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PARTIAL DIFFERENTIAL EQUATIONS

Cauchy Problems for Quasi-Hyperbolic Factorized Even-Order Differential Equations with Smooth Operator Coefficients Having Variable Domains

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Cauchy problems for quasi-hyperbolic factorized operator-differential equations of higher even orders with constant domains were considered in [1]. The Cauchy problem for hyperbolic operator-differential equations with variable domains was investigated in [2, 3] for second-order equations. The present paper deals with the proof of the well-posed solvability in the strong sense of Cauchy problems for some quasi-hyperbolic factorized operator-differential equations of higher even orders with unbounded operator coefficients whose domains vary; mixed problems for some hyperbolic partial differential equations with coefficients in the boundary conditions smoothly depending on time can be reduced to such problems. For the proof, we use modifications and generalizations of the functional method of energy inequalities in [1]. Unlike [1], in the present paper, the derivation of a priori estimates with the use of abstract smoothing operators is generalized to the case of variable domains of variable unbounded operator coefficients; the proof of the solvability by induction, decomposition of operators into operator factors, and the use of the Lemmas 5 and 6 below is a new technique; and a formula for their strong solutions is derived for the first time [see formula (25) below]. This formula generalizes a similar formula for smooth (classical) solutions and shows that, by analogy with smooth solutions, strong solutions of these Cauchy problems can be found in a recursive way on the basis of operator factors. In addition, unlike [1], in the present paper, we do not repeat any steps of the proofs in [4]. In conclusion, we consider an example of new well-posed mixed problems for hyperbolic factorized partial differential equations of even order with coefficients in the boundary conditions depending on t and smooth with respect to t.

1. STATEMENT OF THE CAUCHY PROBLEMS

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. On a bounded interval $]0,T[$, we consider the differential equations

$$\mathscr{L}_m(t)u \equiv \left(d^2/dt^2 + A_m(t)\right)\cdots\left(d^2/dt^2 + A_1(t)\right)u = f, \qquad t \in\,]0,T[, \tag{1}$$

with the initial conditions

$$l_j u \equiv \left(d^j u/dt^j\right)\big|_{t=0} = \varphi_j \in H, \qquad j = 0,\ldots,2m-1, \qquad m = 1,2,\ldots, \tag{2}$$

where u and f are functions of the variable t ranging in H and $A_k(t)$, $t \in [0,T]$, are positive self-adjoint operators in H with domains $D\left(A_k(t)\right)$, $k = 1,\ldots,m$, depending on t.

We assume that all operators $A_k(t)$ satisfy the following conditions.

I. For each $t \in [0,T]$, the operators $A_k(t)$ are the restrictions [to $D\left(A_k(t)\right)$] of some linear unbounded operators $\tilde{A}_k(t)$ in H with domains $D\left(\tilde{A}_k\right)$ independent of t such that $D\left(A_k(t)\right) \subset D\left(\tilde{A}_k\right)$ and $\tilde{A}_k(t)u = A_k(t)u$ for all $u \in D\left(A_k(t)\right)$, $t \in [0,T]$, $k = 1,\ldots,m$.

II. For each $t \in [0,T]$, the inverse operators $A_k^{-1}(t) \in \mathscr{B}([0,T], \mathfrak{L}(H))$, $k = 1, \ldots, m$, have strong derivatives $dA_k^{-1}(t)/dt \in \mathscr{B}([0,T], \mathfrak{L}(H))$ with respect to t [5, p. 22] in H, and these derivatives satisfy the inequalities

$$-\left(\left(dA_k^{-1}(t)/dt\right) g, g\right) \le c_k^{(1)} \left(A_k^{-1}(t) g, g\right) \qquad \forall g \in H, \qquad k = 1, \ldots, m, \tag{3}$$

where $\mathscr{B}([0,T], E)$ is the set of all functions of $t \in [0,T]$ bounded in the norm of a Banach space E and the constants $c_k^{(1)} \ge 0$ are independent of g and t.

III. For all $t \in [0,T]$, the operators $dA_k^{-1}(t)/dt$, $k = 1, \ldots, m$, have strong derivatives

$$d^2 A_k^{-1}(t)/dt^2 \in \mathscr{B}([0,T], \mathfrak{L}(H)) \quad \text{in} \quad H,$$

which satisfy the inequalities

$$\left|\left(\left(d^2 A_k^{-1}(t)/dt^2\right) g, v\right)\right| \le c_k^{(2)} |g| \left|A_k^{-1/2}(t) v\right| \qquad \forall g, v \in H, \qquad k = 1, \ldots, m, \tag{4}$$

where $A_k^{-1/2}(t)$ are the inverses of the square roots $A_k^{1/2}(t)$ of $A_k(t)$ and $c_k^{(2)} \ge 0$ are constants independent of g, v, and t.

IV. For each $t \in [0,T]$ and for all operators $A_k(t)$, $k = 1, \ldots, m$, the norms

$$|A_s(t)u| \sim |A_k(t)u| \sim |A_s(t)u - A_k(t)u|$$
$$\forall u \in D\left(A_k(t)\right), \qquad t \in [0,T], \qquad 1 \le s \ne k \le m,$$

are equivalent, and the domains $D\left(A_k^m(t)\right)$ of their powers $A_k^m(t)$, $t \in [0,T]$, are dense in H.

Following [1] and using induction over i, one can show that the norms

$$\left|A_s^i(t)u\right| \sim \left|A_k^i(t)u\right| \quad \forall u \in D\left(A_k^m(t)\right), \quad t \in [0,T], \quad i = 1, \ldots, m, \quad 1 \le s, k \le m,$$

are equivalent. Hence it follows that $|A_m(t) \cdots A_1(t)u| \sim |A_1^m(t)u|$ for all $u \in D\left(A_1^m(t)\right)$, $t \in [0,T]$. By equipping the domains $D\left(A^{\alpha/2m}(t)\right)$ of positive fractional orders $A^{\alpha/2m}(t)$ of the self-adjoint operators $A(t) = A_1^m(t)$ in H with the norms $|v|_{\alpha,t} = |A_1^{\alpha/2}(t)v|$ for each $t \in [0,T]$, we obtain Hilbert spaces $W^\alpha(t)$, $t \in [0,T]$, $\alpha \le 2m$, $W^0(t) = H$. Obviously, we have the continuous dense embeddings $W^\beta(t) \subset W^\alpha(t)$, $t \in [0,T]$, provided that $\beta > \alpha$. It follows from conditions I and IV that

$$|A_s(t)u - A_k(t)u|_{\alpha,t} \ge c_{s,k} |u|_{\alpha+2,t} \quad \forall u \in W^{\alpha+2}(t), \quad \alpha \le 2m-2, \quad 1 \le s < k \le m, \tag{5}$$

for all $t \in [0,T]$, where $c_{s,k} > 0$ are constants independent of u and t. These inequalities can first be proved for even integer α just as above and then generalized to the remaining values of α with the use of the Heinz inequality [5, pp. 177–179].

V. There exist Banach spaces V^{2i}, $i = 0, \ldots, m$, independent of t and such that $V^0 = H$, $D\left(\tilde{A}_k\right) \subset V^2$, the spaces V^{2j} are continuously embedded in the spaces V^{2i} for $j > i$, the spaces $W^{2i}(t)$ are continuously embedded in the spaces V^{2i}, $t \in [0,T]$, $i = 0, \ldots, m$, and in H, there exist strong t-derivatives [5, p. 218]

$$d^i \tilde{A}_k(t)/dt^i \in \mathscr{B}\left([0,T], \mathfrak{L}\left(V^{2[j/2]+2}, V^{2[j/2]}\right)\right),$$
$$j = 0, \ldots, 2m-2-i, \qquad i = 0, \ldots, 2m-2, \qquad k = 1, \ldots, m,$$

where $[\cdot]$ is the integer part of a number.

VI. If $t \in [0,T]$, then all operators $A_k(t)$, $k = 1, \ldots, m$, satisfy the inequalities

$$\begin{aligned} &|A_s(t)A_k(t)u - A_k(t)A_s(t)u|_{\alpha,t} \le \tilde{c}_{s,k} |u|_{\alpha+3,t} \\ &\forall u \in W^{\alpha+4}(t), \qquad \alpha \le 2m-4, \qquad 1 \le s \ne k \le m, \end{aligned} \tag{6}$$

where $\tilde{c}_{s,k} \ge 0$ are constants independent of u and t.

Condition I is a new (compared with [6, pp. 150–158; 7]) expression of the same specific feature of some differential operators $A_k(t)$: they usually consist of differential expressions $\tilde{A}_k(t)$ and boundary conditions, each of which can have specific independence. In applications, the role of such operators $\tilde{A}_k(t)$ can be played by some elliptic differential operators without boundary conditions, and the role of their restrictions $A_k(t)$ can be played by the same elliptic differential operators but with some t-dependent boundary conditions. Additional conditions imposed on the operators $A_k(t)$ will be stipulated in the statements of lemmas and theorems.

2. AUXILIARY ASSERTION

Throughout the following, in the derivation of a priori estimates for strong solutions and in the proof of the solvability of Cauchy problems, we need the interpolation inequalities (8) (see below) in the positive Hilbert scale of the spaces $\{W^\alpha(t)\}$, $t \in [0,T]$, $\alpha = 0,\dots,2m$, generated by self-adjoint operators with variable domains.

Lemma 1. *Suppose that linear positive self-adjoint operators $A_1(t)$, $t \in [0,T]$, in a Hilbert space H with t-dependent domains $D(A_1(t))$ have inverses $A_1^{-1}(t) \in \mathscr{B}([0,T],\mathfrak{L}(H))$ for which the strong derivative $dA_1^{-1}(t)/dt \in \mathscr{B}([0,T],\mathfrak{L}(H))$ exists in H for all $t \in [0,T]$. If the first strong derivative of the inverse operators $A^{-1}(t) = A_1^{-m}(t)$ of the operators $A(t) = A_1^m(t)$ satisfies the relation*

$$dA^{-1}(t)/dt \in \mathscr{B}\left([0,T],\mathfrak{L}\left(H, W^{2m-1}(t)\right)\right) \tag{7}$$

in H for all $t \in [0,T]$, then the inequalities

$$\left|\frac{d^i u}{dt^i}\right|^2_{m_i,t}\Bigg|_{t=\tau} \le c_1 \int\limits_0^\tau \left|\frac{d^{i+1} u}{dt^{i+1}}\right|^2_{m_{i+1}+1,t} dt + c_1\left(1+2\mathscr{M}_{m_i/(2m)}\right)\int\limits_0^\tau \left|\frac{d^i u}{dt^i}\right|^2_{m_i+1,t} dt$$
$$+ \left|\frac{d^i u}{dt^i}\right|^2_{m_i,t}\Bigg|_{t=0}, \qquad m_i = 2m-2-i, \qquad i = 0,\dots,2m-2, \tag{8}$$

are valid for all $u \in E^m$ (the spaces E^m will be defined in Section 3) and for all $\tau \in [0,T]$, where the constants c_1 and $\mathscr{M}_\gamma$ independent of u and t will be specified below.

Proof. For each $t \in [0,T]$, the operators $\mathscr{A}_\varepsilon(t) = A(t)A_\varepsilon^{-1}(t) = \varepsilon^{-1}\left[I - A_\varepsilon^{-1}(t)\right]$, $\varepsilon > 0$, where $A_\varepsilon^{-1}(t) = (I+\varepsilon A(t))^{-1}$, are bounded, self-adjoint, and positive in H. One can directly show that, for all $t \in [0,T]$, they have the strong derivative

$$d\mathscr{A}_\varepsilon(t)/dt = -\mathscr{A}_\varepsilon(t)\left(dA^{-1}(t)/dt\right)\mathscr{A}_\varepsilon(t) \in \mathscr{B}([0,T],\mathfrak{L}(H))$$

in H satisfying the inequalities

$$\begin{aligned}\left\|\mathscr{A}_\varepsilon^{-\beta}(t)\left(d\mathscr{A}_\varepsilon(t)/dt\right)\mathscr{A}_\varepsilon^{-1}(t)\right\|_{\mathfrak{L}(H)} &= \left\|\mathscr{A}_\varepsilon^{1-\beta}(t)\left(dA^{-1}(t)/dt\right)\right\|_{\mathfrak{L}(H)} \\ &= \left\|A_\varepsilon^{-(1-\beta)}(t)A^{1-\beta}(t)\left(dA^{-1}(t)/dt\right)\right\|_{\mathfrak{L}(H)} \le \mathscr{M},\end{aligned}$$

where $\mathscr{M} = \sup_{0<t<T}\left\|A^{1-\beta}(t)\left(dA^{-1}(t)/dt\right)\right\|_{\mathfrak{L}(H)}$ is a constant and $\beta = 1/(2m)$. Here, for the operators $A_\varepsilon^{-1}(t)$, we have used the estimates $\|A_\varepsilon^{-\varrho}(t)\|_{\mathfrak{L}(H)} \le 1$, $t \in [0,T]$, $\varepsilon > 0$, $0 < \varrho \le 1$, with $\varrho = 1-\beta$. In $H_1 = H$, the operators $\mathscr{A} = \mathscr{B} = \mathscr{A}_\varepsilon^{\beta-1}(t)$ and $\mathscr{T} = \mathscr{A}_\varepsilon^{-\beta}(t)\left(d\mathscr{A}_\varepsilon(t)/dt\right)\mathscr{A}_\varepsilon^{-1}(t)$ satisfy Remark 7.1 in [5, p. 179] for all $t \in [0,T]$ and, in particular, the inequality

$$|\mathscr{B}\mathscr{T}x| = \left|\left(dA^{-1}(t)/dt\right)A^{1-\beta}(t)A_\varepsilon^{-(1-\beta)}(t)\mathscr{A}_\varepsilon^{\beta-1}(t)x\right| \le \mathscr{M}|\mathscr{A}x| \qquad \forall x \in H,$$

since

$$\left\|\overline{\left(dA^{-1}(t)/dt\right)A^{1-\beta}(t)}\right\|_{\mathfrak{L}(H)} = \left\|A^{1-\beta}(t)\left(dA^{-1}(t)/dt\right)\right\|_{\mathfrak{L}(H)}, \qquad t \in [0,T],$$

where the bar stands for the closure of operators by continuity in H [5, p. 228]. By applying the Heinz inequality (7.6) in [5, pp. 177–178] to them, we obtain the inequality

$$\left|\mathscr{A}_\varepsilon^{-\beta-\alpha}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1+\alpha}(t)x\right| \le \mathscr{M}|x| \quad \forall x \in H, \quad \alpha = 0, \dots, 1-\beta, \tag{9}$$

for all $t \in [0, T]$.

By differentiating the integral representation of positive fractional powers of the operators $\mathscr{A}_\varepsilon^{\gamma}(t)$ [5, p. 140] and by using the representation of the resolvents

$$R_\varepsilon(-s) = (1+\varepsilon s)^{-1}(I + \varepsilon A(t))R(-s/(1+\varepsilon s))$$

via the resolvent $R(-r) = (A(t)+r)^{-1}$, the inclusions (7), and the estimates

$$\left\|A^\beta(t)R(-r)\right\|_{\mathscr{L}(H)} \le N_\beta/(1+r)^{1-\beta}, \qquad r > 0, \qquad 0 \le \beta < 1,$$

where N_β are constants known from operator calculus, we obtain the following integral representation of the derivative of these fractional powers for all $t \in [0, T]$:

$$\frac{d\mathscr{A}_\varepsilon^{\gamma}(t)}{dt}x = \frac{\sin\pi\gamma}{\pi}\int_0^{+\infty} s^\gamma R_\varepsilon(-s)\frac{d\mathscr{A}_\varepsilon(t)}{dt}R_\varepsilon(-s)x\,ds \quad \forall x \in W^{2m}(t), \quad 0 < \gamma < 1-\beta, \tag{10}$$

where $R_\varepsilon(-s) = (\mathscr{A}_\varepsilon(t)+s)^{-1}$ and $\varepsilon > 0$. Indeed, by taking into account these representation and estimates, for all $t \in [0, T]$, we obtain the estimates

$$\begin{aligned}
\left|\frac{d\mathscr{A}_\varepsilon^{\gamma}(t)}{dt}x\right| &\le \frac{1}{\pi}\int_0^{+\infty}\frac{s^\gamma}{(1+\varepsilon s)^2}\left\|A^\beta(t)R\left(\frac{-s}{1+\varepsilon s}\right)\right\|\left\|A^{1-\beta}(t)\frac{dA^{-1}(t)}{dt}\right\|\left\|R\left(\frac{-s}{1+\varepsilon s}\right)\right\|ds|A(t)x| \\
&\le \frac{1}{\pi}N_\beta\mathscr{M}N_0\int_0^{+\infty}\frac{s^\gamma}{(1+\varepsilon s)^\beta(1+s+\varepsilon s)^{2-\beta}}ds\,|A(t)x| \\
&\le \frac{1}{\pi}N_\beta\mathscr{M}N_0\int_0^{+\infty}\frac{s^\gamma}{(1+s)^{2-\beta}}ds\,|A(t)x| < +\infty \quad \forall x \in W^{2m}(t)
\end{aligned}$$

uniformly bounded for all $\varepsilon > 0$ if $0 < \gamma < 1-\beta$ and $\beta = 1/(2m)$.

If $Q = \mathscr{A}_\varepsilon^{-\beta}(t)R_\varepsilon(-s)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-\gamma}(t)R_\varepsilon(-s)$ and $x, y \in W^{2m}(t)$, then

$$\begin{aligned}
|(Qx,y)| &= \Big|\Big(\mathscr{A}_\varepsilon^{-\beta-(1-\gamma)/2}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1+(1-\gamma)/2}(t)\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)x, \\
&\qquad \mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\Big)\Big| \\
&\le \left\|\mathscr{A}_\varepsilon^{-\beta-(1-\gamma)/2}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1+(1-\gamma)/2}(t)\right\|_{\mathfrak{L}(H)} \\
&\quad \times \left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)x\right|\,\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\right|.
\end{aligned}$$

By using the integral representation (10) and the estimates (9) for $\gamma \ge 2\beta - 1$, we obtain the inequalities

$$\begin{aligned}
&\left|\left(\mathscr{A}_\varepsilon^{-\beta}(t)\frac{d\mathscr{A}_\varepsilon^{\gamma}(t)}{dt}\mathscr{A}_\varepsilon^{-\gamma}(t)x, y\right)\right| \\
&\quad \le \frac{\sin\pi\gamma}{\pi}\mathscr{M}\int_0^{+\infty}s^\gamma\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)x\right|\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\right|ds \\
&\quad \le \frac{1}{\pi}\mathscr{M}\left(\int_0^{+\infty}s^\gamma\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)x\right|^2 ds\right)^{1/2}\left(\int_0^{+\infty}s^\gamma\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\right|^2 ds\right)^{1/2}.
\end{aligned}$$

For each $\varepsilon > 0$, for the positive definite operators

$$(\mathscr{A}_\varepsilon(t)h, h) \geq c_\varepsilon |h|^2, \qquad h \in H, \qquad c_\varepsilon = \left(\sup_{0<t<T} \left\| A^{-1}(t) \right\|_{\mathfrak{L}(H)} + \varepsilon \right)^{-1},$$

there exists a unique resolution of identity $E_\lambda(\varepsilon)$ such that

$$\int_0^{+\infty} s^\gamma \left| \mathscr{A}_\varepsilon^{(1-\gamma)/2}(t) R_\varepsilon(-s)x \right|^2 ds = \int_0^{+\infty} s^\gamma \int_{c_\varepsilon}^{+\infty} \lambda^{1-\gamma} \frac{1}{(\lambda+s)^2} d\left(E_\lambda(\varepsilon)x, x\right) ds$$

$$= \int_{c_\varepsilon}^{+\infty} \lambda^{1-\gamma} \int_0^{+\infty} \frac{s^\gamma}{(\lambda+s)^2} ds\, d\left(E_\lambda(\varepsilon)x, x\right) = \int_{c_\varepsilon}^{+\infty} \int_0^{+\infty} \frac{(s/\lambda)^\gamma}{(1+s/\lambda)^2} d(s/\lambda) d\left(E_\lambda(\varepsilon)x, x\right)$$

$$= \int_0^{+\infty} \frac{\sigma^\gamma}{(1+\sigma)^2} d\sigma \int_{c_\varepsilon}^{+\infty} d\left(E_\lambda(\varepsilon)x, x\right) = \int_0^{+\infty} \frac{\sigma^\gamma}{(1+\sigma)^2} d\sigma |x|^2 \qquad \forall t \in [0,T].$$

It follows that

$$\left| \mathscr{A}_\varepsilon^{-\beta}(t) \left(d\mathscr{A}_\varepsilon^{\gamma}(t)/dt\right) \mathscr{A}_\varepsilon^{-\gamma}(t)x \right| \leq \mathscr{M}_\gamma |x| \qquad \forall x \in H, \qquad \gamma = \beta, \dots, 1-\beta, \tag{11}$$

for all $t \in [0,T]$, where $\mathscr{M}_\gamma = \pi^{-1}\mathscr{M} \int_0^{+\infty} \sigma^\gamma (1+\sigma)^{-2} d\sigma < +\infty$, if the elements $x \in H$ in (11) are approximated by some sequences $x_n \in W^{2m}(t)$ for each $t \in [0,T]$.

By using the Schwartz and Cauchy–Schwarz inequalities, the estimates (11) and the δ-inequality $2ab \leq \delta a^2 + \delta^{-1} b^2$ for all $\delta > 0$ on the right-hand side of the obvious identities

$$\left| \mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^i u}{dt^i} \right|^2 \Bigg|_{t=\tau} = 2\,\mathrm{Re} \int_0^\tau \left(\mathscr{A}_\varepsilon^{-1/(2m)}(t) \frac{d\mathscr{A}_\varepsilon^{m_i/(2m)}(t)}{dt} \frac{d^i u}{dt^i}, \mathscr{A}_\varepsilon^{(m_i+1)/(2m)}(t) \frac{d^i u}{dt^i} \right) dt$$

$$+ 2\,\mathrm{Re} \int_0^\tau \left(\mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^{i+1} u}{dt^{i+1}}, \mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^i u}{dt^i} \right) dt$$

$$+ \left| \mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^i u}{dt^i} \right|^2 \Bigg|_{t=0} \qquad \forall u \in D\left(L_m\right),$$

we obtain

$$\left| \mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^i u}{dt^i} \right|^2 \Bigg|_{t=\tau} \leq \left(c_1 + \varepsilon^{1/(2m)}\right) \int_0^\tau \left| \mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^{i+1} u}{dt^{i+1}} \right|^2 dt$$

$$+ \left(c_1 + \varepsilon^{1/(2m)}\right) \left(1 + 2\mathscr{M}_{m_i/(2m)}\right) \int_0^\tau \left| \mathscr{A}_\varepsilon^{(m_i+1)/(2m)}(t) \frac{d^i u}{dt^i} \right|^2 dt$$

$$+ \left| \mathscr{A}_\varepsilon^{m_i/(2m)}(t) \frac{d^i u}{dt^i} \right|^2 \Bigg|_{t=0}, \qquad i = 0, \dots, 2m-2,$$

for all $\tau \in]0,T]$, where the constant $c_1 = \sup_{0<t<T} \left\| A_1^{-1}(t) \right\|_{\mathfrak{L}(H)}^{1/2}$ is also independent of ε. By letting ε in the last inequality tend to zero and by using the property $|\mathscr{A}_\varepsilon^{\alpha/(2m)}(t)v - A_1^{\alpha/2}(t)v| \to 0$ for all $v \in W^\alpha(t)$, $t \in [0,T]$, $\alpha \leq 2m$, as $\varepsilon \to 0$, we obtain the estimate (8) for arbitrary functions u in the sets $D\left(L_m\right)$ defined in Section 3. Then the estimate (8) for all $u \in D\left(L_m\right)$ can be generalized to all $u \in E^m$ by passage to the limit.

3. DEFINITION OF STRONG SOLUTIONS OF CAUCHY PROBLEMS

By $\mathscr{H}^\alpha$ we denote the Hilbert spaces $L_2\left(]0,T[,W^\alpha(t)\right)$ with Hermitian norms $\|\cdot\|_\alpha$, $\alpha\le 2m$, $\mathscr{H}^0=\mathscr{H}$. The space $\mathscr{H}^\alpha$ is the set of all measurable functions $u:[0,T]\ni t\to u(t)\in H$ such that $u(t)\in D\left(A^{\alpha/2m}(t)\right)$, $t\in[0,T]$, and $h(t)=A^{\alpha/2m}(t)u(t)\in\mathscr{H}=L_2(]0,T[,H)$.

Let the Hilbert spaces $\mathscr{H}^{p,q}$ be the sets of all functions $u\in\mathscr{H}$ with finite Hermitian norms

$$\|u\|_{p,q}=\left(\sum_{i=0}^{p}\left\|d^iu/dt^i\right\|_{q-i}^2\right)^{1/2};$$

let the Banach spaces $\mathscr{E}^{p,q}$ be the sets of all functions $u\in B([0,T],H)$ with finite norms

$$|||u|||_{p,q}=\left(\sup_{0<t<T}\sum_{i=0}^{p}\left|d^iu(t)/dt^i\right|_{q-i,t}^2\right)^{1/2};$$

and let the Banach space $\mathscr{B}([0,T],H)$ be the set of all bounded functions of $t\in[0,T]$ ranging in H, equipped with the uniform convergence norm $\|\cdot\|_{\mathscr{B}}=\sup_{0<t<T}|\cdot|$. In the definition of the spaces $\mathscr{H}^{p,q}$ and $\mathscr{E}^{p,q}$, the derivative du/dt is treated as a function $du/dt\in W^{q-1}(t)$ such that

$$|\Delta u(t)/\Delta t-du(t)/dt|_{q-1,t}=\left|A_1^{(q-1)/2}(t)[\Delta u(t)/\Delta t-du(t)/dt]\right|\to 0$$

as $\Delta t\to 0$ for all $t\in[0,T]$. Higher derivatives $d^iu/dt^i\in W^{q-i}(t)$ are defined recursively in a similar way. In this definition of the derivative du/dt, we assume that $u(t)\in W^q(t)$ satisfies the inclusions $u(t+\tau)\in W^{q-1}(t)$ for all sufficiently small $\tau>0$, which are not necessarily valid for any operator $A_1(t)$ with variable domain $D\left(A_1(t)\right)$. In the case of variable domains $D\left(A_1(t)\right)$, sufficient conditions for the well-posedness of this definition and the existence of all derivatives in the below-introduced spaces $\mathscr{H}^{2m,2m}$ and $\mathscr{E}^{2m-1,2m-1}$ will be given in Lemma 1. The continuous embeddings [1] $\mathscr{H}^{p,q}\subset\mathscr{E}^{p-1,q-1}$ are usually valid for constant domains of the operators $A_1(t)$. These embeddings are not necessarily valid for variable domains of the operators $A_1(t)$.

Let $\mathscr{H}^{2m,2m}$ and $\mathscr{E}^{2m-1,2m-1}$ be the closures of the above-defined sets $D\left(L_m\right)$, whose definition contains the requirement of the corresponding smoothness of the operators $\tilde{A}_k(t)$ with respect to t, in the norms $\|u\|_{2m,2m}$ and $|||u|||_{2m-1,2m-1}$, respectively. The following assertion describes the case of variable domains $D\left(A_1(t)\right)$ for which the continuous embeddings $\mathscr{H}^{2m,2m}\subset\mathscr{E}^{2m-1,2m-1}$ are valid.

Assertion 1. *If the assumptions of Lemma* 1 *are valid, then*

$$\left|\frac{d^iu(t)}{dt^i}\right|_{m_i+1,t}^2\le c_1\left\|\frac{d^{i+1}u}{dt^{i+1}}\right\|_{m_1+1}^2+c_1\left(1+2\mathscr{M}_{(m_i+1)/(2m)}+T^{-1}c_1\right)\left\|\frac{d^iu}{dt^i}\right\|_{2m-i}^2,\quad i=0,\dots,2m-1,$$

for all $u\in\mathscr{H}^{2m,2m}$ *and all* $t\in[0,T]$.

Proof. The proof of the assertion is similar to that of Lemma 1 with the only difference: in the integration with respect to t from 0 to τ (see the end of the proof of Lemma 1), one should take the variable lower integration limit $s<\tau$ instead of the lower integration limit $t=0$ and then perform the estimate with the use of additional integration with respect to s from 0 to T.

However, condition (7) is quite restrictive for the differential operators $A_1(t)$ for which the dependence of the domains on t is caused by the dependence of the coefficients in the boundary conditions on t. Examples of operators $A_1(t)$ with variable domains $D\left(A_1(t)\right)$ that satisfy or do not satisfy this condition will be given in Section 6. Therefore, in the following, we simply assume where necessary (see Remark 1) that $D\left(L_m\right)\subset\mathscr{E}^{2m-1,2m-1}$, i.e.,

$$A_1^{(m_i+1)/2}(t)\left(d^iu/dt^i\right)\in B([0,T],H),\qquad 0\le i\le 2m-1,\qquad\forall u\in D\left(L_m\right).\tag{12}$$

As spaces of strong solutions of the Cauchy problems (1), (2), we take the Banach spaces E^m that are the closures of the sets

$$D(L_m) = \left\{ u \in \tilde{D}(L_m) : \; d^s u/dt^s \in \mathscr{H}^{2m-2[(s+1)/2]}, \; s = 0, \ldots, 2m-1 \right\},$$

where

$$\tilde{D}(L_m) = \left\{ u \in \mathscr{H} : \; \frac{d^s u}{dt^s} \in L_2\left(]0, T[, V^{2m-2[(s+1)/2]}\right), \; s = 0, \ldots, 2m; \right.$$
$$\frac{d^{2m-2}u}{dt^{2m-2}}, \; \prod_{j=1}^{p} \frac{d^{\alpha_j} \tilde{A}_{k_j}(t)}{dt^{\alpha_j}} \frac{d^{2m-2p-2-|\alpha(p)|}u}{dt^{2m-2p-2-|\alpha(p)|}} \in \mathscr{H}^2, \; |\alpha(p)| \le 2m-2p-2,$$
$$\left. 1 \le p \le m-1, \; 1 \le k_1, \; \ldots, \; k_p \le m, \; k_i \ne k_j \right\},$$

$[\cdot]$ is the integer part of a number, $\alpha(p) = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p$, and $|\alpha(p)| = \alpha_1 + \cdots + \alpha_p$, in the norms

$$|||u|||_m = \left\{ \sup_{0<t<T} \sum_{i=0}^{2m-1} \left| \frac{d^i u(t)}{dt^i} \right|^2_{m_i+1,t} \right\}^{1/2}.$$

For the spaces of right-hand sides of Eqs. (1) and the initial conditions (2), we take the Hilbert spaces $F^m = \mathscr{H} \times W^{2m-1}(0) \times \cdots \times H$, that is, the sets of all elements $\mathscr{F} = \{f, \varphi_0, \ldots, \varphi_{2m-1}\} \in F^m$ with finite Hermitian norm

$$\langle\|\mathscr{F}\|\rangle_m = \left\{ \|f\|_0^2 + \sum_{j=0}^{2m-1} |\varphi_j|^2_{m_j+1,0} \right\}^{1/2}.$$

The Cauchy problems (1), (2) correspond to the linear unbounded operators

$$L_m \equiv \{\mathscr{L}_m(t), l_0, \ldots, l_{2m-1}\} : \; E^m \supset D(L_m) \to F^m$$

with dense domains $D(L_m)$, $m = 1, 2, \ldots$ In forthcoming considerations, we use the following sufficient conditions for their closability.

Lemma 2. *If Conditions* I, II [*without inequality* (3)], IV, *and* V *and condition* (12) *are satisfied, then the operators* L_m, $m = 1, 2, \ldots,$ *admit closure.*

Proof. First, let us show that the set $\mathscr{H}_0^{1,1} = \{v \in \mathscr{H}^{1,1} : \; v(0) = v(T) = 0\}$ is dense in $\mathscr{H}$. Suppose the contrary: there exists some function $0 \ne w \in \mathscr{H}$ such that $\int_0^T (v, w)dt = 0$ for all $v \in \mathscr{H}_0^{1,1}$. In this integral, we set $v = A_{1,\varepsilon}^{-1}(t)h$, where $A_{1,\varepsilon}^{-1}(t) = (I + \varepsilon A_1(t))^{-1}$, $\varepsilon > 0$, $h, dh/dt \in \mathscr{H}$, and $h(0) = h(T) = 0$; then we obtain the relation

$$\int_0^T \left(A_{1,\varepsilon}^{-1}(t)h, w\right) dt = 0,$$

where $A_{1,\varepsilon}^{-1}(t)$ are operators satisfying the properties 1 and 2 at the beginning of the proof of Theorem 1. By using the known property (15), we pass to the limit in this relation as $\varepsilon \to 0$, use the passage to the limit to generalize the resulting relation to all $h \in \mathscr{H}$, set $h = w$, and obtain $\|w\|_0^2 = 0$, i.e., $w = 0$.

Let us now verify the validity of the closability criterion for the linear operators L_m; this criterion says that if $u_n \in D(L_m)$, $u_n \to 0$ in E^m, and $L_m u_n = \{\mathscr{L}_m(t)u_n, l_0 u_n, \ldots, l_{2m-1}u_n\} \to \mathscr{F} =$

$\{f, \varphi_0, \ldots, \varphi_{2m-1}\}$ in F^m as $n \to \infty$, then $\mathscr{F} = 0$. Since $l_j : E^m \to W^{2m-1-j}(0)$, $j = 0, \ldots, 2m-1$, are bounded operators, it follows that $\varphi_j = 0$, $j = 0, \ldots, 2m-1$, and consequently, after integration by parts with respect to t, we have

$$\int_0^T (f, v)dt = \lim_{n\to\infty} \int_0^T (\mathscr{L}_m(t)u_n, v)\,dt = -\lim_{n\to\infty} \int_0^T \left(\frac{d}{dt}\mathscr{M}_{m-1}(t)\cdots\mathscr{M}_1(t)u_n, \frac{dv}{dt}\right) dt$$

$$+ \lim_{n\to\infty} \int_0^T \left(A_m^{1/2}(t)\mathscr{M}_{m-1}(t)\cdots\mathscr{M}_1(t)u_n, A_m^{1/2}(t)v\right) dt = 0,$$

$$\mathscr{M}_k(t) = \frac{d^2}{dt^2} + \tilde{A}_k(t)$$

for all $v \in \mathscr{H}_0^{1,1}$. It follows that $f = 0$, since $\mathscr{H}_0^{1,1}$ is dense in $\mathscr{H}$. The proof of the lemma is complete.

Then we construct the closures $\bar{L}_m : E^m \supset D\left(\bar{L}_m\right) \to F^m$ of the operators L_m, $m = 1, 2, \ldots$ The domains $D\left(\bar{L}_m\right)$ of the operators $\bar{L}_m$ are defined to contain all functions $u \in E^m$ for each of which there exists a sequence $u_n \in D\left(L_m\right)$ and an element $\mathscr{F} \in F^m$ such that $|||u_n - u|||_m \to 0$ and $\langle \| L_m u_n - \mathscr{F} \| \rangle_m \to 0$ as $n \to \infty$, $m = 1, 2, \ldots$ In this connection, we assume that $\bar{L}_m u = \lim_{n\to\infty} L_m u_n = \mathscr{F}$, $m = 1, 2, \ldots$

Definition 1. Solutions $u \in D\left(\bar{L}_m\right)$ [respectively, $u \in D\left(L_m\right)$] of the operator equations $\bar{L}_m u = \mathscr{F}$, $\mathscr{F} \in F^m$, $m = 1, 2, \ldots$ [respectively, $L_m u = \mathscr{F}$, $\mathscr{F} \in R\left(L_m\right) = L_m\left(D\left(L_m\right)\right)$, $m = 1, 2, \ldots$] are referred to as *strong* (respectively, *smooth*) solutions of the Cauchy problems (1), (2).

4. UNIQUENESS THEOREM FOR CAUCHY PROBLEMS

First, we derive a priori estimates for smooth solutions of the Cauchy problems (1), (2).

Theorem 1. *If Conditions* I, II, IV–VI *and condition* (7) *are satisfied for* $m > 1$, *then there exist constants* $c_0(m) > 0$ *independent of* u *such that*

$$|||u|||_m^2 \le c_0(m) \langle \| L_m u \| \rangle_m^2 \qquad \forall u \in D\left(L_m\right), \qquad m = 1, 2, \ldots \tag{13}$$

Proof. In view of Conditions I and V, we set

$$\mathscr{M}_k(t) = d^2/dt^2 + \tilde{A}_k(t), \qquad \mathscr{L}_k^{(n,s)}(t) = \mathscr{M}_n(t) \times \cdots \times \mathscr{M}_{k+1}(t) \times \mathscr{M}_{k-1}(t) \times \cdots \times \mathscr{M}_s(t),$$

$1 \le s \le k \le n \le m$, and $\mathscr{L}_k^{(k,k)}(t) = I$ and write

$$\mathscr{L}_m(t) = \mathscr{M}_k(t)\mathscr{L}_k^{(m,1)}(t) + \mathscr{P}_{k,m}(t),$$

where, by virtue of (6),

$$\left|\mathscr{P}_{k,m}(t)u\right|^2 \le \tilde{c}_k \sum_{i=0}^{2m-3} \left|\frac{d^i u}{dt^i}\right|^2_{m_i+1,t} \qquad \forall u \in D\left(L_m\right) \tag{14}$$

for all $t \in [0, T]$ with constants $\tilde{c}_k \ge 0$ independent of u and t. The smoothing operators $A_{k,\varepsilon}^{-1}(t) = \left(I + \varepsilon A_k(t)\right)^{-1}$, $\varepsilon > 0$, $k = 1, \ldots, m$, have the following properties [2].

1. We have

$$\left|A_{k,\varepsilon}^{-1}(t)v - v\right| \to 0 \quad \forall v \in H \tag{15}$$

for all $t \in [0, T]$ as $\varepsilon \to 0$.

2. The operators $A_{k,\varepsilon}^{-1}(t)$ have the strong derivatives $dA_{k,\varepsilon}^{-1}(t)/dt \in \mathscr{B}([0,T],\mathfrak{L}(H))$ in H for all $t \in [0,T]$.

By integrating by parts, we obtain the identities

$$\begin{aligned}
&\left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)\mathscr{L}_k^{(m,1)}(t)u\right)\Big|_{t=\tau} \\
&\quad = 2\,\mathrm{Re}\int\limits_0^\tau \left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right)dt \\
&\quad\quad + \int\limits_0^\tau \left(\frac{d\left(A_k(t)A_{k,\varepsilon}^{-1}(t)\right)}{dt}\mathscr{L}_k^{(m,1)}(t)u, \mathscr{L}_k^{(m,1)}(t)u\right)dt \\
&\quad\quad + \left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)\mathscr{L}_k^{(m,1)}(t)u\right)\Big|_{t=0}, \qquad m = 1,2,\ldots,
\end{aligned}$$

for all $u \in \tilde{D}\left(L_m\right)$. By using the formulas [2]

$$d\left(A_k(t)A_{k,\varepsilon}^{-1}(t)\right)/dt = -A_k(t)A_{k,\varepsilon}^{-1}(t)\left(dA_k^{-1}(t)/dt\right)A_k(t)A_{k,\varepsilon}^{-1}(t)$$

and inequalities (3) in the second integral on the right-hand side, we obtain the inequalities

$$\begin{aligned}
&\left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)\mathscr{L}_k^{(m,1)}(t)u\right)\Big|_{t=\tau} \\
&\quad \le 2\,\mathrm{Re}\int\limits_0^\tau \left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right)dt \\
&\quad\quad + c_k^{(1)}\int\limits_0^\tau \left(A_{k,\varepsilon}^{-1}(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)A_k(t)\mathscr{L}_k^{(m,1)}(t)u\right)dt \\
&\quad\quad + \left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, A_{k,\varepsilon}^{-1}(t)\mathscr{L}_k^{(m,1)}(t)u\right)\Big|_{t=0}.
\end{aligned}$$

In these relations, we use property (15) and pass to the limit as $\varepsilon \to 0$; this yields

$$\begin{aligned}
&\left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, \mathscr{L}_k^{(m,1)}(t)u\right)\Big|_{t=\tau} \\
&\quad \le 2\,\mathrm{Re}\int\limits_0^\tau \left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, \frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right)dt + c_k^{(1)}\int\limits_0^\tau \left(\mathscr{L}_k^{(m,1)}(t)u, A_k(t)\mathscr{L}_k^{(m,1)}(t)u\right)dt \\
&\quad\quad + \left(A_k(t)\mathscr{L}_k^{(m,1)}(t)u, \mathscr{L}_k^{(m,1)}(t)u\right)\Big|_{t=0}.
\end{aligned} \tag{16}$$

By integrating by parts, we obtain the identities

$$\begin{aligned}
\left|\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right|^2\Bigg|_{t=\tau} &= 2\,\mathrm{Re}\int\limits_0^\tau \left(\frac{d^2}{dt^2}\mathscr{L}_k^{(m,1)}(t)u, \frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right)dt \\
&\quad + \left|\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right|^2\Bigg|_{t=0} \qquad \forall u \in \tilde{D}\left(L_m\right).
\end{aligned}$$

By adding these identities to inequalities (16), we obtain

$$\left[\left|\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right|^2+\left|A_k^{1/2}(t)\mathscr{L}_k^{(m,1)}(t)u\right|^2\right]\Bigg|_{t=\tau}$$
$$\leq 2\operatorname{Re}\int_0^\tau\left(\mathscr{L}_m(t)u,\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right)dt+\int_0^\tau\Phi_k(u,u)dt$$
$$+\left[\left|\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right|^2+\left|A_k^{1/2}(t)\mathscr{L}_k^{(m,1)}(t)u\right|^2\right]\Bigg|_{t=0}\qquad\forall u\in\tilde{D}\left(L_m\right),\tag{17}$$

where

$$\Phi_k(u,u)=c_k^{(1)}\left(\mathscr{L}_k^{(m,1)}(t)u,A_k(t)\mathscr{L}_k^{(m,1)}(t)u\right)-2\operatorname{Re}\left(\mathscr{R}_{k,m}(t)u,\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right).$$

By virtue of Condition IV, the left-hand sides of inequalities (17) do not exceed the quantity

$$c_2\left[\left|\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right|^2+\left|\mathscr{L}_k^{(m,1)}(t)u\right|_{1,t}^2\right]\Bigg|_{t=\tau},\tag{18}$$

where $c_2>0$ is a constant independent of u, t, and k. By using inequalities (5), one can justify the following assertion.

Lemma 3. *If the assumptions of Theorem* 1 *are valid, then there exist constants* $c_3>0$ *and* $c_4\geq 0$ *independent of* u *and* t *such that*

$$\sum_{k=1}^m\left(\left|\frac{d}{dt}\mathscr{L}_k^{(m,1)}(t)u\right|^2+\left|\mathscr{L}_k^{(m,1)}(t)u\right|_{1,t}^2\right)\geq c_3\sum_{i=0}^{2m-1}\left|\frac{d^iu}{dt^i}\right|_{m_i+1,t}^2-c_4\sum_{i=0}^{2m-3}\left|\frac{d^iu}{dt^i}\right|_{m_i,t}^2\tag{19}$$

for all $u\in D\left(L_m\right)$ *and* $t\in[0,T]$.

Proof. We prove the desired assertion by induction on m. If $m=1$, then Lemma 3 is valid. Suppose that it holds for $m-1$ distinct factors $\mathscr{M}_k(t)$; in particular, inequalities of the form (19) are valid for two sums,

$$\mathscr{S}_{m-j,2-j}(v)=\sum_{k=2-j}^{m-j}\left(\left|\frac{d\mathscr{L}_k^{(m-j,2-j)}(t)v}{dt}\right|^2+\left|\mathscr{L}_k^{(m-j,2-j)}(t)v\right|_{1,t}^2\right),\qquad j=0,1.$$

By denoting the sum on the left-hand side in (19) by $\mathscr{S}_{m,1}(u)$, we find that it satisfies the relation

$$\mathscr{S}_{m,1}(u)=(1/3)\left[\mathscr{S}_{m,2}\left(\mathscr{M}_1(t)u\right)+\mathscr{S}_{m-1,1}\left(\mathscr{M}_m(t)u\right)\right]+\mathscr{P}_m(t)u;$$

when estimating it from below, the last four nonnegative terms in the expression

$$\mathscr{P}_m(t)u=\frac{1}{3}\Bigg\{2\sum_{k=1}^{m-1}\left(\left|\frac{d\mathscr{M}_m(t)\mathscr{L}_k^{(m-1,1)}(t)u}{dt}\right|^2+\left|\mathscr{M}_m(t)\mathscr{L}_k^{(m-1,1)}(t)u\right|_{1,t}^2\right)$$
$$-\mathscr{S}_{m-1,1}\left(\mathscr{M}_m(t)u\right)+2\left|\frac{d\mathscr{L}_m^{(m,1)}(t)u}{dt}\right|^2+2\left|\mathscr{L}_m^{(m,1)}(t)u\right|_{1,t}^2$$
$$+\left|\frac{d\mathscr{L}_1^{(m,1)}(t)u}{dt}\right|^2+\left|\mathscr{L}_1^{(m,1)}(t)u\right|_{1,t}^2\Bigg\}$$

can be omitted. If in the remaining terms of this expression $\mathscr{P}_m(t)u$, we use an estimate of the form $(1+\delta)|x|^2 - |y|^2 \geq -\left(1+\delta^{-1}\right)|x-y|^2$, $x, y \in H$, $\delta > 0$, with $\delta = 1$, then, by virtue of inequalities (6), we obtain the estimate

$$\mathscr{P}_m(t)u \geq -\tilde{c}_4^{(1)} \sum_{i=0}^{2m-3} \left|\frac{d^i u}{dt^i}\right|^2_{m_i,t} \qquad \forall u \in D\left(L_m\right)$$

for all $t \in [0,T]$ with a constant $\tilde{c}_4^{(1)} \geq 0$ independent of u and t. Therefore, by the induction assumption, we have

$$\mathscr{S}_{m,1}(u) \geq \tilde{c}_3^{(1)} \sum_{i=0}^{2m-3} \left(\left|\frac{d^i \mathscr{M}_1(t)u}{dt^i}\right|^2_{m_i-1,t} + \left|\frac{d^i \mathscr{M}_m(t)u}{dt^i}\right|^2_{m_i-1,t} \right) - \tilde{c}_4^{(2)} \sum_{i=0}^{2m-3} \left|\frac{d^i u}{dt^i}\right|^2_{m_i,t},$$

where $\tilde{c}_3^{(1)} > 0$ and $\tilde{c}_4^{(2)} \geq 0$ are constants independent of u and t.

Since, by virtue of Condition V, the differentiation of the operators $\tilde{A}_1(t)$ and $\tilde{A}_m(t)$ with respect to t, the evaluation of their restrictions and the restrictions of their derivatives with respect to t with $\tilde{D}\left(L_m\right)$ to $D\left(L_m\right)$, and elementary estimates lead to the inequality

$$\left|\mathscr{M}_1(t)\frac{d^i u}{dt^i}\right|^2_{m_i-1,t} + \left|\mathscr{M}_m(t)\frac{d^i u}{dt^i}\right|^2_{m_i-1,t} \leq 2\left|\frac{d^i \mathscr{M}_1(t)u}{dt^i}\right|^2_{m_i-1,t} + 2\left|\frac{d^i \mathscr{M}_m(t)u}{dt^i}\right|^2_{m_i-1,t}$$
$$+ \tilde{c}_4^{(3)} \sum_{i=0}^{2m-3} \left|\frac{d^i u}{dt^i}\right|^2_{m_i,t}, \qquad \tilde{c}_4^{(3)} \geq 0,$$

it follows from the identity $|x+y|^2 = |x|^2 + |y|^2 + 2\,\mathrm{Re}\,(x,y)$ and the Schwarz inequality that the left-hand side of the last inequality is not less than the quantity

$$2\left|\frac{d^{i+2}u}{dt^{i+2}}\right|^2_{m_i-1,t} + \left(|A_1(t)w|^2_{m_i-1,t} + |A_m(t)w|^2_{m_i-1,t}\right)\left|\frac{d^i u}{dt^i}\right|^2_{m_i-1,t}$$
$$- 2\left|A_1(t)w + A_m(t)w\right|_{m_i-1,t}\left|\frac{d^{i+2}u}{dt^{i+2}}\right|_{m_i-1,t}\left|\frac{d^i u}{dt^i}\right|_{m_i+1,t}, \quad w = \frac{d^i u}{dt^i}\Big/\left|\frac{d^i u}{dt^i}\right|_{m_i+1,t},$$

which, in turn, by virtue of the δ-inequality, is not less than the quantity

$$(2-\delta)\left|\frac{d^{i+2}u}{dt^{i+2}}\right|^2_{m_i-1,t} + \frac{|A_1(t)w|^2_{m_i-1,t} + |A_m(t)w|^2_{m_i-1,t}}{\delta}$$
$$\times \left(\delta - \frac{|A_1(t)w + A_m(t)w|^2_{m_i-1,t}}{|A_1(t)w|^2_{m_i-1,t} + |A_m(t)w|^2_{m_i-1,t}}\right)\left|\frac{d^i u}{dt^i}\right|^2_{m_i+1,t}.$$

Since, by virtue of the parallelogram identity and inequalities (5),

$$\delta_0 = \sup_w \frac{|A_1(t)w + A_m(t)w|^2_{m_i-1,t}}{|A_1(t)w|^2_{m_i-1,t} + |A_m(t)w|^2_{m_i-1,t}} = 2 - \inf_w \frac{|A_1(t)w - A_m(t)w|^2_{m_i-1,t}}{|A_1(t)w|^2_{m_i-1,t} + |A_m(t)w|^2_{m_i-1,t}} < 2,$$

it follows that δ can be chosen so as to ensure that $\delta_0 < \delta < 2$. This implies that there exist constants $c_3 > 0$ and $c_4 \geq 0$ independent of u and t such that inequalities (19) are valid for m distinct factors $\mathscr{M}_k(t)$, $k = 1, \ldots, m$. The proof of Lemma 3 is complete.

By summing inequalities (17) in view of the estimates (18) and by using the estimates (19) on the left-hand sides of the resulting inequalities and the estimates (14), the Schwarz inequality, the

Cauchy–Schwarz inequality, the δ-inequality, and elementary estimates on the right-hand sides, we find constants $c_5, c_6, c_7 > 0$ independent of u and t such that

$$c_2 c_3 \sum_{i=0}^{2m-1} \left| \frac{d^i u}{dt^i} \right|^2_{m_i+1,t} \Bigg|_{t=\tau} \le c_5 \int_0^\tau \sum_{i=0}^{2m-1} \left| \frac{d^i u}{dt^i} \right|^2_{m_i+1,t} dt + c_6 \int_0^\tau |\mathscr{L}_m(t) u|^2 \, dt + c_7 \sum_{i=0}^{2m-1} |l_j u|^2_{m_j+1,0} + c_2 c_4 \sum_{i=0}^{2m-3} \left| \frac{d^i u}{dt^i} \right|^2_{m_i,t} \Bigg|_{t=\tau} \quad \forall u \in D(L_m). \tag{20}$$

In the last sum in inequality (20), we use the interpolation inequalities (8) and find a constant $c_8 > 0$ independent of u and t such that

$$c_2 c_3 \sum_{i=0}^{2m-1} \left| \frac{d^i u}{dt^i} \right|^2_{m_i+1,t} \Bigg|_{t=\tau} \le c_8 \int_0^\tau \sum_{i=0}^{2m-1} \left| \frac{d^i u}{dt^i} \right|^2_{m_i+1,t} dt + c_6 \int_0^\tau |\mathscr{L}_m(t) u|^2 \, dt + \left(c_7 + c_1^2 c_2 c_4\right) \sum_{j=0}^{2m-1} |l_j u|^2_{m_j+1,0} \qquad \forall u \in D(L_m).$$

Then, in these inequalities, we use the following Gronwall lemma [4, p. 23 of the Russian translation].

Lemma 4. *If v and g are nonnegative functions on $[0,T]$, v is integrable, and g is nondecreasing, then the inequality $v(\tau) \le c \int_0^\tau v(t)dt + g(\tau)$ implies the inequality $v(\tau) \le e^{c\tau} g(\tau)$, $\tau \in [0,T]$.*

Then we obtain the inequalities

$$c_2 c_3 \sum_{i=0}^{2m-1} \left| \frac{d^i u}{dt^i} \right|^2_{m_i+1,t} \Bigg|_{t=\tau} \le e^{c_9 \tau} \left(c_6 \int_0^\tau |\mathscr{L}_m(t) u|^2 \, dt + \left(c_7 + c_1^2 c_2 c_4\right) \sum_{j=0}^{2m-1} |l_j u|^2_{m_j+1,0} \right), \tag{21}$$

where $c_9 = c_8/(c_2 c_3)$. By evaluating the least upper bound with respect to τ in inequality (21), we obtain (13) for all $u \in D(L_m)$ with constants $c_0(m) = \exp(c_9 T) \max\{c_6, c_7 + c_1^2 c_2 c_4\}/(c_2 c_3)$. The proof of Theorem 1 is complete.

Remark 1. In general, if the energy inequalities (13) can be derived without using the inclusion (12), then condition (12) is satisfied. If, in the derivation of the energy inequalities (13), instead of the integration with respect to t from 0 to τ, we use the integration with respect to t from $s < \tau$ to τ and then the additional integration with respect to s from 0 to T, then

$$|||u|||^2_{2m-1,2m-1} \le \tilde{c}_0(m) ||u||^2_{2m,2m}, \qquad \tilde{c}_0(m) > 0, \qquad m = 1, 2, \ldots,$$

for all functions $u \in D(L_m)$.

Now let us derive a priori estimates for strong solutions of the Cauchy problems (1), (2). The following assertion is a straightforward consequence of Theorem 1.

Corollary 1. *If the assumptions of Theorem 1 are valid, then the energy inequalities*

$$|||u|||^2_m \le c_0(m) \left\langle \left\| \bar{L}_m u \right\| \right\rangle^2_m$$

are valid for all $u \in D(\bar{L}_m)$, $m = 1, 2, \ldots$

Proof. By taking into account Remark 1, we find that condition (12) and Lemma 2 on the closeness of the operators L_m, $m = 1, 2, \ldots$, are valid. Therefore, by using passage to the limit, one can generalize the energy inequalities (13) from smooth solutions $u \in D(L_m)$ to all strong solutions $u \in D(\bar{L}_m)$ of the Cauchy problems (1), (2).

5. EXISTENCE THEOREM FOR CAUCHY PROBLEMS

From Theorem 1 and Corollary 1, we find that if a strong solution of the Cauchy problems (1), (2) exists, then it is unique and continuously depends on the data f and φ_j, $j = 0, \dots, 2m-1$. The following assertion justifies the strong solvability of the Cauchy problems (1), (2) for all $\mathscr{F} \in F^m$.

Theorem 2. *Let Conditions* I–VI *be satisfied. If condition* (7) *is valid for* $m > 1$, *then for any* $f \in \mathscr{H}$ *and* $\varphi_j \in W^{m_j+1}(0)$, $j = 0, \dots, 2m-1$, *there exists a strong solution* $u \in E^m$ *of the Cauchy problems* (1), (2).

Proof. We perform the proof by induction on m. If $m = 1$, then, by virtue of inequalities (3) and (4), the equation $\bar{L}_1 u = \mathscr{F}$ is solvable [2, 3] for any $\mathscr{F} \in F^1$. By induction, we suppose that the equations $\bar{L}_{m-1} u = \mathscr{F}$ are solvable for any $\mathscr{F} \in F^{m-1}$ and for any set and order of $m-1$ distinct factors $\mathscr{M}_k(t)$ in $\mathscr{L}_{m-1}(t)$ and prove the solvability of the equations $\bar{L}_m u = \mathscr{F}$ for all $\mathscr{F} \in F^m$, $m = 2, \dots$

Consider the linear operators $L_k^{(m,1)} \equiv \{\mathscr{L}_k^{(m,1)}(t), l_0, \dots, l_{2m-3}\} : E^{m-1} \supset D(L_{m-1}) \to F^{m-1}$ in other spaces, $L_k^{(m,1)} \colon E^m \supset D(L_m) \to E^{1,m}$, $k = 1, \dots, m$, where $E^{1,m} = E^1 \times W^{2m-2}(0) \times \dots \times W^1(0)$ are the Banach spaces with norms

$$|||u|||_{1,m} = \left(|||u|||_1^2 + \sum_{j=0}^{2m-3} |l_j u|^2_{m_j,0} \right)^{1/2}.$$

The closure $\widehat{L_k^{(m,1)}}$ of the last operators is given by the restriction of the closure of the first operators $\overline{L_k^{(m,1)}}$ to E^m; i.e., $\widehat{L_k^{(m,1)}} = \overline{L_k^{(m,1)}}\Big|_{E^m}$, since $\widehat{L_k^{(m,1)}} \subset \overline{L_k^{(m,1)}}$ and the $L_k^{(m,1)} : E^m \supset D(L_m) \to E^{1,m}$, $k = 1, \dots, m$, are continuous operators. In addition, consider the linear operators

$$M_k = \left\{ \mathscr{M}_k(t), A_1^{-1/2}(0), \dots, A_1^{-1/2}(0), l_0, l_1 \right\} : E^{1,m} \supset D(L_1) \times W^{2m-2}(0) \times \dots \times W^1(0) \to F^m,$$

whose closures $\bar{M}_k$, by [2, 3[, have bounded inverses $\bar{M}_k^{-1} : F^m \to E^{1,m}$, $k = 1, \dots, m$. By virtue of the estimates (14), the solutions of the equations $\bar{L}_m u = \mathscr{F}$ for $\mathscr{F} \in F^m$ are simultaneously solutions of the equations $\overline{M_k L_k^{(m,1)}} u = \mathscr{F}_k$ for

$$\mathscr{F}_k = \mathscr{F} + \left\{ \left[\mathscr{M}_k(t) \mathscr{L}_k^{(m,1)}(t) - \mathscr{L}_m(t) \right] u, 0, \dots, 0 \right\} \in F^m, \qquad k = 1, \dots, m.$$

Lemma 5. *Let* X, Y, *and* Z *be Banach spaces. If* $S : X \to Y$ *is a linear operator closable by continuity to a bounded operator* $\bar{S}$ *and* $P : Y \to Z$ *is a linear operator that admits closure* $\bar{P}$, *then the product* $P \cdot S : X \to Z$ *admits closure* $\overline{P \cdot S}$, *and* $\overline{P \cdot S} \subset \bar{P} \cdot \bar{S}$.

Proof. By the closability criterion for linear operators in Banach spaces, to prove that the product $P \cdot S$ is closable in $X \times Z$, we show that if $u_n \in D(P \cdot S)$, $u_n \to 0$ in X, and $(P \cdot S)u_n = P(Su_n) \to g$ in Z as $n \to \infty$, then $g = 0$. It follows from the assumptions of this criterion that $v_n = Su_n \in D(P)$, $v_n \to 0$ in Y since $\bar{S}$ is a bounded operator, and $Pv_n \to g$ in Z as $n \to \infty$. Consequently, $g = 0$, since P admits closure in $Y \times Z$.

It remains to prove the algebraic embedding $\overline{P \cdot S} \subset \bar{P} \cdot \bar{S}$. Let $\left(\overline{P \cdot S}\right) u = g$; i.e., there exist $u_n \in D(P \cdot S)$ such that $u_n \to u$ in X and $(P \cdot S)u_n \to g$ in Z as $n \to \infty$. Then, obviously, $v_n = Su_n \in D(P)$, $v_n \to \bar{S}u$ in Y since $\bar{S}$ is a bounded operator, and $Pv_n \to g$ in Z as $n \to \infty$. Hence it follows that $\bar{S}u \in D\left(\bar{P}\right)$ and $\left(\bar{P} \cdot \bar{S}\right) u = g$. The proof of Lemma 5 is complete.

By applying Lemma 5 to the operators $S = L_k^{(m,1)}$ and $P = M_k$ in the spaces $X = E^m$, $Y = E^{1,m}$, and $Z = F^m$, we obtain the embeddings $\overline{M_k L_k^{(m,1)}} \subset \bar{M}_k \overline{L_k^{(m,1)}}$, $k = 1, \dots, m$. Hence we find that the equations $\overline{M_k L_k^{(m,1)}} u = \mathscr{F}_k$ for all $u \in D\left(\bar{L}_m\right)$ can be represented in the form $\bar{M}_k \overline{L_k^{(m,1)}} u = \mathscr{F}_k$,

$k=1,\dots,m$. By [2, 3], the last equations have solutions $\overline{L_k^{(m,1)}}u=\bar M_k^{-1}\mathscr{F}_k\in E^{1,m}$ for all $\mathscr{F}_k\in F^m$, and, by the induction assumption, they have solutions $u=\overline{L_k^{(m,1)}}^{-1}\bar M_k^{-1}\mathscr{F}_k\in E^{m-1}$, where $\overline{L_k^{(m,1)}}^{-1}$ are the inverses of the operators $\overline{L_k^{(m,1)}}$, $k=1,\dots,m$. It remains to justify the inclusion $u\in E^m$.

Let us show that the smoothness of this solution $u\in E^{m-1}$ can be increased by unity owing to the smoothness of the right-hand side, $\bar M_k^{-1}\mathscr{F}_k\in E^{1,m}$ instead of $\bar M_k^{-1}\mathscr{F}_k\in F^{m-1}$, $k=1,\dots,m$. Lemma 3, together with Lemmas 1 and 4, implies the inequalities

$$c_{10}|||u|||_m^2\le\sum_{k=1}^m\left|\left|\left|L_k^{(m,1)}u\right|\right|\right|_{1,m}^2,\qquad c_{10}>0,\qquad\forall u\in D\left(L_m\right),$$

which, by passage to the limit, can be generalized to all functions $u\in E^m$ in the domains of the closure $\widehat{L_k^{(m,1)}}$. These inequalities imply that the operators $\overline{L_k^{(m,1)}}$ are homeomorphic mappings of the space E^m onto their ranges $R\left(\widehat{L_m^{(m,1)}}\right)$ equipped with the norm of the space $E^{1,m}$. To complete the proof, it remains to show that $R\left(\widehat{L_m^{(m,1)}}\right)=E^{1,m}$. In turn, to this end, it suffices to show that the equations $\overline{L_k^{(m,1)}}v=\Phi_k$, $k=1,\dots,m$, with an arbitrary right-hand side Φ_k in a set dense in $E^{1,m}$ have solutions in $v\in E^m$. By virtue of the estimate (14) with $m-1$ instead of m, the solutions $v\in E^{m-1}$ of the equations $\overline{L_k^{(m,1)}}v=\Phi_k\in F^{m-1}$ are simultaneously solutions of the equations $\overline{L_{m-2}^{(k)}M_{k-1}}v=\tilde\Phi_{k-1}$, where $L_{m-2}^{(k)}=L_k^{(m,k)}L_{k-1}^{(k-1,1)}$ and

$$\tilde\Phi_{k-1}=\Phi_k+\left\{\left[\mathscr{L}_k^{(m,k)}(t)\mathscr{L}_{k-1}^{(k-1,1)}(t)\mathscr{M}_{k-1}(t)-\mathscr{L}_k^{(m,1)}(t)\right]v,0,\dots,0\right\}\in F^{m-1}$$

for $k=2,\dots,m$ and the equation $\overline{L_{m-2}^{(1)}M_m}v=\tilde\Phi_m$, where $L_{m-2}^{(1)}=L_m^{(m,2)}$ and

$$\tilde\Phi_m=\Phi_1+\left\{\left[\mathscr{L}_m^{(m,2)}(t)\mathscr{M}_m(t)-\mathscr{L}_1^{(m,1)}(t)\right]v,0,\dots,0\right\}\in F^{m-1}$$

for $k=1$. Below we increase the smoothness of solutions of the operators $\widehat{L_k^{(m,1)}}$ owing to increasing the smoothness of solutions of the operators $\bar M_k$, $k=1,\dots,m$.

Let the Hilbert space $\tilde{\mathscr{H}}^{2,2}$ be defined as the set $D\left(L_1\right)$ equipped with the Hermitian norm

$$\langle|u|\rangle_{2,2}=\left(\left|\left|d^2u/dt^2\right|\right|_0^2+||du/dt||_0^2+||u||_2^2\right)^{1/2}.$$

To shorten the notation, we set $\mathscr{M}_0(t)=\mathscr{M}_m(t)$, $M_0=M_m$, and $\tilde\Phi_0=\tilde\Phi_m$. If the function $v\in E^{m-1}$ is a solution of the equations $\overline{L_{m-2}^{(k)}M_{k-1}}v=\tilde\Phi_{k-1}$ for $\tilde\Phi_{k-1}\in F^{2,m}=\tilde{\mathscr{H}}^{2,2}\times W^{2m-1}(0)\times\dots\times W^2(0)$, then $M_k\overline{L_{m-2}^{(k)}M_{k-1}}v=M_k\tilde\Phi_{k-1}\in F^m$, where $M_k\equiv\{\mathscr{M}_k(t),I,\dots,I,l_0,l_1\}:F^{2,m}\to F^m$, $k=1,\dots,m$, are bounded linear operators.

Lemma 6. *Let X, Y, and Z be Banach spaces. If $P:X\to Y$ is a linear operator that admits a closure $\bar P$, $S:Y\to Z$ is a linear bounded operator, and their product $S\cdot P:X\to Z$ admits a closure $\overline{S\cdot P}$, then $S\cdot\bar P\subset\overline{S\cdot P}$.*

Proof. Let $\left(S\cdot\bar P\right)u=g$. Then $u\in D\left(\bar P\right)$; i.e., there exist $u_n\in D(P)$ such that $u_n\to u$ in X and $Pu_n\to\bar Pu$ in Y as $n\to\infty$. Since the operator S is bounded, we have $S\left(Pu_n\right)\to S\left(\bar Pu\right)$ in Z as $n\to\infty$. Hence it follows that $u_n\in D(S\cdot P)$, $u_n\to u$ in X, and $S\left(Pu_n\right)=(S\cdot P)u_n\to g$ in Z as $n\to\infty$; i.e., $u\in D\left(\overline{S\cdot P}\right)$ and $\left(\overline{S\cdot P}\right)u=g$. The proof of the lemma is complete.

By applying Lemma 6 to the operators $P = L_{m-2}^{(k)} M_{k-1}$ and $S = M_k$ in the spaces $X = E^m$, $Y = F^{2,m}$, and $Z = F^m$, we obtain the embeddings

$$M_k \overline{L_{m-2}^{(k)} M_{k-1}} \subset \overline{M_k \left(L_{m-2}^{(k)} M_{k-1} \right)}, \qquad k = 1, \ldots, m. \tag{22}$$

By applying Lemma 5 to the operators $S = M_{k-1} \equiv \{\mathscr{M}_{k-1}(t), l_0, l_1\} : E^m \supset D(L_m) \to E^{m-1} \times W^{2m-2}(0) \times W^{2m-3}(0)$ and $P = M_k L_{m-2}^{(k)} \equiv \{\mathscr{M}_k(t) \mathscr{L}_{m-2}^{(k)}(t), A_1^{-1/2}(0), A_1^{-1/2}(0), l_0, \ldots, l_{2m-3}\} : E^{m-1} \times W^{2m-2}(0) \times W^{2m-3}(0) \supset D(L_{m-1}) \times W^{2m-2}(0) \times W^{2m-3}(0) \to F^m$, where $\mathscr{L}_{m-2}^{(k)}(t)$ are the first operator coordinates of the vector operators $L_{m-2}^{(k)}$, we obtain the embeddings

$$\overline{\left(M_k L_{m-2}^{(k)} \right) M_{k-1}} \subset \overline{M_k L_{m-2}^{(k)}} \, \overline{M_{k-1}}, \qquad k = 1, \ldots, m. \tag{23}$$

From (22) and (23), we have the equations $\overline{M_k L_{m-2}^{(k)}} \, \overline{M_{k-1}} v = M_k \tilde{\Phi}_{k-1}$, $k = 1, \ldots, m$, which, by the induction assumption, have the solutions $\overline{\mathscr{M}_n(t)} v \in E^{m-1}$, $n = 1, \ldots, m$, for $n = k - 1$.

Lemma 7. *If the assumptions of Theorem* 1 *are valid, then there exist constants* $c_{11} > 0$ *and* $c_{12} \geq 0$ *independent of* v *and* t *such that*

$$\sum_{k=1}^{m} \sum_{i=0}^{2m-3} \left| \frac{d^i \mathscr{M}_k(t) v}{dt^i} \right|^2_{m_i - 1, t} \geq c_{11} \sum_{i=0}^{2m-1} \left| \frac{d^i v}{dt^i} \right|^2_{m_i + 1, t} - c_{12} \sum_{i=0}^{2m-3} \left| \frac{d^i v}{dt^i} \right|^2_{m_i, t}, \qquad m = 2, 3, \ldots, \tag{24}$$

for all $v \in D(L_m)$ *and* $t \in [0, T]$.

Proof. Lemma 7 can be proved by induction over m by analogy with the proof of Lemma 3.

By using Lemmas 1 and 4, from (24), we obtain

$$c_{13} |||v|||_m^2 \leq \sum_{k=1}^{m} |||\mathscr{M}_k(t) v|||_{m-1}^2 + \sum_{j=0}^{2m-3} |l_j v|_{m_j, 0}, \qquad c_{13} > 0.$$

By passing to the limit, we generalize these inequalities from the solutions $v \in D(L_m)$ to solutions of the desired equations with right-hand sides $M_k \tilde{\Phi}_{k-1}$ such that $\overline{\mathscr{M}_k(t)} v \in E^{m-1}$, $k = 1, \ldots, m$, and obtain $v \in E^m$, since $v \in E^{m-1}$. Therefore, the earlier-found solution $u \in E^{m-1}$ of the original equation with arbitrary right-hand side $\mathscr{F} \in F^m$ indeed belongs to the space E^m, $m = 2, \ldots$

By induction over m, hence we obtain

$$u = \bar{L}_m^{-1} \mathscr{F} = \bar{M}_1^{-1} \cdots \bar{M}_m^{-1} \mathscr{F} \in E^m \qquad \forall \mathscr{F} \in F^m, \qquad m = 1, 2, \ldots \tag{25}$$

Remark 2. In the same way, the assertions of Theorem 1 and Corollary 1 [possibly, with larger values of the constants $c_0(m)$] and the assertion of Theorem 2 (with the use of continuation with respect to a parameter) can be generalized to the equations with lower terms

$$\mathscr{L}_m(t) u + \sum_{k=0}^{2m-1} B_k(t) \frac{d^k u}{dt^k} = f, \qquad t \in]0, T[, \qquad m = 1, 2, \ldots, \tag{26}$$

if $B_k(t) \in \mathscr{B}([0, T], \mathfrak{L}(W^{m_k+1}(t), H))$, $k = 0, \ldots, 2m - 1$. The lower terms of Eqs. (26) should be treated not only as additional terms subjected to the leading terms of these equations but also as the factorization remainder of arbitrary quasi-hyperbolic even-order differential-operator equations, i.e., as terms preserved under the reduction of quasi-hyperbolic even-order differential-operator equations to their factorized (divergent) form (1).

Remark 3. The analysis of the proof of Theorem 1, Corollary 1, and Theorem 2, shows that if all operators $\tilde{A}_k(t) = \tilde{A}_k$ are independent of t and commute with each other, then the interpolation inequalities (8) become unnecessary, and consequently, in these assertions, the sufficient condition (7) for $m > 1$ is unnecessary even if the domains $D(A_k(t))$ of the restrictions $A_k(t)$ depend on t. Note that, as a rule, this condition is satisfied even if all $\tilde{A}_k(t)$ smoothly depend on t and do not commute with each other, but the domains $D(A_k)$ of the operators $A_k(t)$ are independent of t [1], i.e., if $A_k(t) = \tilde{A}_k(t)$. In addition, such variable operators $A_k(t)$ with constant domains $D(A_k)$ were subjected in [1] to the more restrictive condition $dA^{-1}(t)/dt \in \mathscr{B}([0,T], \mathfrak{L}(H, W^{2m}(0)))$ for all $m \geq 1$.

6. EXAMPLE OF MIXED PROBLEMS

In the bounded domain $G =]0,T[\, \times\,]0,l[$ of the variables t and x, we consider the following mixed problems: the differential equations

$$\left(\partial^2/\partial t^2 - a_m^2 \partial^2/\partial x^2\right) \cdots \left(\partial^2/\partial t^2 - a_1^2 \partial^2/\partial x^2\right) u(t,x) = f(t,x), \qquad (t,x) \in G, \tag{27}$$

where $a_k > 0$ are distinct constants and $a_1 = 1$, with the boundary conditions

$$\begin{aligned} &\partial^{2i+1} u(t,0)/\partial x^{2i+1} - \beta(t) \partial^{2i} u(t,0)/\partial x^{2i} = 0, \\ &\partial^{2i+1} u(t,l)/\partial x^{2i+1} + \tilde{\beta}(t) \partial^{2i} u(t,l)/\partial x^{2i} = 0, \qquad t \in [0,T], \qquad i = 0, \ldots, m-1, \end{aligned} \tag{28}$$

where $\beta(t)$ and $\tilde{\beta}(t)$ are nonnegative functions that do not simultaneously vanish for any $t \in [0,T]$ and are twice continuously differentiable with respect to t, with the initial conditions

$$\partial^j u(0,x)/\partial t^j = \varphi_j(x), \qquad x \in]0,l[\,, \qquad j = 0, \ldots, 2m-1, \qquad m = 1, 2, \ldots \tag{29}$$

Let us show that, by Remark 3, the differential operators $A_k(t)$ obtained as the restriction of the differential expressions $\tilde{A}_k u(t,x) = -a_k^2 \partial^2 u(t,x)/\partial x^2$, $t \in [0,T]$, to the domains $D(A_k(t)) = \{u \in L_2(0,l) : \ u(x) \in (26) \text{ for } m = 1;\ \tilde{A}_k(t) u(x) \in L_2(0,l)\}$, $t \in [0,T]$, satisfy the sufficient assumptions of Theorems 1 and 2 in the Hilbert space $H = L_2(0,l)$. The operators $A_1(t)$, $t \in [0,T]$, are self-adjoint in $L_2(0,l)$, since they are obviously symmetric in $L_2(0,l)$ and have bounded inverses

$$A_1^{-1}(t) g = -\int_0^x (x-s) g(s) ds + (\mathscr{A}_1(t) + \mathscr{B}_1(t) x) \int_0^l g(s) ds + (\mathscr{C}_1(t) + \mathscr{D}_1(t) x) \int_0^l (l-s) g(s) ds$$

on $L_2(0,l)$, where $\mathscr{A}_1(t) = 1/\big(\beta + \tilde{\beta} + l\beta\tilde{\beta}\big)$, $\mathscr{B}_1(t) = \beta \mathscr{A}_1(t)$, $\mathscr{C}_1(t) = \tilde{\beta}\mathscr{A}_1(t)$, and $\mathscr{D}_1(t) = \beta\tilde{\beta}\mathscr{A}_1(t)$. Obviously, they are positive in $L_2(0,l)$. Their boundedness in $L_2(0,l)$ follows from the inequalities

$$\left\| A_1^{-1}(t) g \right\|_{0,\Omega}^2 \leq c_{13} \|g\|_{0,\Omega}^2 \qquad \forall g \in L_2(0,l), \qquad t \in [0,T],$$

where $\|\cdot\|_{0,\Omega}$ is the norm in $L_2(\Omega)$, $\Omega =]0,l[$, and

$$c_{13} = l^2 \max_{0 \leq t \leq T} \left[l^2 + 3(1 + l\beta)^2 \mathscr{A}_1^2(t) + l^2 (1 + l\beta)^2 \mathscr{C}_1^2(t) \right].$$

The operators $A_1^{-1}(t)$, $t \in [0,T]$, have the strong derivative

$$\frac{dA_1^{-1}(t)}{dt} g = \left(\dot{\mathscr{A}}_1(t) + \dot{\mathscr{B}}_1(t) x \right) \int_0^l g(s) ds + \left(\dot{\mathscr{C}}_1(t) + \dot{\mathscr{D}}_1(t) x \right) \int_0^l (l-s) g(s) ds$$

in $L_2(0,l)$, where the dots above functions stand for the first derivatives with respect to t. This strong derivative is a bounded operator in $L_2(0,l)$, since $\left\| \left(dA_1^{-1}(t)/dt \right) g \right\|_{0,\Omega}^2 \leq c_{14} \|g\|_{0,\Omega}^2$ for all $t \in [0,T]$, all $g \in L_2(0,l)$, $t \in [0,T[$, where

$$c_{14} = 4l^2 \max_{0 \leq t \leq T} \left[\dot{\mathscr{A}}_1^2(t) + \left(l^2/3\right) \dot{\mathscr{B}}_1^2(t) + \left(l^2/3\right) \dot{\mathscr{C}}_1^2(t) + \left(l^4/9\right) \dot{\mathscr{D}}_1^2(t) \right],$$

and satisfies inequality (3), since

$$-\left(\left(dA_1^{-1}(t)/dt\right)A_1(t)u, A_1(t)u\right)_{0,\Omega} \le c_{15}\left\|A_1^{1/2}(t)u\right\|_{0,\Omega}^2$$

for all $u \in D\left(A_1(t)\right)$, $t \in [0,T]$, where $(\cdot\,,\cdot)_{0,\Omega}$ is the inner product in $L_2(0,l)$ and

$$c_{15} = \max_{0\le t\le T}\Big\{\left(2\left|\dot{\mathscr{A}}_1(t)\right| + \sqrt{l}(1+l)\left|\dot{\mathscr{B}}_1(t)\right| + \sqrt{l}\left|\dot{\mathscr{C}}_1(t)\right| + l^2\left|\dot{\mathscr{D}}_1(t)\right|\right)\left(1+\tilde{\beta}\right),$$
$$\left(2\left|\dot{\mathscr{A}}_1(t)\right| + \sqrt{l}\left|\dot{\mathscr{B}}_1(t)\right| + \sqrt{l}(1+l)\left|\dot{\mathscr{C}}_1(t)\right| + l^2\left|\dot{\mathscr{D}}_1(t)\right|\right)(1+\beta),$$
$$\sqrt{l}\left|\dot{\mathscr{B}}_1(t)\right| + \sqrt{l}\left|\dot{\mathscr{C}}_1(t)\right| + 2l\left|\dot{\mathscr{D}}_1(t)\right|\Big\}.$$

Here we have used the relation

$$\left\|A_1^{1/2}(t)u\right\|_{0,\Omega}^2 = \frac{1}{1+\tilde{\beta}(t)}\left(\left|\frac{\partial u(t,l)}{\partial x}\right|^2 + \tilde{\beta}(t)|u(t,l)|^2\right) + \frac{1}{1+\beta(t)}\left(\left|\frac{\partial u(t,0)}{\partial x}\right|^2 + \beta(t)|u(t,0)|^2\right) + \left\|\frac{\partial u}{\partial x}\right\|_{0,\Omega}^2, \qquad t \in [0,T]. \tag{30}$$

The operators $dA_1^{-1}(t)/dt$ have the strong derivative

$$\frac{d^2A_1^{-1}(t)}{dt^2}g = \left(\ddot{\mathscr{A}}_1(t) + \ddot{\mathscr{B}}_1(t)x\right)\int_0^l g(s)ds + \left(\ddot{\mathscr{C}}_1(t) + \ddot{\mathscr{D}}_1(t)x\right)\int_0^l (l-s)g(s)ds$$

in $L_2(0,l)$ for all $t \in [0,T]$, which is bounded in $L_2(0,l)$, since

$$\left\|\left(d^2A_1^{-1}(t)/dt^2\right)g\right\|_{0,\Omega}^2 \le c_{16}\|g\|_{0,\Omega}^2 \qquad \forall g \in L_2(0,l)$$

for all $t \in [0,T]$, where the constant c_{16} is obtained from the constant c_{14} by the replacement of functions with a single dot by the same functions with double dots standing for their second derivatives with respect to t. The operators $d^2A_1^{-1}(t)/dt^2$ satisfy inequalities (4), since

$$\left|\left(\left(d^2A_1^{-1}(t)/dt^2\right)g, A_1(t)u\right)_{0,\Omega}\right| \le c_{17}||g||_{0,\Omega}\left\|A_1^{1/2}(t)u\right\|_{0,\Omega} \quad \forall g \in L_2(0,l), \quad \forall u \in D\left(A_1(t)\right)$$

for all $t \in [0,T]$, where

$$c_{17} = \sqrt{l}\sup_{0<t<T}\Big\{\left(\sqrt{3}\left|\ddot{\mathscr{A}}_1(t)\right| + l\left|\ddot{\mathscr{C}}_1(t)\right| + \sqrt{3}l\left|\ddot{\mathscr{B}}_1(t)\right| + l^2\left|\ddot{\mathscr{D}}_1(t)\right|\right)\sqrt{1+\tilde{\beta}},$$
$$\left(\sqrt{3}\left|\ddot{\mathscr{A}}_1(t)\right| + l\left|\ddot{\mathscr{C}}_1(t)\right|\right)\sqrt{1+\beta}, \sqrt{3}l\left|\ddot{\mathscr{B}}_1(t)\right| + l^{3/2}\left|\ddot{\mathscr{D}}_1(t)\right|\Big\}.$$

Obviously, the operators $A_k(t)$, $t \in [0,T]$, also satisfy Conditions IV–VI. Moreover, the Banach spaces V^{2k} are just the Sobolev spaces $W_2^{2k}(0,l)$ with their ordinary norms $\|\cdot\|_{2k,\Omega}$, $k = 0,\dots,m$, $V^2 = D(\tilde{A}_1)$, $V^0 = L_2(0,l)$. The Hilbert spaces $W^{2k}(t)$ are the closed subspaces $W_{2,\Delta(t)}^{2k}(0,l)$ of the Sobolev spaces $W_2^{2k}(0,l)$, namely, the sets

$$\left\{u \in W_2^{2k}(0,l):\ u \in (28),\ t \in [0,T],\ i = 0,\dots,k-1\right\}$$

equipped with the Hermitian norms $\|\cdot\|_{2k,t,\Omega}$ inherited from $W_2^{2k}(0,l)$, $k=1,\ldots,m$. The Hilbert spaces $W^{2k+1}(t)$, $t\in[0,T]$, are the spaces $W_{2,\Delta(t)}^{2k+1}(0,l)$, which are the closures of the sets

$$\left\{u\in W_2^{2k+2}(0,l):\ u\in(28),\ t\in[0,T],\ i=0,\ldots,k\right\}$$

in the Hermitian norms

$$\|u(t,x)\|_{2k+1,t,\Omega}=\left\|A_1^{1/2}(t)\partial^{2k}u(t,x)/\partial x^{2k}\right\|_{0,\Omega},\qquad k=0,\ldots,m-1.$$

The Hilbert spaces $W^{\alpha}(t)$, $t\in[0,T]$, for noninteger $\alpha\in\,]0,2m[$ are defined in a similar way.

For the mixed problems (27)–(29), for the Banach spaces $\mathscr{E}^m(G)$ of their strong solutions, we take the closures of the intersections of the closed subspaces of the Sobolev–Slobodetskii spaces $\mathscr{D}(L_m)=\{u\in\tilde{\mathscr{D}}(L_m):u\in(28)\}$, where

$$\tilde{\mathscr{D}}(L_m)=\left\{u\in\bigcap_{i=0}^{2m}W_2^{i,2m-2[(i+1)/2]}(G):\ v_s\equiv\frac{\partial^{2m-2-2[(s+1)/2]+s}u}{\partial t^s\partial x^{2m-2-2[(s+1)/2]}},\right.$$
$$\left.\frac{\partial v_s(t,0)}{\partial x}-\beta(t)v_s(t,0)=\frac{\partial v_s(t,l)}{\partial x}+\tilde{\beta}(t)v_s(t,l)=0,\ t\in[0,T],\ s=0,\ldots,2m-2\right\},$$

in the norms

$$|||u(t,x)|||_m=\left\{\sup_{0<t<T}\sum_{i=0}^{2m-1}\left\|\frac{\partial^i u(t,x)}{\partial t^i}\right\|^2_{m_i+1,t,\Omega}\right\}^{1/2}.$$

For the mixed problems (27)–(29), for the spaces of the right-hand sides $f(t,x)$ and the initial data $\varphi_j(x)$, we take the Hilbert spaces $\mathscr{F}^m(G)=L_2(G)\times W_{2,\Delta(0)}^{2m-1}(0,l)\times\cdots\times L_2(0,l)$ of functions $\mathscr{F}(t,x)=\{f(t,x),\,\varphi_0(x),\,\ldots,\,\varphi_{2m-1}(x)\}$ with Hermitian norms

$$\langle\|\mathscr{F}(t,x)\|\rangle_m=\left\{\int_0^T\|f(t,x)\|^2_{0,\Omega}dt+\sum_{j=0}^{2m-1}\|\varphi_j\|^2_{m_j+1,0,\Omega}\right\}^{1/2},$$

where, just as above, the Hilbert spaces $W_{2,\Delta(0)}^s(0,l)$ are the closures of the sets of all functions $u(x)$ in the Sobolev spaces $W_2^{2[(s+1)/2]}(0,l)$ satisfying condition (28) for $t=0$ and $i=0,\ldots,[(s+1)/2]$ in the Hermitian norms

$$\|u(x)\|_{s,0,\Omega}=\left\|A_1^{(s-2[s/2])/2}(0)\partial^{2[s/2]}u(x)/\partial x^{2[s/2]}\right\|_{0,\Omega},\qquad s=1,\ldots,2m-1.$$

By taking into account Remark 3 saying that condition (7) with $m>1$ becomes unnecessary in the case of constant coefficients a_k, from Theorem 1, Corollary 1, and Theorem 2, one can obtain the following theorem on the existence and uniqueness of strong solutions of problems (27)–(29) and their continuous dependence on the right-hand sides of the equations.

Theorem 3. *If the coefficients β and $\tilde{\beta}$ satisfy the above-mentioned conditions for the functions $f(t,x)\in L_2(G)$ and $\varphi_j(x)\in W_{2,\Delta(0)}^{m_j+1}(0,l)$, $j=0,\ldots,2m-1$, then the mixed problems (27)–(29) have a unique strong solution $u(t,x)\in C^{(2m-1)}([0,T],L_2(0,l))\cap\mathscr{E}^m(G)$ such that*

$$\begin{gathered}|||u(t,x)|||_m^2\le c_0(m)\langle\|\mathscr{F}(t,x)\|\rangle_m^2,\\ \mathscr{F}(t,x)=\{f(t,x),\varphi_0(x),\ldots,\varphi_{2m-1}(x)\},\qquad m=1,2,\ldots\end{gathered}\tag{31}$$

Note that Theorem 3 was announced in [8].

Remark 4. In these mixed problems, the requirement that the functions β and $\tilde{\beta}$ do not simultaneously vanish for any $t \in [0, T]$ is not important and is caused only by the fact that the inverse operators $A_1^{-1}(t)$ are bounded in $L_2(0, l)$ to simplify the verification of the assumptions of Theorems 1 and 2 and Corollary 1. If, in the proof of Theorem 3, we choose $(A_1(t) + \delta_1 I)^{-1}$, $\delta_1 > 0$, instead of $A_1^{-1}(t)$, then in this case, the functions β and $\tilde{\beta}$ can simultaneously vanish, and the assertion of Theorem 3 remains valid in this case.

7. DISCUSSION OF THE CAUCHY PROBLEMS

In the Hilbert space $H = L_2(\mathbb{R}^n)$, $n \geq 1$, condition (7) with $m > 1$ in Theorems 1 and 2 is valid for the differential operators

$$A(t) = (I - \Delta_x)^{p(t)}, \qquad p(t) > n/2, \qquad p(t) \in C^{(1)}[0, T],$$

where Δ_x is the Laplace operator with respect to $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with t-dependent domains $D(A(t)) = \left\{u(x) \in L_2(\mathbb{R}^n) : \ (I - \Delta_x)^{p(t)} u(x) \in L_2(\mathbb{R}^n)\right\}$, $t \in [0, T]$. Here fractional-order partial derivatives and fractional-order powers of the given operators $A(t)$ in the definition of the spaces $W^\alpha(t)$ are given by the expressions $A^{\alpha/(2m)}(t)u = F^{-1}\left[(1 + |\xi|^2)^{p(t)\alpha/(2m)} F[u]\right]$, $\alpha > 0$, for all $u \in D\left(A^{\alpha/(2m)}(t)\right)$ and $A^{-\alpha/(2m)}(t)g = F^{-1}\left[(1 + |\xi|^2)^{-p(t)\alpha/(2m)}\right] * g$, $\alpha > 0$, for all $g \in L_2(\mathbb{R}^n)$, where F and F^{-1} are the direct and inverse Fourier–Plancherel integral transforms and $*$ stands for the convolution of functions. By using the properties of the Fourier–Plancherel transforms, we find that $A^{1-1/(2m)}(t)\,(dA^{-1}(t)/dt) \in \mathscr{B}([0, T], \mathscr{L}(L_2(\mathbb{R}^n)))$ for $m > 1$, since

$$\begin{aligned}\left\|A^{1-1/(2m)}(t)\left(dA^{-1}(t)/dt\right) g\right\|^2 &= (p'(t))^2 \left\|F^{-1}\left[\left(1 + |\xi|^2\right)^{-p(t)/(2m)} \ln\left(1 + |\xi|^2\right) F[g]\right]\right\|^2 \\ &\leq \frac{(p'(t))^2}{(\varrho e)^2 (2\pi)^n} \left\|\left(1 + |\xi|^2\right)^{-p(t)/(2m)+\varrho} F[g]\right\|^2 \\ &\leq \frac{(p'(t))^2}{(\varrho e)^2 (2\pi)^n} \|F[g]\|^2 = \frac{(p'(t))^2}{(\varrho e)^2} \|g\|^2 \qquad \forall g \in L_2(\mathbb{R}^n)\end{aligned}$$

for all t provided that the parameter satisfying the estimate $\ln z \leq (1/\varrho e) z^\varrho$ for all $z \geq 1$ is $0 < \varrho \leq \min_{[0,T]} p(t)/(2m)$.

Unfortunately, condition (7) with $m > 1$, which provides the interpolation inequalities (8), is rarely valid for the elliptic differential operators $A_1(t)$ with t-dependent coefficients in boundary conditions. We show that it fails for the operators $A_1(t)$ in the mixed problems (27)–(29) with $m = 2$. For them, this condition with $m = 2$ is equivalent to the condition

$$A_1^{3/2}(t)\left(dA_1^{-2}(t)/dt\right) \in \mathscr{B}([0, T], \mathfrak{L}(L_2(0, l))),$$

which is not necessarily valid. On the right-hand sides of the relation

$$A_1^{3/2}(t) \frac{dA_1^{-2}(t)}{dt} = A_1^{1/2}(t) \frac{dA_1^{-1}(t)}{dt} + A_1^{3/2}(t) \frac{dA_1^{-1}(t)}{dt} A_1^{-1}(t), \tag{32}$$

the terms are not necessarily bounded operators if at least one of the coefficients $\beta(t)$ and $\tilde{\beta}(t)$ depends on t, since in this case the derivative $dA_1^{-1}(t)/dt$ can "lose" boundary conditions necessary for the square root $A_1^{1/2}(t)$. If the coefficients $\beta(t)$ and $\tilde{\beta}(t)$ in (28) with $m = 1$ vanish in some open neighborhood V_0 of the point t_0, then it is well known that the boundary conditions $\partial v(x)/\partial x|_{x=0} = 0$ and $\partial v(x)/\partial x|_{x=l} = 0$ for $t \in V_0$ are not necessarily satisfied by all functions $v(x) \in D\left(A_1^{1/2}(t)\right)$ in the domains of the operators $A_1^{1/2}(t)$, since in this case their graph norm is equivalent to the norm of the Sobolev space $W_2^1(0, l)$ [see (28) and (30)]. However, if there exists a $t_0 \in [0, T]$ such that $\beta(t_0) \neq 0$ and $\tilde{\beta}(t_0) \neq 0$, then, by virtue of their continuity, there exists a neighborhood V_0 of this point t_0 such that $\beta(t), \tilde{\beta}(t) \neq 0$ for all $t \in V_0$. Then, from the formula for the graph norm of the operators $A_1^{1/2}(t)$, the first boundary condition (28) for $m = 1$, and formula (30), we find that, for each $t \in V_0$, the functions $v(x) \in D\left(A_1^{1/2}(t)\right)$ are continuous with respect to x, and the derivatives $\partial v(x)/\partial x$ belong to the space $L_2(0, l)$ and have trace for $x = 0$ in the generalized sense as the limit

$$\partial v(x)/\partial x|_{x=0} = \lim_{n\to\infty} \partial v_n(x)/\partial x|_{x=0} = \beta(t) \lim_{n\to\infty} v_n(x)|_{x=0} = \beta(t)v(0)$$

of some functions $v_n(x) \in D\left(A_1(t)\right) = \{w \in W_2^2(0, l) : \ w \in (28) \text{ for } m = 1\}$ in $\mathbb{R}$ by virtue of the definition of the square root $A_1^{1/2}(t)$. The above-mentioned assertions are valid for the trace $\partial v(x)/\partial x|_{x=l}$ of all functions $v(x) \in D\left(A_1^{1/2}(t)\right)$, $t \in V_0$. Therefore, in this sense, for each $t \in V_0$, the boundary conditions $[\partial v(x)/\partial x - \beta(t)v(x)]|_{x=0} = 0$ and $\left[\partial v(x)/\partial x + \tilde{\beta}(t)v(x)\right]\Big|_{x=l} = 0$ are preserved for all functions $v(x) \in D\left(A_1^{1/2}(t)\right)$, while they can fail for the derivative $dA_1^{-1}(t)/dt$ in the case of the coefficients $\beta(t)$ and (or) $\tilde{\beta}(t)$ depending on $t \in V_0$. Indeed, if, for example, $\beta(t) = t$, $\tilde{\beta}(t) = 1$, and $l = 1$, then the functions

$$v(x) = \frac{dA_1^{-1}(t)}{dt} g = \frac{x-2}{(2t+1)^2} \int_0^1 (2-s)g(s)ds \qquad \forall g \in L_2(0,1)$$

do not satisfy the t-dependent boundary condition

$$\left[\frac{\partial v(x)}{\partial x} - tv(x)\right]\Bigg|_{x=0} = \frac{1}{2t+1} \int_0^1 (2-s)g(s)ds = 0 \qquad \forall t > 0,$$

for example, for $g(x) = 1$. Therefore, for such $\beta(t)$, $\tilde{\beta}(t)$, and l, the first and, all the more, second products of operators on the right-hand side in (32) cannot be bounded in $L_2(0, l)$; in addition, this disadvantage cannot cancel by summation. Note that, for the above-mentioned values of $\beta(t) = t$, $\tilde{\beta}(t) = 1$, and $l = 1$, the functions $v(x) = \left(dA_1^{-1}(t)/dt\right) g$ satisfy the t-independent second boundary condition $[\partial v(x)/\partial x + v(x)]|_{x=1} = 0$, $t \in [0, T]$, for all $g \in L_2(0, 1)$. How to eliminate or, at least, weaken condition (7) for $m > 1$ in the Cauchy problems (1), (2) for the case in which the operators $A_k(t)$ depend on t, have t-dependent domains, and do not commute with each other?

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