \in \chi'(\mathbb{B}^u_1 \times \mathbb{B}^s_1) \times M$. By Proposition 1.2, the point $\beta' = \Sigma^n_\varepsilon(\beta)$ belongs to a leaf $\Gamma'_\varepsilon \subset \Sigma^n_\varepsilon(\Gamma)$ of the form $\Gamma'_\varepsilon \sim (y'_\varepsilon(x'), z'_\varepsilon(x'), \chi')$, where $x' \in \mathbb{B}^u_5$; moreover, β' has a coordinate $x' \in \mathbb{B}^u_1$. We construct such a leaf for every point

$\beta \in \Gamma^*$. Since the linear dimensions of the set $\Sigma_\varepsilon^n(\Gamma^*)$ are bounded, the number of such leaves must be finite and bounded irrespective of the choice of the standard leaf Γ. We denote these leaves by $\Gamma'_{l\varepsilon} \sim (y'_{l\varepsilon}(x'), z'_{l\varepsilon}(x'), \chi'_l)$, where $x' \in \mathbb{B}_5^u$ and $l = 1, \dots, m$. Let $\rho_{l\varepsilon}(x')\,dx'$ denote the Riemannian volume on $\Gamma'_{l\varepsilon}$. For every point $x'_0 \in \mathbb{B}_4^u$, consider the standard leaf $\Gamma'_{l\varepsilon}(x'_0) \sim (y'_{l\varepsilon}(x'), z'_{l\varepsilon}(x'), \chi'_l, x'_0)$ centered at $(\chi'_l(x'_0, y'_{l\varepsilon}(x'_0)),\ z'_{l\varepsilon}(x'_0))$.

Let us fix an infinitely differentiable nonnegative function ζ on $\mathbb{R}^u$ that takes value 1 on $\mathbb{B}_2^u$ and vanishes outside $\mathbb{B}_3^u$. On each leaf $\Gamma'_{l\varepsilon}$, we define a function $\zeta_l = \zeta_l(x')$, where x' is the coordinate on $\Gamma'_{l\varepsilon}$. Since $\Gamma'_{l\varepsilon} \subset \Sigma_\varepsilon^n(\Gamma)$, this function can be extended by zero to $\Sigma_\varepsilon^n(\Gamma) \setminus \Gamma'_{l\varepsilon}$. Consider the function $\overline{\zeta}_\varepsilon = \sum_{l=1}^m \zeta_l$ on $\Sigma_\varepsilon^n(\Gamma)$. We denote its restriction to the leaf $\Gamma'_{l\varepsilon}$ by $\overline{\zeta}_{l\varepsilon} = \overline{\zeta}_{l\varepsilon}(x')$. By construction, $\Sigma_\varepsilon^n(\Gamma^*) \subset \bigcup_l \operatorname{supp}\zeta_l$. Therefore, the functions $\zeta_l/\overline{\zeta}_{l\varepsilon}$ form a smooth partition of unity on the set $\Sigma_\varepsilon^n(\Gamma^*)$.

Let us define a function $\xi_{l\varepsilon} = \xi_{l\varepsilon}(x')$ on $\Gamma'_{l\varepsilon}$ in such a way that $(\Gamma'_{l\varepsilon}, \xi_{l\varepsilon}) \subset \Sigma_\varepsilon^n(\Gamma, \xi_\Gamma)$. Then, formula (42) can be reduced to the form

$$(A_{\varepsilon,n}g)(\Gamma, \Phi) = \sum_{l=1}^m \int_{\mathbb{B}_5^u} \xi_{l\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{l\varepsilon}(x')} g(\Gamma'_{l\varepsilon}(x'), \Phi'_\varepsilon) \rho_{l\varepsilon}(x')\,dx'. \tag{44}$$

Now, suppose that, instead of a fixed leaf Γ, a standard deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ is given. If the range of the parameter t is small, then, for Γ_t, we can construct standard deformations $\Gamma'_{lt\varepsilon} \sim (y'_{lt\varepsilon}(x'), z'_{lt\varepsilon}(x'), \chi'_l)$ and deformations of the functions ξ_{Γ_t}, ρ_t, $\xi_{lt\varepsilon}$, $\overline{\zeta}_{lt\varepsilon}$, $\rho_{lt\varepsilon}$, and $\Phi'_{t\varepsilon}$ in precisely the same way as the leaves $\Gamma'_{l\varepsilon}$ and the functions ξ_Γ, ρ, $\xi_{l\varepsilon}$, $\overline{\zeta}_{l\varepsilon}$, $\rho_{l\varepsilon}$, and Φ'_ε were constructed above (but the function ζ_l remains independent of t and ε). Accordingly, formula (44) takes the form

$$(A_{\varepsilon,n}g)(\Gamma_t, \Phi) = \sum_{l=1}^m \int_{\mathbb{B}_5^u} \xi_{lt\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{lt\varepsilon}(x')} g(\Gamma'_{lt\varepsilon}(x'), \Phi'_{t\varepsilon}) \rho_{lt\varepsilon}(x')\,dx'. \tag{45}$$

To differentiate (45) with respect to t and ε, we need three lemmas.

Lemma 6.1. *There exists a large B_7 such that if a standard deformation of leaves Γ_t is majorized by a pair (v, η) and has the form $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$, then*

$$\left\| \frac{\partial^{i+|j|}\rho_t(x)}{\partial x^i \partial t^j} \right\| \le B_7 v^j, \qquad \left\| \frac{\partial^{i+|j|}\xi_{\Gamma_t}(x)}{\partial x^i \partial t^j} \right\| \le B_7 v^j, \qquad i + |j| \le N - 1.$$

Proof. Recall that $\rho_t(x)\,dx$ is the Riemannian volume on Γ_t and $\xi_{\Gamma_t}(x) = \xi_{\Gamma_t}(\alpha_t, \beta_t(x))$, where $\alpha_t = (\chi(x_0, y_t(x_0)), z_t(x_0))$ and $\beta_t(x) = (\chi(x, y_t(x)), z_t(x))$. The density $\rho_t(x)$ is a smooth function of the variables x, $y_t(x)$, $z_t(x)$, $dy_t(x)/dx$, and $dz_t(x)/dx$. This readily implies bounds for its derivatives. The bounds for the derivatives of $\xi_{\Gamma_t}(x)$ are obtained by substituting the expressions for α_t, $\beta_t(x)$, and the measure $\rho_t(x)\,dx$ into equality (43) and differentiating the result.

Lemma 6.2. *There exists a large B_8 independent of the choice of the deformations Γ_t and $\Gamma'_{lt\varepsilon}$ such that the relation $\Gamma_t \prec (v, \eta)$ implies the estimates*

$$\left\| \frac{\partial^{i+|j|+k}}{(\partial x')^i \partial t^j \partial \varepsilon^k} \left(\xi_{lt\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{lt\varepsilon}(x')} \rho_{lt\varepsilon}(x') \right) \right\| \le B_8 v^j, \qquad i + |j| + k \le N - 1.$$

Proof. By Lemma 6.1 and Theorem 5.2, we have

$$\left\| \frac{\partial^{i+|j|+k}\xi_{lt\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^k} \right\| \le C_n B_7 v^j, \qquad i + |j| + k \le N - 1. \tag{46}$$

By Theorem 1.4, the deformation $\Gamma'_{lt\varepsilon} \sim (y'_{lt\varepsilon}(x'), z'_{lt\varepsilon}(x'), \chi'_l)$ obeys the estimates

$$\left\| \frac{\partial^{i+|j|+k} y'_{lt\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^k} \right\| \le C_n v^j, \qquad \left\| \frac{\partial^{i+|j|+k} z'_{lt\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^k} \right\| \le C_n v^j, \qquad 1 \le i+|j|+k \le N.$$

Therefore, the function

$$\rho_{lt\varepsilon}(x') = \rho_l\left(x', y'_{lt\varepsilon}(x'), z'_{lt\varepsilon}(x'), \frac{dy'_{lt\varepsilon}(x')}{dx'}, \frac{dz'_{lt\varepsilon}(x')}{dx'}\right)$$

is estimated as

$$\left\| \frac{\partial^{i+|j|+k} \rho_{lt\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \varepsilon^k} \right\| \le C v^j, \qquad i+|j|+k \le N-1. \tag{47}$$

The function $\overline{\zeta}_{lt\varepsilon}$ has the form $\overline{\zeta}_{lt\varepsilon}(x') = \sum_{k=1}^m \zeta \circ (\chi'_k)^{-1} \circ \chi'_l(x', y'_{lt\varepsilon}(x'))$. By construction, all derivatives of the functions $(\chi'_k)^{-1} \circ \chi'_l$ are bounded. Therefore, the derivatives of $\overline{\zeta}_{lt\varepsilon}$ obey the estimates of the form

$$\left\| \frac{\partial^{i+|j|+k}}{(\partial x')^i \partial t^j \partial \varepsilon^k} \left(\frac{\zeta_l(x')}{\overline{\zeta}_{lt\varepsilon}(x')}\right) \right\| \le C v^j, \qquad i+|j|+k \le N-1. \tag{48}$$

Inequalities (46)–(48) imply the assertion of the lemma.

Suppose that a standard deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ of skew leaves is majorized by a pair (v, σ), where $\sigma \in [0, \sigma_0]$. Consider the standard deformation $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$ of straight leaves and the homotopy $\Gamma_{t\tau} \sim (y_t(x),\, \tau z_t(x) + (1-\tau) z_t(x_0),\, \chi, x_0)$. If the number σ_0 is sufficiently small, then we can replace the index t in (45) by the pair (t, τ). We obtain

$$(A_{\varepsilon,n} g)(\Gamma_{t\tau}, \Phi) = \sum_{l=1}^m \int\limits_{\mathbb{B}_5^u} \xi_{lt\tau\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{lt\tau\varepsilon}(x')}\, g(\Gamma'_{lt\tau\varepsilon}(x'), \Phi'_{t\tau\varepsilon}) \rho_{lt\tau\varepsilon}(x')\, dx'. \tag{49}$$

For short, we set

$$\Psi_{lt\tau\varepsilon}(x') = \xi_{lt\tau\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{lt\tau\varepsilon}(x')} \rho_{lt\tau\varepsilon}(x'). \tag{50}$$

Lemma 6.3. *There exists a large B_9 independent of the deformations Γ_t and $\Gamma'_{lt\tau\varepsilon}$ such that the relation $\Gamma_t \prec (v, \sigma)$ implies the estimates*

$$\left\| \frac{\partial^{i+|j|+k+m} \Psi_{lt\tau\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \tau^k \partial \varepsilon^m} \right\| \le B_9 v^j \sigma^k, \qquad i+|j|+k+m \le N-1. \tag{51}$$

Proof. By Lemma 4.1, $\Gamma_t \prec (v, \sigma)$ implies the relation $\Gamma_{t\tau} \prec ((v, B_1\sigma), \eta)$. Therefore, estimates (51) follow from Lemma 6.2.

Now, we can proceed to the proof of Theorem 3.2. We have to show that the function $d^m A_{\varepsilon,n} g / d\varepsilon^m$ belongs to $\mathcal{F}^{p+1+m, q-4-m}$ for any foliated function $g \in \mathcal{F}^{pq}$. Let us verify that this function satisfies all conditions of Definition 3.3 of the space $\mathcal{F}^{p+1+m,q-4-m}$.

A. It follows directly from the definition of the operator $A_{\varepsilon,n}$ that $A_{\varepsilon,n} g(\Gamma, \Phi)$ linearly depends on Φ, and the values of $A_{\varepsilon,n} g$ on equivalent traces coincide. The same is true for the derivatives $d^m A_{\varepsilon,n} g(\Gamma, \Phi)/d\varepsilon^m$ (if they exist). Thus, conditions (a) and (b) of Definition 3.3 hold.

B. Let us verify point (c). Suppose that $\Gamma_{tz} \sim (y_t(x), z, \chi, x_0)$ and $y_t \prec v$. We will calculate $A_{\varepsilon,n}g(\Gamma_{tz}, \Phi)$ by (45), replacing the index t in this formula with the pair (t, z). We obtain

$$(A_{\varepsilon,n}g)(\Gamma_{tz}, \Phi) = \sum_{l=1}^{m} \int\limits_{\mathbb{B}_5^u} \xi_{ltz\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{ltz\varepsilon}(x')} \, g(\Gamma'_{ltz\varepsilon}(x'), \Phi'_{tz\varepsilon}) \rho_{ltz\varepsilon}(x') \, dx'. \tag{52}$$

We assume that Γ_{tz} depends on the composite parameter (t, z). Definition 1.4 of majorizing pairs implies that $\Gamma_{tz} \prec ((v, (\theta\eta)^{-1}e), 0)$, where the vector $e = (1, \dots, 1)$ has the same dimension as z. Theorem 1.4 implies a relation of the form $\Gamma'_{ltz\varepsilon}(x') \prec ((Cv, Ce, C), \eta)$ with respect to the composite parameter (t, z, ε), where the constant C does not depend on the choice of the deformations Γ_{tz} and $\Gamma'_{ltz\varepsilon}(x')$. Applying Theorem 5.2 with $\beta = \|\Phi\|_q$, we obtain the estimates

$$\left\| \frac{\partial^{i+|j|+|k|+m} \Phi'_{tz\varepsilon}(x')}{(\partial x')^i \partial t^j \partial z^k \partial \varepsilon^m} \right\| \le C_n \|\Phi\|_q v^j (\theta\eta)^{-|k|}, \qquad i + |j| + |k| + m \le q.$$

By Lemma 4.6, for all $|j| + |k| + m \le q - 4$, we have

$$\left| \frac{\partial^{|j|+|k|+m}}{\partial t^j \partial z^k \partial \varepsilon^m} \, g(\Gamma'_{ltz\varepsilon}(x'), \Phi'_{tz\varepsilon}) \right| \le B_6(1+\eta) \|g\|_{pq} C_n \|\Phi\|_{p+1+|j|+|k|+m} (Cv)^j C^{|k|+m}. \tag{53}$$

Applying Lemma 6.2 with t replaced by (t, z), we obtain

$$\left| \frac{\partial^{i+|j|+|k|+m}}{(\partial x')^i \partial t^j \partial z^k \partial \varepsilon^m} \left(\xi_{ltz\varepsilon}(x') \frac{\zeta_l(x')}{\overline{\zeta}_{ltz\varepsilon}(x')} \rho_{ltz\varepsilon}(x') \right) \right| \le B_8 v^j (\theta\eta)^{-|k|} \tag{54}$$

for all $i + |j| + |k| + m \le N - 1$. Finally, the differentiation of (52) with regard to (53) and (54) yields an estimate of the form

$$\left| \frac{\partial^{|j|+|k|+m}}{\partial t^j \partial z^k \partial \varepsilon^m} A_{\varepsilon,n} g(\Gamma_{tz}, \Phi) \right| \le C \|g\|_{pq} v^j \|\Phi\|_{p+1+|j|+|k|+m}, \qquad |j| + |k| + m \le q - 4.$$

It is an analogue of estimate (25) for the function $d^m A_{\varepsilon,n} g / d\varepsilon^m$ as an element of the space $\mathcal{F}^{p+1+m, q-4-m}$.

C. Let us verify that the function $d^m A_{\varepsilon,n} g / d\varepsilon^m$ satisfies condition (d) of Definition 3.3. Consider a standard deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ of skew leaves, the corresponding deformation $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$ of straight leaves, and the homotopy $\Gamma_{t\tau} \sim (y_t(x),\ \tau z_t(x) + (1-\tau) z_t(x_0),\ \chi, x_0)$. Suppose that $\Gamma_t \prec (v, \sigma)$. Then, by Lemma 4.1, $\Gamma_{t\tau} \prec ((v, B_1\sigma), \eta)$. Theorem 1.4 implies a relation of the form $\Gamma'_{lt\tau\varepsilon}(x') \prec ((Cv, C\sigma, C), \eta)$ with respect to the composite parameter (t, τ, ε), where the constant C does not depend on the choice of the deformations Γ_t and $\Gamma'_{lt\tau\varepsilon}(x')$. By Theorem 5.2, we have

$$\left\| \frac{\partial^{i+|j|+k+m} \Phi'_{t\tau\varepsilon}(x')}{(\partial x')^i \partial t^j \partial \tau^k \partial \varepsilon^m} \right\| \le C_n \|\Phi\|_q v^j (B_1\sigma)^k, \qquad i + |j| + k + m \le q.$$

By Lemma 4.6,

$$\left| \frac{\partial^{|j|+k+m}}{\partial t^j \partial \tau^k \partial \varepsilon^m} g(\Gamma'_{lt\tau\varepsilon}(x'), \Phi'_{t\tau\varepsilon}) \right| \le B_6(1+\eta) \|g\|_{pq} C_n \|\Phi\|_{p+1+|j|+k+m} (Cv)^j (C\sigma)^k C^m \tag{55}$$

for all $|j| + k + m \le q - 4$. The differentiation of (49) with regard to (51) and (55) yields an estimate of the form

$$\left| \frac{\partial^{|j|+m}}{\partial t^j \partial \varepsilon^m} \big(A_{\varepsilon,n} g(\Gamma_t, \Phi) - A_{\varepsilon,n} g(\overline{\Gamma}_t, \Phi) \big) \right| = \left| \int\limits_0^1 \frac{\partial^{|j|+m+1}}{\partial t^j \partial \tau \partial \varepsilon^m} A_{\varepsilon,n}(\Gamma_{t\tau}, \Phi) \right| \le C \|g\|_{pq} \sigma v^j \|\Phi\|_{p+2+|j|+m}$$

for $|j| + m \le -5$. It is an analogue of estimate (26) for the function $d^m A_{\varepsilon,n} g/d\varepsilon^m$ as an element of the space $\mathcal{F}^{p+1+m,q-4-m}$.

D. The continuity of the derivatives of the function $d^m A_{\varepsilon,n} g/d\varepsilon^m \in \mathcal{F}^{p+1+m,q-4-m}$, which is required by condition (e) of Definition 3.3, follows from the method for evaluating these derivatives described above. This completes the proof of Theorem 3.2.

7. FOLIATED FUNCTIONS ON STRAIGHT TRACES

Any foliated function can be restricted to (a) the set of straight standard traces and (b) the set of straight standard traces contained in a layer $z = \text{const}$. For these restrictions, we use the same term "foliated functions." For $\varepsilon = 0$, the averaged weighted shift operator $A_{0,n}$ is well defined in the spaces of such functions. Below, we give the corresponding formal definitions.

Definition 7.1. A foliated function of type (p,q) (where $p \ge 1$, $q \ge 1$, and $p + q \le N - 1$) on a layer $z = \text{const}$ is an arbitrary real-valued function $g_z(\Gamma, \Phi)$ that is defined on the set of straight standard traces contained in the layer $z = \text{const}$ and has the following properties:

(a) it linearly depends on the density Φ;

(b) its values on equivalent traces coincide;

(c) there exists a positive number c such that if a straight standard deformation $\Gamma_{tz} \sim (y_t(x), z, \chi, x_0)$ obeys the estimate $y_t \prec v$, then, for any density $\Phi = \Phi(x)$,

$$\left| \frac{\partial^{|j|} g_z(\Gamma_{tz}, \Phi)}{\partial t^j} \right| \le c v^j \|\Phi\|_{p+|j|}, \qquad |j| \le q; \tag{56}$$

(d) all partial derivatives in (56) are continuous with respect to t and vary continuously under a sliding deformation of the trace (Γ_{tz}, Φ).

We denote the minimum c for which estimates (56) hold by $\|g_z\|_{pq}$ and the space of all functions satisfying the conditions of Definition 7.1 by $\mathcal{F}_z^{pq}$. Let $A_{z,n}$ be the restriction of the operator $A_{0,n}$ to $\mathcal{F}_z^{pq}$. The properties of the operator $A_{z,n}$ were thoroughly studied in [2] in a more general situation when the mapping S_z varies with time. In the case under consideration, S_z does not depend on time, and all the results obtained in [2] become substantially simpler. In particular, Theorem 6.3 and Propositions 2.4 and 3.1 from [2] readily imply the following theorem.

Theorem 7.1. *The operator $A_{z,n}$ maps continuously every space $\mathcal{F}_z^{pq}$ to itself. If the weight function J is positive and bounded away from zero and a positive integer n is sufficiently large, then, for any $z \in M$, there exist a positive function $h_z \in \mathcal{F}_z^{1,N-2}$, a positive linear functional $\nu_z \colon \mathcal{F}_z^{N-2,1} \to \mathbb{R}$, and a number λ_z such that*

(a) $A_{z,n} h_z = e^{\lambda_z} h_z$, $\nu_z \circ A_{z,n} = e^{\lambda_z} \nu_z$, *and* $\nu_z(h_z) = 1$;

(b) *the sequence of operators $[e^{-\lambda_z} A_{z,n}]^m$ in the space $\mathcal{F}_z^{pq}$ converges to the projector $\overline{A}_{z,n} g = \nu_z(g) h_z$ in the uniform operator norm as $m \to \infty$;*

(c) *there exists a large C_0 independent of z such that $h_z(\Gamma_{z1}, 1) \le C_0 h_z(\Gamma_{z2}, 1)$ for any two leaves Γ_{z1} and Γ_{z2}.*

Definition 7.2. The space $\mathcal{F}_M^{pq}$, where $p \ge 1$, $q \ge 2$, and $p + q \le N - 1$, consists of all possible parametric families $\{g_z \in \mathcal{F}_z^{pq} \mid z \in M\}$ for each of which there exists a number $c \ge 0$ such that, for any standard deformation $\Gamma_{tz} \sim (y_t(x), z, \chi, x_0)$ of straight leaves and any density $\Phi = \Phi(x)$, the estimate $y_t \prec v$ implies the inequalities

$$\left| \frac{\partial^{|j|+|k|} g_z(\Gamma_{tz}, \Phi)}{\partial t^j \partial z^k} \right| \le c v^j \|\Phi\|_{p+|j|+|k|}, \qquad |j| + |k| \le q, \quad |k| \le q - 2, \tag{57}$$

and all derivatives in these inequalities exist and are continuous with respect to t and z.

If $g = \{g_z\} \in \mathcal{F}_M^{pq}$, then we denote by $\|g\|_{pq}$ the minimum c for which inequalities (57) hold. Obviously, the space $\mathcal{F}_M^{pq}$ with this norm is a Banach space. It is a module over $C^{q-2}(M)$ and over $C^{p+q}(W \times M)$.

The spaces $\mathcal{F}_z^{pq}$ with different $z \in M$ are naturally identified with each other. Therefore, we can assume that all of them coincide with some fixed space $\mathcal{F}_0^{pq}$. Then, any function $g \in \mathcal{F}_M^{pq}$ is identified with the parametric family $\{g_z \in \mathcal{F}_0^{pq} \mid z \in M\}$. It is seen from Definitions 7.1 and 7.2 that a family $g = \{g_z\}$ belongs to $\mathcal{F}_M^{pq}$ if and only if $\partial^{|k|} g_z / \partial z^k \in \mathcal{F}_0^{p+|k|, q-|k|}$ for all $|k| \le q-2$, and

$$\|g\|_{pq} = \sup_{z\in M} \sup_{|k|\le q-2} \left\| \frac{\partial^{|k|} g_z}{\partial z^k} \right\|_{p+|k|, q-|k|}. \tag{58}$$

In [2], such parametric families were called *quasidifferentiable.* It was shown in [2] that the parametric family of operators $A_{z,n}\colon \mathcal{F}_0^{pq} \to \mathcal{F}_0^{p+l, q-l}$ is $l-1$ times continuously differentiable with respect to z, and the operators $\partial^{|k|} A_{z,n}/\partial z^k$ map continuously $\mathcal{F}_0^{pq}$ to $\mathcal{F}_0^{p+|k|, q-|k|}$. Obviously, the operator $A_{0,n}$ acts fiberwise in $\mathcal{F}_M^{pq}$; namely, if $g = \{g_z\}$ and $h = \{h_z\}$, then $h = A_{0,n} g$ if and only if $h_z = A_{z,n} g_z$ for all $z \in M$. Proposition 4.2 from [2] implies that $A_{0,n}$ maps continuously $\mathcal{F}_M^{pq}$ to itself. Let us define an operator $L_{z,n}\colon \mathcal{F}^{pq} \to \mathcal{F}^{pq}$ by the formula $L_{z,n} = A_{z,n} - e^{\lambda_z} \overline{A}_{z,n}$, where $\overline{A}_{z,n} g = \nu_z(g) h_z$ is the projector mentioned in Theorem 7.1. Then, Lemma 4.6 from [2] takes the following form in the situation under consideration.

Theorem 7.2. *Under the conditions of Theorem* 7.1, *the families $h = \{h_z\}$ and $\nu = \{\nu_z\}$ can be chosen so that $h \in \mathcal{F}_M^{1, N-2}$ and ν determines a $C(M)$-linear functional $\nu\colon \mathcal{F}_M^{N-3,2} \to C(M)$ that maps continuously each space $\mathcal{F}_M^{pq}$ to $C^{q-2}(M)$. There exist numbers C and $\Lambda \in (0,1)$ such that, for any $|k| \le q-1$ and any positive integer m, the norm of the operator $\partial^{|k|}[L_{z,n}]^m/\partial z^k\colon \mathcal{F}_0^{pq} \to \mathcal{F}_0^{p+|k|, q-|k|}$ does not exceed $C\Lambda^m e^{m\lambda_z}$.*

Corollary 7.2.1. *The family $\lambda = \{\lambda_z\}$, where $\lambda_z = \ln \nu_z(A_{z,n} h_z)$ are as in Theorem* 7.1, *belongs to the space $C^{N-4}(M)$.*

Indeed, the families $h = \{h_z\}$ and $A_{0,n} h = \{A_{z,n} h_z\}$ belong to $\mathcal{F}_M^{1,N-2}$. Therefore, $e^\lambda = \nu(A_{0,n} h) \in C^{N-4}(M)$.

Corollary 7.2.2. *If $p \ge 1$, $q \ge 2$, and $p + q \le N - 1$, then the operator $e^{-\lambda} A_{0,n}$ maps continuously the space $\mathcal{F}_M^{pq}$ to itself. The sequence $[e^{-\lambda} A_{0,n}]^m$ converges to the projector $\overline{A}_{0,n}$ defined by the formula $\overline{A}_{0,n} g = \nu(g) h$.*

Proof. As mentioned, $\mathcal{F}_M^{pq}$ is a module over $C^{q-2}(M)$, and $A_{0,n}$ maps continuously this module to itself. Obviously, $q \le N - 1 - p \le N - 2$. Since $e^{-\lambda} \in C^{N-4}(M)$ and the functional $\nu\colon \mathcal{F}_M^{pq} \to C^{q-2}(M)$ is continuous, the operators $e^{-\lambda} A_{0,n}$ and $\overline{A}_{0,n}$ are continuous on $\mathcal{F}_M^{pq}$. By construction, these operators leave the function h fixed. Therefore, it is sufficient to show that the norm of the operator $[e^{-\lambda} A_{0,n} - \overline{A}_{0,n}]^m$ tends to zero as $m \to \infty$. Let L_n denote the operator that maps each foliated function $g = \{g_z\} \in \mathcal{F}_M^{pq}$ to $f = \{f_z\}$, where $f_z = L_{z,n} g_z$. Then, $e^{-\lambda} A_{0,n} - \overline{A}_{0,n} = e^{-\lambda} L_n$. Now, suppose that $f = [e^{-\lambda} L_n]^m g$, where $f = \{f_z\}$ and $g = \{g_z\}$. Differentiating by the Leibniz formula, we obtain

$$\frac{\partial^{|k|} f_z}{\partial z^k} = \sum_{k_1+k_2+k_3=k} \frac{k!}{k_1!\, k_2!\, k_3!} \frac{\partial^{|k_1|} e^{-m\lambda_z}}{\partial z^{k_1}} \frac{\partial^{|k_2|} [L_{z,n}]^m}{\partial z^{k_2}} \frac{\partial^{|k_3|} g_z}{\partial z^{k_3}}.$$

This equality, identity (58), and the estimates of Theorem 7.2 imply that $\|f\|_{pq}$ is bounded by a quantity of order $m^q \Lambda^m \|g\|_{pq}$. This proves the corollary.

8. SEMINORMS ON THE SPACES $\mathcal{F}^{pq}$

Consider a foliated function $h \in \mathcal{F}_M^{1,N-2}$, the $C(M)$-linear functional $\nu\colon \mathcal{F}_M^{N-3,2} \to C(M)$ from Theorem 7.2, and the function $\lambda \in C^{N-4}(M)$ from Corollary 7.2.1. Take a family of straight standard leaves $\Gamma_z \sim (y(x), z, \chi, x_0)$. We normalize the function h by the condition $h(\Gamma_z, 1) \equiv 1$ and the functional ν by the condition $\nu(h) \equiv 1$. Instead of the linear operator $A_{0,n}$, we consider the operator $e^{-\lambda}A_{0,n}$. Obviously, this is equivalent to the replacement of the weight function J_0 by $J_0 e^{-\lambda/n}$. This operator satisfies the equalities $e^{-\lambda}A_{0,n}h = h$ and $\nu \circ (e^{-\lambda}A_{0,n}) = \nu$.

There exists a natural restriction operator $\pi\colon \mathcal{F}^{pq} \to \mathcal{F}_M^{pq}$ that maps every foliated function $g \in \mathcal{F}^{pq}$ to its restriction πg to the set of straight standard traces. It is seen from the definitions of the spaces $\mathcal{F}^{pq}$ and $\mathcal{F}_M^{pq}$ that $\|\pi g\|_{pq} \le \|g\|_{pq}$. Therefore, the functional $\nu \circ \pi$ induced by ν on $\mathcal{F}^{pq}$ is continuous.

Recall that a density $\Phi = \Phi(x)$ of smoothness $q \le N-1$ is said to be flowing if $\|d^i\Phi(x)/dx^i\| \le \beta_{i0}\Phi(x)$ for all x and $i = 1, \dots, q$. Hence, there exists C such that all flowing densities satisfy the inequality $\|\Phi\|_q \le C \inf \Phi$. Recall also that the constants β_{ik} in Theorem 5.1 were chosen in such a way that $\beta_{ik} \ge 2$.

Lemma 8.1. *There exists a large B_{10} such that, for any standard straight trace (Γ, Φ) with flowing density $\Phi = \Phi(x)$ of smoothness $q \in [1, N-1]$, the following inequalities hold*:

$$B_{10}^{-1}\|\Phi\|_q \le h(\Gamma, \Phi) \le B_{10} \inf \Phi. \tag{59}$$

Proof. The normalization condition $h(\Gamma_z, 1) \equiv 1$ and assertion (c) of Theorem 7.1 imply that $C_0^{-1} \le h(\Gamma, 1) \le C_0$. Since h is positive, we have $C_0^{-1} \inf \Phi \le h(\Gamma, \Phi) \le C_0 \sup \Phi$. These inequalities and $\sup \Phi \le \|\Phi\|_q \le C \inf \Phi$ imply (59).

Definition 8.1. For any foliated function $g \in \mathcal{F}^{pq}$, we define a seminorm $|g|_{pq}^h$ as the minimum number satisfying the following condition: if a standard deformation $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ of skew leaves is majorized by a pair (v, σ), where $\sigma \in [0, \sigma_0]$, and the deformation $\overline{\Gamma}_t$ of straight leaves has the form $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$, then, for any flowing density $\Phi = \Phi(x)$ of smoothness $p+1+|j|$,

$$\left| \frac{\partial^{|j|} g(\Gamma_t, \Phi)}{\partial t^j} - \frac{\partial^{|j|} g(\overline{\Gamma}_t, \Phi)}{\partial t^j} \right| \le |g|_{pq}^h \sigma v^j h(\overline{\Gamma}_t, \Phi), \qquad |j| \le q-4. \tag{60}$$

Obviously, if $p' \ge p$ and $q' \le q$, then $|g|_{p'q'}^h \le |g|_{pq}^h$.

Lemma 8.2. *Any foliated function $g \in \mathcal{F}^{pq}$ obeys the estimates*

$$B_{10}^{-1}|g|_{pq}^h \le \|g\|_{pq} \le 5B_{10}|g|_{pq}^h + \|\pi g\|_{pq}.$$

Proof. The definition of the norm $\|g\|_{pq}$ (to be more precise, estimate (26)) and Lemma 8.1 imply $|g|_{pq}^h \le B_{10}\|g\|_{pq}$. On the other hand, any density $\Phi(x)$ of smoothness $p+1+|j|$ can be represented as a difference of two flowing densities, i.e., as $\Phi = \Phi_1 - \Phi_2$, where $\Phi_1(x) = \Phi(x) + 2\|\Phi\|_{p+1+|j|}$ and $\Phi_2 = 2\|\Phi\|_{p+1+|j|}$. Therefore, (59) and (60) imply

$$\left| \frac{\partial^{|j|} g(\Gamma_t, \Phi)}{\partial t^j} - \frac{\partial^{|j|} g(\overline{\Gamma}_t, \Phi)}{\partial t^j} \right| \le |g|_{pq}^h \sigma v^j \bigl(h(\overline{\Gamma}_t, \Phi_1) + h(\overline{\Gamma}_t, \Phi_2)\bigr) \le |g|_{pq}^h \sigma v^j \cdot 5B_{10}\|\Phi\|_{p+1+|j|}. \tag{61}$$

Comparing this estimate with (26) and definition (57) of the norm $\|\pi g\|_{pq}$ with (25), we obtain $\|g\|_{pq} \le 5B_1|g|_{pq}^h + \|\pi g\|_{pq}$.

Lemma 8.3. *There exists a large* B_{11} *such that, for any deformation* $\overline{\Gamma}_t \sim (y_t(x), z_t, \chi, x_0)$ *of standard straight leaves and any family of densities* $\Phi_t(x)$*, the relation* $\overline{\Gamma}_t \prec (v, 0)$ *and the estimates*

$$\left\| \frac{\partial^{i+|k|}\Phi_t(x)}{\partial x^i \partial t^k} \right\| \le \beta_{i|k|} v^k \Phi_t(x), \qquad i + |k| \le p + |j|,$$

which hold for some fixed multiindex j*, imply the inequalities*

$$\left| \frac{\partial^{|j|} g(\overline{\Gamma}_t, \Phi_t)}{\partial t^j} \right| \le B_{11} \|\pi g\|_{pq} v^j h(\overline{\Gamma}_t, \Phi_t), \qquad |j| \le q - 2.$$

Proof. Obviously, under the conditions of the lemma, the density Φ_t is flowing. By Lemma 8.1, we have $\|\Phi_t\|_{p+|j|} \le B_{10} h(\overline{\Gamma}_t, \Phi_t)$. In estimates (37) in Lemma 4.6, we can replace $\|g\|_{pq}$ by $\|\pi g\|_{pq}$. Setting $\beta = \sup \beta_{i|k|} \|\Phi_t\|_{p+|j|}$ in these estimates and replacing $\|\Phi_t\|_{p+|j|}$ by $B_{10} h(\overline{\Gamma}_t, \Phi_t)$, we obtain the assertion of Lemma 8.3.

As above, suppose that the weight function J_0 is bounded from below by a positive constant. Then, the following theorem is valid.

Theorem 8.4. *There exist a large integer* $n \in \mathbb{N}$*, a small number* $\sigma_0 \in (0, \eta]$*, and a large constant* C *such that, for any standard leaf deformations* $\Gamma_t \sim (y_t(x), z_t(x), \chi, x_0)$ *and* $\overline{\Gamma}_t \sim (y_t(x), z_t(x_0), \chi, x_0)$*, the following assertion holds. If standard trace deformations* $(\Gamma'_t, \Phi'_t) \subset (\Sigma_0 J_0 e^{-\lambda/n})^n (\Gamma_t, \Phi)$ *and* $(\overline{\Gamma}'_t, \overline{\Phi}'_t) \subset (\Sigma_0 J_0 e^{-\lambda/n})^n (\overline{\Gamma}_t, \Phi)$ *have the forms* $\Gamma'_t \sim (y'_t(x'), z'_t(x'), \chi', x'_0)$ *and* $\overline{\Gamma}'_t \sim (\overline{y}'_t(x'), z_t(x_0), \chi', x'_0)$ *and the density* $\Phi = \Phi(x)$ *of smoothness* $p + 1 + |j|$ *is flowing, then, for any foliated function* $g \in \mathcal{F}^{pq}$*, the relation* $\Gamma_t \prec (v, \sigma)$*, where* $\sigma \in [0, \sigma_0]$*, implies the estimates*

$$\left| \frac{\partial^{|j|} g(\Gamma'_t, \Phi'_t)}{\partial t^j} - \frac{\partial^{|j|} g(\overline{\Gamma}'_t, \overline{\Phi}'_t)}{\partial t^j} \right| \le \left(\frac{1}{4} |g|^h_{pq} + C |g|^h_{p,q-1} + C \|\pi g\|_{pq} \right) \sigma v^j h(\overline{\Gamma}'_t, \overline{\Phi}'_t)$$

for all $|j| \le q - 4$*. If* $q = 4$*, it is assumed that* $|g|^h_{p,q-1} = 0$ *in this inequality.*

Proof. As n, we can take any positive integer satisfying the inequality

$$(1 - b)^n e^{cN\eta} < 1/16 \tag{62}$$

with c defined in Corollary 1.3.1. Consider the homotopy $\Gamma_{t\tau} \sim (y_t(x), \tau z_t(x) + (1-\tau) z_t(x_0), \chi, x_0)$, where $\tau \in [0, 1]$, as a deformation depending on the composite parameter (t, τ). By Lemma 4.1, the relation $\Gamma_t \prec (v, \sigma)$ implies

$$\Gamma_{t\tau} \prec \big((v, B_1\sigma), \eta\big). \tag{63}$$

Suppose that $\Gamma'_{t\tau} \sim (y'_{t\tau}(x'), z'_{t\tau}(x'), \chi', x'_0)$ is a deformation of standard leaves such that $\Gamma'_{t\tau} \subset \Sigma_0^n(\Gamma_{t\tau})$. For this deformation, consider the standard deformation $\overline{\Gamma}'_{t\tau} \sim (y'_{t\tau}(x'), z'_{t\tau}(x'_0), \chi', x'_0)$ of straight leaves and the trace deformation $(\Gamma'_{t\tau}, \Phi'_{t\tau}) \subset (\Sigma_0 J_0 e^{-\lambda/n})^n (\Gamma_{t\tau}, \Phi)$. By Corollary 1.3.1,

$$\Gamma'_{t1} \prec \big(e^{c\sigma} v, (1-b)^n \sigma\big), \tag{64}$$

$$\Gamma'_{t\tau} \prec \big((e^{c\eta} v, e^{c\eta} B_1 \sigma), (1-b)^n \eta\big), \tag{65}$$

whence

$$\overline{\Gamma}'_{t\tau} \prec \big((e^{c\eta} v, e^{c\eta} B_1 \sigma), 0\big). \tag{66}$$

Applying Corollary 5.1.2 (with $\nu = 0$), we obtain

$$\left\| \frac{\partial^{i+|k|+l} \Phi'_{t\tau}(x')}{(\partial x')^i \partial t^k \partial \tau^l} \right\| \le \beta_{i,|k|+l} (C_n v)^k (C_n B_1 \sigma)^l \Phi'_{t\tau}(x'), \qquad i + |k| + l \le p + 1 + |j|. \tag{67}$$

These inequalities with $i=0$, $k=0$, and $l=1$ yield $\Phi'_{t\tau} \le e^{\beta_{01}C_nB_1\sigma}\Phi'_{t0}$ for all $\tau\in[0,1]$. Take σ_0 so small that $e^{\beta_{01}C_nB_1\sigma}\le 2$ for all $\sigma\in[0,\sigma_0]$. Then, $\Phi'_{t\tau}\le 2\Phi'_{t0}$ and, since the foliated function h is positive, we have $h(\overline{\Gamma}'_{t\tau},\Phi'_{t\tau})\le 2h(\overline{\Gamma}'_{t\tau},\Phi'_{t0})$. Estimate (66) and Lemmas 4.2 and 8.1 imply

$$\left|\frac{\partial h(\overline{\Gamma}'_{t\tau},\Phi'_{t0})}{\partial\tau}\right| \le B_2\|h\|_{1,N-2}e^{c\eta}B_1\sigma\|\Phi'_{t0}\|_2 \le \|h\|_{1,N-2}e^{c\eta}B_1B_2\sigma B_{10}h(\overline{\Gamma}'_{t\tau},\Phi'_{t0}),$$

whence

$$h(\overline{\Gamma}'_{t\tau},\Phi'_{t\tau}) \le 2h(\overline{\Gamma}'_{t\tau},\Phi'_{t0}) \le 2\exp\bigl(\|h\|_{1,N-2}e^{c\eta}B_1B_2B_{10}\sigma\bigr)h(\overline{\Gamma}'_{t0},\Phi'_{t0}).$$

Take σ_0 so small that, for all $\sigma\in[0,\sigma_0]$, this inequality implies

$$h(\overline{\Gamma}'_{t\tau},\Phi'_{t\tau}) \le 4h(\overline{\Gamma}'_{t0},\Phi'_{t0}). \tag{68}$$

Consider the difference $g(\Gamma'_t,\Phi'_t)-g(\overline{\Gamma}'_t,\overline{\Phi}'_t)=D_1(t)+D_2(t)$, where $D_1(t)=g(\Gamma'_{t1},\Phi'_{t1})-g(\overline{\Gamma}'_{t1},\Phi'_{t1})$ and $D_2(t)=g(\overline{\Gamma}'_{t1},\Phi'_{t1})-g(\overline{\Gamma}'_{t0},\Phi'_{t0})$. Let us calculate separately the derivatives of $D_i(t)$. Differentiating by the Leibniz formula, we obtain

$$\frac{\partial^{|j|}D_1(t)}{\partial t^j} = \frac{\partial^{|j|}}{\partial t^j}\bigl(g(\Gamma'_{t1},\Phi'_{T1})-g(\overline{\Gamma}'_{t1},\Phi'_{T1})\bigr)\Big|_{T=t} + \sum_{0\ne k\le j}C_j^k\frac{\partial^{|j|}}{\partial t^{j-k}\partial T^k}\bigl(g(\Gamma'_{t1},\Phi'_{T1})-g(\overline{\Gamma}'_{t1},\Phi'_{T1})\bigr)\Big|_{T=t}.$$

The application of estimates (64), (60), and (61) to this equality yields

$$\left|\frac{\partial^{|j|}D_1(t)}{\partial t^j}\right| \le |g|^h_{pq}(1-b)^n\sigma(e^{c\sigma}v)^jh(\overline{\Gamma}'_{t1},\Phi'_{t1}) + \sum_{0\ne k\le j}C_j^k|g|^h_{p,q-|k|}(1-b)^n\sigma(e^{c\sigma}v)^{j-k}\cdot 5B_{10}\left\|\frac{\partial^{|k|}\Phi'_{t1}}{\partial t^k}\right\|_{p+1+|j-k|}.$$

Substituting (62), (67), and (59) into this inequality, we obtain

$$\left|\frac{\partial^{|j|}D_1(t)}{\partial t^j}\right| \le \left(\frac{1}{16}|g|^h_{pq}+C'|g|^h_{p,q-1}\right)\sigma v^jh(\overline{\Gamma}'_{t1},\Phi'_{t1}), \tag{69}$$

where C' is a constant independent of the traces under consideration and the foliated function g. Similarly, applying Lemma 8.3 and estimates (66) and (67), we derive an estimate of the form

$$\left|\frac{\partial^{|j|}D_2(t)}{\partial t^j}\right| \le \int_0^1\left|\frac{\partial^{|j|+1}g(\overline{\Gamma}'_{t\tau},\Phi'_{t\tau})}{\partial t^j\partial\tau}\right|d\tau \le C'\|\pi g\|_{pq}\sigma v^jh(\overline{\Gamma}'_{t\tau},\Phi'_{t\tau}). \tag{70}$$

The substitution of (68) into (69) and (70) gives

$$\left|\frac{\partial^{|j|}}{\partial t^j}\bigl(D_1(t)+D_2(t)\bigr)\right| \le \left(\frac14|g|^h_{pq}+4C'|g|^h_{p,q-1}+4C'\|\pi g\|_{pq}\right)\sigma v^jh(\overline{\Gamma}'_{t0},\Phi'_{t0}),$$

as required.

9. PROOF OF THEOREM 3.3

Theorem 9.1. *Under the conditions of Theorem 7.2, there exist a large $n\in\mathbb{N}$, a small $\sigma_0\in(0,\eta]$, and a large C such that, for any foliated function $g\in\mathcal{F}^{pq}$,*

$$|e^{-\lambda}A_{0,n}g|^h_{pq} \le \frac12|g|^h_{pq}+C|g|^h_{p,q-1}+C\|\pi g\|_{pq}.$$

For $q=4$, it is assumed that $|g|^h_{p,q-1}=0$ in this inequality.

First, we derive Theorem 3.3 from this theorem.

A. Let π be the natural projector from $\mathcal{F}^{pq}$ to $\mathcal{F}^{pq}_M$, and let $\nu\colon \mathcal{F}^{pq}_M \to C^{q-2}(M)$ be the functional specified in Theorem 7.2. Then, the composition $\nu\circ\pi$ maps continuously every space $\mathcal{F}^{pq}$ to $C^{q-2}(M)$. This proves assertion (a) of Theorem 3.3.

B. *There exist a norm* $\|\cdot\|^\delta_{pq}$ *on* $\mathcal{F}^{pq}$ *equivalent to* $\|\cdot\|_{pq}$ *and a number* $\Lambda\in(0,1)$ *such that* $\|e^{-\lambda}A_{0,n}g\|^\delta_{pq}\le\Lambda\|g\|^\delta_{pq}$ *for all foliated functions* $g\in\mathcal{F}^{pq}\cap\ker\nu\circ\pi$.

By virtue of Corollary 7.2.2, the sequence of restrictions of the operators $(e^{-\lambda}A_{0,n})^m$ to $\mathcal{F}^{pq}_M\cap\ker\nu$ converges to the zero operator in the uniform norm. Hence, there exist a norm $\|\cdot\|'_{pq}$ on $\mathcal{F}^{pq}_M$ equivalent to $\|\cdot\|_{pq}$ and a number $\Lambda'\in(0,1)$ such that $\|e^{-\lambda}A_{0,n}g\|'_{pq}\le\Lambda'\|g\|'_{pq}$ for all $g\in\mathcal{F}^{pq}_M\cap\ker\nu$.

Consider the following norm on $\mathcal{F}^{pq}$:

$$\|g\|^\delta_{pq}=\sum_{k=4}^{q}\delta^k|g|^h_{pk}+\|\pi g\|'_{pq}.$$

By Lemma 8.2, this norm is equivalent to $\|\cdot\|_{pq}$. By Theorem 9.1, for all $g\in\mathcal{F}^{pq}\cap\ker\nu\circ\pi$, we have

$$\begin{aligned}\|e^{-\lambda}A_{0,n}g\|^\delta_{pq}&\le\sum_{k=4}^{q}\delta^k\left(\frac12|g|^h_{pk}+C|g|^h_{p,k-1}+C\|\pi g\|_{pk}\right)+\Lambda'\|\pi g\|'_{pq}\\&\le\sum_{k=4}^{q}\delta^k\left(\frac12+C\delta\right)|g|^h_{pk}+Cq\delta^4\|\pi g\|_{pq}+\Lambda'\|\pi g\|'_{pq}.\end{aligned}$$

Now, it suffices to take an arbitrary $\Lambda\in(\Lambda',1)$ and δ so small that $\frac12+C\delta\le\Lambda$ and, simultaneously, $Cq\delta^4\|\cdot\|_{pq}\le(\Lambda-\Lambda')\|\cdot\|'_{pq}$.

C. *There exists a foliated function* $h\in\mathcal{F}^{1,N-2}$ *such that* $e^{-\lambda}A_{0,n}h=h$ *and* $\nu\circ\pi(h)\equiv1$.

Take an arbitrary foliated function $g\in\mathcal{F}^{1,N-2}$ for which the function $\nu\circ\pi(g)\in C^{N-4}(M)$ is bounded from below by a positive number. The foliated function $h_0=(\nu\circ\pi(g))^{-1}g$ belongs to $\mathcal{F}^{1,N-2}$, and $\nu\circ\pi(h_0)\equiv1$. Consider the sequence $h_m=(e^{-\lambda}A_{0,n})^m h_0$. By construction, we have $\nu\circ\pi\circ e^{-\lambda}A_{0,n}=\nu\circ\pi$. Therefore, $\nu\circ\pi(h_m)\equiv1$ and, hence, $h_{m+1}-h_m\in\ker\nu\circ\pi$. According to assertion B, the sequence h_m converges at an exponential rate to a certain foliated function $h\in\mathcal{F}^{1,N-2}$. This function is the required one.

D. *The sequence of operators* $(e^{-\lambda}A_{0,n})^m$ *on* $\mathcal{F}^{pq}$ *converges to the projector* $(\nu\circ\pi)\otimes h$ *in the uniform operator norm.*

This is so because any foliated function $g\in\mathcal{F}^{pq}$ decomposes into a sum as $g=\nu\circ\pi(g)h+(g-\nu\circ\pi(g)h)$, where the second term belongs to $\mathcal{F}^{pq}\cap\ker\nu\circ\pi$. This completes the proof of Theorem 3.3.

We proceed to prove Theorem 9.1. Suppose that a standard deformation $\Gamma_t\sim(y_t(x),z_t(x),\chi,x_0)$ of skew leaves is majorized by a pair (v,σ), where $\sigma\in[0,\sigma_0]$, and $\Phi=\Phi(x)$ is a flowing density of smoothness $p+1+|j|$. Consider the standard deformation $\overline{\Gamma}_t\sim(y_t(x),z_t(x_0),\chi,x_0)$ of straight leaves and the homotopy $\Gamma_{t\tau}\sim(y_t(x),\,\tau z_t(x)+(1-\tau)z_t(x_0),\,\chi,x_0)$. We will calculate the value $(e^{-\lambda}A_{0,n}g)(\Gamma_{t\tau},\Phi)$ by the same method as in Section 6. By (49), we have

$$(e^{-\lambda}A_{0,n}g)(\Gamma_{t\tau},\Phi)=\sum_{l=1}^{m}\int\limits_{\mathbb{B}^u_5}\Psi_{lt\tau}(x')g(\Gamma'_{lt\tau}(x'),\Phi'_{t\tau})\,dx',\tag{71}$$

where

$$\Psi_{lt\tau}(x')=\xi_{lt\tau}(x')\frac{\zeta_l(x')}{\bar\zeta_{lt\tau}(x')}\rho_{lt\tau}(x'),\qquad(\Gamma'_{lt\tau}(x'),\Phi'_{t\tau})\subset(\Sigma_0J_0e^{-\lambda/n})^n(\Gamma_{t\tau},\Phi).$$

Lemma 9.2. *There exists a large B_{12} independent of Γ_t and $\Gamma'_{lt\tau}(x')$ such that, for any foliated function $g \in \mathcal{F}^{pq}$ and any flowing density Φ of smoothness $p+1+|j|$, the relation $\Gamma_t \prec (v,\sigma)$ implies the estimate*

$$\left|\frac{\partial^{|j|}}{\partial t^j} g(\Gamma'_{lt0}(x'), \Phi'_{t0})\right| \le B_{12}\|\pi g\|_{pq} v^j h(\Gamma'_{lt0}(x'), \Phi'_{t0}), \qquad |j| \le q-2.$$

Proof. Obviously, $\overline{\Gamma}_t \prec (v,0)$. Hence, $\Gamma'_{lt0}(x') \prec (v,0)$ by Corollary 1.3.1. According to Corollary 5.1.2, we have

$$\left\|\frac{\partial^{i+|k|}\Phi'_{t0}(x')}{(\partial x')^i \partial t^k}\right\| \le \beta_{i|k|}(C_n v)^k \Phi'_t(x'), \qquad i+|k| \le p+1+|j|.$$

Thus, Lemma 9.2 follows from Lemma 8.3.

Let us represent the difference

$$d_l(t,x') = \Psi_{lt1}(x') g(\Gamma'_{lt1}(x'), \Phi'_{t1}) - \Psi_{lt0}(x') g(\Gamma'_{lt0}(x'), \Phi'_{t0}) \tag{72}$$

as a sum of three terms: $d_l(t,x') = d_{l1}(t,x') + d_{l2}(t,x') + d_{l3}(t,x')$, where

$$d_{l1}(t,x') = \Psi_{lt0}(x')\big(g(\Gamma'_{lt1}(x'), \Phi'_{t1}) - g(\Gamma'_{lt0}(x'), \Phi'_{t0})\big),$$
$$d_{l2}(t,x') = \big(\Psi_{lt1}(x') - \Psi_{lt0}(x')\big)\big(g(\Gamma'_{lt1}(x'), \Phi'_{t1}) - g(\Gamma'_{lt0}(x'), \Phi'_{t0})\big),$$
$$d_{l3}(t,x') = \big(\Psi_{lt1}(x') - \Psi_{lt0}(x')\big) g(\Gamma'_{lt0}(x'), \Phi'_{t0}).$$

By construction, $d_i(t,x') = 0$ for $x' \notin \mathbb{B}^u_3$. Let us differentiate the terms separately:

$$\frac{\partial^{|j|} d_{l1}(t,x')}{\partial t^j} = \Psi_{lt0}(x')\frac{\partial^{|j|}}{\partial t^j}\big(g(\Gamma'_{lt1}(x'), \Phi'_{t1}) - g(\Gamma'_{lt0}(x'), \Phi'_{t0})\big)$$
$$+ \sum_{0\ne k\le j} C^k_j \frac{\partial^{|k|}\Psi_{lt0}(x')}{\partial t^k}\frac{\partial^{|j-k|}}{\partial t^{j-k}}\big(g(\Gamma'_{lt1}(x'), \Phi'_{t1}) - g(\Gamma'_{lt0}(x'), \Phi'_{t0})\big),$$

$$\frac{\partial^{|j|} d_{l2}(t,x')}{\partial t^j} = \sum_{k\le j} C^k_j \frac{\partial^{|k|}}{\partial t^k}\big(\Psi_{lt1}(x') - \Psi_{lt0}(x')\big)\frac{\partial^{|j-k|}}{\partial t^{j-k}}\big(g(\Gamma'_{lt1}(x'), \Phi'_{t1}) - g(\Gamma'_{lt0}(x'), \Phi'_{t0})\big),$$

$$\frac{\partial^{|j|} d_{l3}(t,x')}{\partial t^j} = \sum_{k\le j} C^k_j \frac{\partial^{|k|}}{\partial t^k}\big(\Psi_{lt1}(x') - \Psi_{lt0}(x')\big)\frac{\partial^{|j-k|}}{\partial t^{j-k}} g(\Gamma'_{lt0}(x'), \Phi'_{t0})\big).$$

Recall that $|g|^h_{pq} \ge |g|^h_{pq'}$ for $q \ge q'$. Applying Theorem 8.4, Lemma 6.3 (with $m = 0$), and Lemma 9.2 to these three equalities, we obtain estimates of the form

$$\left|\frac{\partial^{|j|} d_{l1}(t,x')}{\partial t^j}\right| \le \Psi_{lt0}(x')\left(\frac{1}{4}|g|^h_{pq} + C|g|^h_{p,q-1} + C\|\pi g\|_{pq}\right)\sigma v^j h(\Gamma'_{lt0}(x'), \Phi'_{t0})$$
$$+ \sum_{0\ne k\le j} C^k_j B_9 v^k (C+1)\left(|g|^h_{p,q-|k|} + \|\pi g\|_{p,q-|k|}\right)\sigma v^{j-k} h(\Gamma'_{lt0}(x'), \Phi'_{t0}),$$

$$\left|\frac{\partial^{|j|} d_{l2}(t,x')}{\partial t^j}\right| \le \sum_{k\le j} C^k_j B_9 v^j \sigma (C+1)\left(|g|^h_{p,q-|k|} + \|\pi g\|_{p,q-|k|}\right)\sigma v^{j-k} h(\Gamma'_{lt0}(x'), \Phi'_{t0}),$$

$$\left|\frac{\partial^{|j|} d_{l3}(t,x')}{\partial t^j}\right| \le \sum_{k\le j} C^k_j B_9 v^k \sigma B_{12}\|\pi g\|_{p,q-|k|} v^{j-k} h(\Gamma'_{lt0}(x'), \Phi'_{t0}).$$

Summing them, we obtain

$$\left|\frac{\partial^{|j|} d_l(t,x')}{\partial t^j}\right| \le \Psi_{lt0}(x')\left(\frac{1}{4}|g|^h_{pq} + C\|\pi g\|_{pq}\right)\sigma v^j h(\Gamma'_{lt0}(x'),\Phi'_{t0}) + C'\left(|g|^h_{p,q-1} + \sigma|g|^h_{pq} + \|\pi g\|_{pq}\right)\sigma v^j h(\Gamma'_{lt0}(x'),\Phi'_{t0}) \tag{73}$$

for a sufficiently large C'.

Formula (71) implies that

$$\sum_{l=1}^{m}\int\limits_{\mathbb{B}^u_5}\Psi_{lt0}(x')h(\Gamma'_{lt0}(x'),\Phi'_{t0})\,dx' = (e^{-\lambda}A_{0,n}h)(\overline{\Gamma}_t,\Phi) = h(\overline{\Gamma}_t,\Phi).$$

The functions $\Psi_{lt0}(x')$ are nonnegative; for some l and x', they are positive and bounded away from zero. Hence, there exists a constant C'' independent of the choice of the leaves Γ_t and $\Gamma'_{lt\tau}(x')$ and of the flowing density Φ, such that

$$\sum_{l=1}^{m}\int\limits_{\mathbb{B}^u_3} h(\Gamma'_{lt0}(x'),\Phi'_{t0})\,dx' \le C''\sum_{l=1}^{m}\int\limits_{\mathbb{B}^u_5}\Psi_{lt0}(x')h(\Gamma'_{lt0}(x'),\Phi'_{t0})\,dx' = C''h(\overline{\Gamma}_t,\Phi).$$

Finally, (71)–(73) imply the estimate

$$\left|\frac{\partial^{|j|}}{\partial t^j}(e^{-\lambda}A_{0,n}g)(\Gamma_t,\Phi) - \frac{\partial^{|j|}}{\partial t^j}(e^{-\lambda}A_{0,n}g)(\overline{\Gamma}_t,\Phi)\right| \le \sum_{l=1}^{m}\int\limits_{\mathbb{B}^u_5}\left|\frac{\partial^{|j|} d_l(t,x')}{\partial t^j}\right|dx'$$

$$\le \left(\frac{1}{4}|g|^h_{pq} + C\|\pi g\|_{pq}\right)\sigma v^j h(\overline{\Gamma}_t,\Phi) + C'C''\big(|g|^h_{p,q-1} + \sigma|g|^h_{pq} + \|\pi g\|_{pq}\big)\sigma v^j h(\overline{\Gamma}_t,\Phi).$$

Taking $\sigma\in[0,\sigma_0]$ with a sufficiently small σ_0, we obtain the assertion of Theorem 9.1.

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