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PARTIAL DIFFERENTIAL EQUATIONS

A Generalization of the Lions Theory for First-Order Evolution Differential Equations with Smooth Operator Coefficients: II

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The present paper is a continuation of [1], where Theorem 1.1 in [2, p. 129] on the existence and uniqueness of solutions of the Cauchy problem for first-order operator-differential equations with variable domains of symmetric leading parts was generalized to the case of nonsymmetric leading parts of smooth operator coefficients and a new class of even-order partial differential operators with symmetric leading parts and variable domains satisfying the assumptions of Theorems 1, 3, and 4 in [1] was constructed. The well-posed solvability of mixed problems was proved for partial differential equations with symmetric and nonsymmetric leading parts and time-dependent boundary conditions. We continue the numbering of sections, assertions, remarks, formulas, and constants in [1].

Continuing Section 5 in [1], we first give a second example of variable domains $D(A(t))$ for odd-order partial differential operators $A(t)$ satisfying property III_1 in Theorem 4 in [1].

2. In the theory of boundary value problems, we use this example for nonsymmetric leading parts of the operators $A(t)$. If all coefficients $a_{i,j}$ belong to $B\left([0,T],\mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)\right)$, then the boundary conditions

$$\begin{gathered}\Gamma_j(t)u \equiv \gamma_j u\left(x'\right) - \sum_{i\in J_{-(2m+1)}}^{i<j} a_{i,j}(t)\gamma_i u\left(x'\right) = 0, \quad x'\in S, \quad j\in J_{2m+1},\\ \Gamma_{j,k}(t)u \equiv \gamma_j u\left(x'\right) - \sum_{i\in J_{-(2m+1)}}^{i<j} a_{i,j}(t)\gamma_i u\left(x'\right) = 0, \quad x'\in S_k^-, \quad 1\le k\le n, \quad j\in J_{2m+1}^-,\\ t\in[0,T], \quad m=0,1,\dots,\end{gathered} \tag{23}$$

make sense in the Sobolev space $W_2^{2m+1}(\Omega)$, where

$$\begin{aligned}J_{2m+1} &= \{j_s\in[0,\dots,2m]:\ 1\le s\le q\},\\ J_{2m+1}^- &= \{j_s\in([0,\dots,2m]\backslash J_{2m+1}):\ q<s\le m+1\},\\ J_{-(2m+1)} &= [0,\dots,2m]\backslash\left(J_{2m+1}\cup J_{2m+1}^-\right),\end{aligned}$$

and S_k^- are the sets of all points x' of the boundary S with negative direction cosines of the angles between the outward normal ν to S and the axes Ox_k, $1\le k\le n$. We set

$$D(A(t)) = \left\{u(t)\in W_2^{2m+1}(\Omega):\ u(t)\in(23),\ \tilde{A}(t)u(t)\in L_2(\Omega)\right\},$$

where $\tilde{A}(t)=\sum_{|\alpha|\le 2m+1} a_\alpha(t,x)D_x^\alpha$. Consider the operators

$$A(t): L_2(\Omega)\supset D(A(t))\ni u\to\tilde{A}(t)u\in L_2(\Omega), \qquad t\in[0,T].$$

For each $t \in [0,T]$, the domains $D(A(t))$ are closed in the space $W_2^{2m+1}(\Omega)$. The Sobolev space $W_2^{2m+1}(\Omega)$ can be expanded in the direct sum

$$W_2^{2m+1}(\Omega) = \mathring{W}_2^{2m+1}(\Omega) \oplus \mathscr{W}_2^{2m+1}(\Omega).$$

The projection operators are defined with the use of the boundary operators $\Gamma_j(t)$, $j \in J_{2m+1}$, and $\Gamma_{j,k}(t)$, $1 \le k \le n$, $j \in J^-_{2m+1}$.

Definition 5. The projection of a function $u \in W_2^{2m+1}(\Omega)$ are defined by $P(t)u = u(t)$, where $u(t) \in (23)$ and $\inf_{v(t)} \|u - v(t)\|_{2m+1,\Omega} = \|u - u(t)\|_{2m+1,\Omega}$, the infimum being taken over all $v(t) \in D(A(t))$, $t \in [0,T]$.

The following assertion describes the action of the projections $P(t)$.

Lemma 5. *Let the projection operators $P(t)$ be defined on $D(A(t))$ in the space $W_2^{2m+1}(\Omega)$. For each function $u \in W_2^{2m+1}(\Omega)$, there exists a unique function $\tilde{u}(t) \in D(A(t))$ with boundary values*

$$\begin{aligned}
&\gamma_j \tilde{u}(t) = \sum_{i \in J_{-(2m+1)}}^{i<j} a_{i,j}(t)\gamma_i u, \quad j \in J_{2m+1},\\
&\gamma_j \tilde{u}(t) = \gamma_j u, \quad j \notin J_{2m+1} \cup J^-_{2m+1}, \quad \text{on} \quad S,\\
&\gamma_j \tilde{u}(t) = \sum_{i \in J_{-(2m+1)}}^{i<j} a_{i,j}(t)\gamma_i u \quad \text{on} \quad S_k^-,\\
&\gamma_j \tilde{u}(t) = \gamma_j u \quad \text{on} \quad S_k^+ = S \backslash S_k^-, \quad 1 \le k \le n, \quad j \in J^-_{2m+1},
\end{aligned} \tag{24}$$

in $W_2^{2m-j+1/2}(S)$, $0 \le j \le 2m$, such that $P(t)u = \tilde{u}(t)$ in $W_2^{2m+1}(\Omega)$ for all $t \in [0,T]$ provided that

$$\gamma_j \tilde{u}(t) \in W_2^{2m-j+1/2}(S) \quad \forall j \in J^-_{2m+1} \tag{25}$$

in the data (24) for all $u \in W_2^{2m+1}(\Omega)$.

Proof. The proof is similar to that of Lemma 1 in [1].

For each function $u \in W_2^{2m+1}(\Omega)$, the function $P(t)u = \tilde{u}(t)$ is strongly continuous with respect to t on $[0,T]$ in $W_2^{2m+1}(\Omega)$ provided that the coefficients $a_{i,j}$ are strongly continuous with respect to t in $\mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)$, $i \in J_{-(2m+1)}$, $j \in J_{2m+1}$. The projections $P(t)$ have a weak derivative with respect to t in $W_2^{2m+1}(\Omega)$ provided that the coefficients $a_{i,j}$ are weakly differentiable with respect to t in $\mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)$, $i \in J_{-(2m+1)}$, $j \in J_{2m+1}$.

Lemma 6. *Let the coefficients $a_{i,j}$ be strongly continuous with respect to t in $\mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)$ and have weak derivatives $a'_{i,j} \in L_\infty\left(]0,T[, \mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)\right)$, $i \in J_{-(2m+1)}$, $j \in J_{2m+1}$, with respect to t. For each function $u \in W_2^{2m+1}(\Omega)$, for almost all t, there exists a function $\tilde{w}(t) \in \mathscr{W}_2^{2m+1}(\Omega)$ with boundary values*

$$\begin{aligned}
&\gamma_j \tilde{u}'(t) = \sum_{i \in J_{-(2m+1)}}^{i<j} a'_{i,j}(t)\gamma_i u, \qquad j \in J_{2m+1},\\
&\gamma_j \tilde{u}'(t) = 0, \qquad j \notin J_{2m+1} \cup J^-_{2m+1}, \quad \text{on} \quad S,\\
&\gamma_j \tilde{u}'(t) = \sum_{i \in J_{-(2m+1)}}^{i<j} a'_{i,j}(t)\gamma_i u \quad \text{on} \quad S_k^-,\\
&\gamma_j \tilde{u}'(t) = 0 \quad \text{on} \quad S_k^+, \quad 1 \le k \le n, \quad j \in J^-_{2m+1},
\end{aligned} \tag{26}$$

in $W_2^{2m-j+1/2}(S)$, $0 \le j \le 2m$, *where* $\tilde{u}(t)$ *is the function in Lemma* 5 *such that*

$$P'(t)u = \tilde{w}(t) \quad \text{for almost all} \quad t \tag{27}$$

in the space $W_2^{2m+1}(\Omega)$ *provided that*

$$\gamma_j \tilde{u}'(t) \in W_2^{2m-j+1/2}(S) \quad \forall j \in J_{2m+1}^- \tag{28}$$

in the data (26) *for all* $u \in W_2^{2m+1}(\Omega)$.

The proof is similar to that of Lemma 2 in [1]. By using Lemmas 5 and 6, one can prove the following assertion.

Theorem 6. *Let the coefficients* $D_x^\beta a_\alpha \in C([0,T] \times \Omega)$, $|\beta| \le |\alpha|$, *have the derivative* $\partial a_\alpha/\partial t \in L_\infty(]0,T[\times\Omega)$, $|\alpha| \le 2m+1$, *for almost all* t, *let the differential expressions* $\tilde{A}(t)$ *with the boundary conditions* (23) *satisfy properties* I_1 *and* II_1 *in Theorem* 4, *and let* $\tilde{A}_0(t) = \tilde{A}(t) + c_0 I$ *be coercive in* $W_2^{2m+1}(\Omega)$ $\big($*i.e., the inequality*

$$\|u(t)\|_{2m+1,\Omega} \le c_9 \left\|\tilde{A}_0(t)u(t)\right\|_{0,\Omega} \quad \forall u(t) \in D(A(t)) \tag{29}$$

is valid for all $t \in [0,T]$, *where* $c_9 > 0$ *is a constant independent of* u *and* $t\big)$. *Let the adjoint operators in* $L_2(\Omega)$ *be given by some differential expressions*

$$\tilde{A}^*(t) = \sum_{|\alpha| \le 2m+1} a_\alpha^*(t,x) D_x^\alpha, \qquad \tilde{A}_0^*(t) = \tilde{A}^*(t) + c_0 I,$$

with coefficients $D_x^\beta a_\alpha^* \in C([0,T] \times \Omega)$, $\partial a_\alpha^*/\partial t \in L_\infty(]0,T[\,\times\Omega)$, $|\beta| \le |\alpha| \le 2m+1$, *and some boundary conditions* $\left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}$ *satisfying the inequalities*

$$\left\|\left(\partial \tilde{A}^*(t)/\partial t\right) v(t)\right\|_{0,\Omega} \le c_{10} \left\|\tilde{A}_0^*(t) v(t)\right\|_{0,\Omega} \quad \forall v(t) \in D\left(A^*(t)\right), \tag{30}$$

where $c_{10} \ge 0$ *is a constant independent of* v *and* t. *If all coefficients* $a_{i,j}$ *are strongly continuous with respect to* t *in* $\mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)$ *and have weak derivatives* $a'_{i,j} \in L_\infty\left(]0,T[, \mathfrak{L}\left(W_2^{2m-i+1/2}(S)\right)\right)$ *with respect to* t *in* $W_2^{2m-i+1/2}(S)$ *for almost all* t *such that*

$$\begin{aligned} \left\|a'_{i,j}(t)\gamma_i u\right\|_{0,S} &\le c'_{i,j}[u]_{(t)} \quad \forall u \in D(A(t)), \quad i \in J_{-(2m+1)}, \quad j \in J_{2m+1}, \\ \left\|a'_{i,j}(t)\gamma_i u\right\|_{0,S_k^-} &\le c_{i,j}^{(k)}[u]_{(t)} \quad \forall u \in D(A(t)), \quad i \in J_{-(2m+1)}, \quad 1 \le k \le n, \quad j \in J_{2m+1}^-, \end{aligned} \tag{31}$$

where $c'_{i,j}, c_{i,j}^{(k)} \ge 0$ *are constants independent of* u *and* t, *and the continuation conditions* (25) *and* (28) *are valid, then the projections* $P(t) : W_2^{2m+1}(\Omega) \to D(A(t))$ *satisfy property* III_1 *in Theorem* 4.

The proof is similar to that of Theorem 5 in [1]. We only indicate the crucial points.

1. Property (i) of the projections $P(t)$ in [1] follows from Lemmas 5 and 6.

2. The weak continuity of the function $u(t) = A_0^{-1}(t)g$ with respect to t on the interval $[0,T]$ in $W_2^{2m+1}(\Omega)$ for each $g \in L_2(\Omega)$ follows from the estimates (29) and the continuity of the coefficients a_α^* of the adjoint differential operators with respect to t. This permits one to prove the strong continuity of the functions $u(t)$ with respect to t on $[0,T]$ in $W_2^{2m+1}(\Omega)$ with the use of the continuity of the coefficients a_α of the original differential operators with respect to t. One can readily show that $\tau^{-1}P(t+\tau)(u(t+\tau) - u(t)) \to w(t)$ weakly in $W_2^{2m+1}(\Omega)$ for almost all t as $\tau \to 0$; therefore, by Lemma 6, there exists a weak derivative of $u'(t)$ with respect to t in $W_2^{2m+1}(\Omega)$. In this connection, Lemma 4, which is valid in $W_2^{2m+1}(\Omega)$ as well, is essentially used in the proof. The proof of property (ii) of the projections $P(t)$ in [1] is complete.

Let us compute the weak derivative of $u'(t)$ with respect to t.

Lemma 7. *Let the assumptions of Theorem* 6 [*without conditions* (30) *and* (31)] *be satisfied. The functions* $u(t) = A_0^{-1}(t)g$ *have the weak derivative*

$$u'(t) = -A_0^{-1}(t)\left(\partial\tilde{A}(t)/\partial t\right)u(t) - A_0^{-1}(t)\tilde{A}_0(t)\tilde{w}(t) + \tilde{w}(t), \tag{32}$$

$$P(t)u'(t) = -A_0^{-1}(t)\left(\partial\tilde{A}(t)/\partial t\right)u(t) - A_0^{-1}(t)\tilde{A}_0(t)\tilde{w}(t) \tag{33}$$

with respect to t *in* $W_2^{2m+1}(\Omega)$ *for all* $g \in L_2(\Omega)$ *for almost all* t, *where the function* $\tilde{w}(t) \in \mathscr{W}_2^{2m+1}(\Omega)$ *has the boundary data* (26) *with* $u = u(t)$ *for almost all* t.

The proof is similar to that of Lemma 3 in [1].

3. By taking into account (27) and (33), by using inequalities (30), (31) and analogs of inequalities (21) and (22) in [1], and by following the first example of domains $D(A(t))$ in [1], we can derive the estimate (8) from [1]. The proof of property (iii) of the projections $P(t)$ in [1] is complete.

Remark 8. Theorem 6 generalizes Theorem 6.1 in [2, p. 142] for symmetric operators to the case of nonsymmetric operators. If all coefficients $a_{i,j}$ of the boundary conditions are independent of t, then it follows from Eq. (14) in [1] and (26) that $\tilde{w} = 0$; consequently, by Eq. (15) in [1] and (32), $u'(t) = -A_0^{-1}(t)\left(\partial\tilde{A}(t)/\partial t\right)u(t)$ for almost all t in both examples of domains $D(A(t))$. If, in addition, the coefficients a_α in the equations are independent of t, then $u'(t) = 0$ for all $t \in [0, T]$.

6. APPLICATIONS

Let us apply the abstract results obtained in the preceding and this papers to the analysis of well-posed solvability of boundary value problems for linear partial differential equations with variable boundary conditions. Let us indicate two new classes of well-posed mixed problems for partial differential equations. The first class for parabolic equations with time-dependent boundary conditions contains mixed problems not studied by J.-L. Lions. The second class for nonclassical equations with time-dependent boundary conditions has not been considered by anybody yet.

1. Let us analyze the first class of mixed problems for even-order parabolic partial differential equations under time-dependent boundary conditions. In the cylinder $G =]0, T[\times\Omega$ of the variables t and $x = (x_1, \dots, x_n)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $S \in C^\infty$, consider the parabolic equations

$$\frac{\partial u(t,x)}{\partial t} + \sum_{|\alpha|\le 2m} a_\alpha(t,x)D_x^\alpha u(t,x) = f(t,x), \qquad m = 1, 2, \dots, \tag{34}$$

with the boundary conditions (9) on S for $t \in [0, T]$ and the initial condition

$$u(0,x) = u_0(x), \qquad x \in \Omega. \tag{35}$$

Let the differential expressions $\tilde{A}(t) = \sum_{|\alpha|\le 2m} a_\alpha(t,x)D_x^\alpha$ with the boundary conditions (9) for $t \in [0, T]$ satisfy the following conditions.

I_2. The coefficients a_α in the equations satisfy the inclusion $D_x^\beta a_\alpha \in C(\Omega)$, $|\beta| \le |\alpha| \le 2m$, for each $t \in [0, T]$, and the system $\{\Gamma_j(t)\}_{J_{2m}}$ of boundary operators induced by the boundary conditions (9) is normal; i.e., all orders $j_s \in J_{2m}$ are distinct, and the boundary S is not their characteristic surface [no boundary operator $\Gamma_j(t)$ is tangential to S].

For all $t \in [0, T]$, the differential operators $\tilde{A}(t)$ and the system $\{\Gamma_j(t)\}_{J_{2m}}$ of boundary operators form an elliptic self-adjoint boundary value problem $\left(\tilde{A}(t), \{\Gamma_j(t)\}_{J_{2m}}\right)$ in $\bar{\Omega}$. The ellipticity of this problem means that the operators $\tilde{A}(t)$ are properly elliptic in $\bar{\Omega}$ and are related to $\{\Gamma_j(t)\}_{J_{2m}}$ by the well-known Shapiro–Lopatinskii condition. The self-adjointness of this problem means that the operators $\tilde{A}(t)$ coincide with their formal adjoints in $L_2(\Omega)$ on the sets

$$W_2^{2m}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m}}\right) = \left\{u(t) \in W_2^{2m}(\Omega):\ u(t) \in (9)\right\}, \qquad t \in [0, T].$$

The operators $\tilde{A}_0(t) = \tilde{A}(t) + c_0 I$ are coercive in $L_2(\Omega)$ on $W_2^{2m}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m}}\right)$ for some $c_0 \geq 0$ and for each $t \in [0, T]$; i.e., inequality (10) is valid.

III_2. All coefficients a_α of the equations belong to $C(G)$, and their derivative $\partial a_\alpha / \partial t$ belongs to $L_\infty(G)$. All coefficients $a_{i,j}$ of the boundary conditions belong to $C[0, T]$, their derivative $\partial a_{i,j}/\partial t$ belong to $L_\infty(0, T)$, and they satisfy the estimates (12).

Since inequalities (5) and (11) are necessarily valid for self-adjoint coercive differential operators $\tilde{A}_0(t)$ on $D(A(t))$, it follows that conditions I_2 and III_2 provide the validity of properties I_1–III_1 in $V = W_2^{2m}(\Omega)$ and hence of conditions I–III in $H = L_2(\Omega)$. This, together with Theorems 1–5, implies the following assertion.

Theorem 7. *If conditions* I_2 *and* III_2 *are satisfied, then for each*

$$f \in L_2\left(]0, T[, W_2^{-m}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m}}\right)\right)$$

and for $u_0 \in L_2(\Omega)$, *there exists the unique weak solution* $u \in L_2(G)$ *of the mixed problem* (34), (9), *and* (35).

Here $W_2^{-m}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m}}\right)$ are the antidual spaces of the Hilbert spaces $W_2^{m}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m}}\right)$, which are obtained by the closure of the sets $W_2^{2m}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m}}\right)$ in the Hermitian norms $[\cdot]_{(t)}$ corresponding to these problems [see (3)].

A special case of problem (34), (9), (35) is given by the mixed problem

$$\frac{\partial u}{\partial t} - a(t)\frac{\partial^6 u}{\partial x^6} = f(t, x), \qquad 0 < t < T, \qquad 0 < x < l, \tag{34_1}$$

$$\begin{gathered}
\frac{\partial^5 u(0)}{\partial x^5} = a_1(t) u(0), \qquad \frac{\partial^4 u(0)}{\partial x^4} = -a_2(t)\frac{\partial u(0)}{\partial x}, \\
\frac{\partial^3 u(0)}{\partial x^3} = a_3(t)\frac{\partial^2 u(0)}{\partial x^2}, \qquad \frac{\partial^5 u(l)}{\partial x^5} = -a_1(t) u(l), \\
\frac{\partial^4 u(l)}{\partial x^4} = a_2(t)\frac{\partial u(l)}{\partial x}, \qquad \frac{\partial^3 u(l)}{\partial x^3} = -a_3(t)\frac{\partial^2 u(l)}{\partial x^2}, \qquad 0 \leq t \leq T,
\end{gathered} \tag{9_1}$$

$$u(0, x) = u_0(x), \qquad 0 < x < l, \tag{35_1}$$

where $a \in C[0, T]$ is a strictly positive coefficient, $\partial a/\partial t \in L_\infty(0, T)$, the coefficients $a_i \in C[0, T]$ are nonnegative, and $\partial a_i/\partial t \in L_\infty(0, T)$, $i = 1, 2, 3$.

The differential operators $\tilde{A}(t) = -a(t)\partial^6/\partial x^6$ with the boundary conditions (9_1) are self-adjoint in $L_2(0, l)$. If $c_0 = 1$ and $t \in [0, T]$ is arbitrary, then, obviously, the operators $\tilde{A}_0(t) = \tilde{A}(t) + c_0 I$ are coercive in $L_2(0, l)$, i.e., satisfy inequality (10). We have inequality (12), since, by virtue of the boundary conditions (9_1),

$$\begin{aligned}
[u]^2_{(t)} &= \left(\tilde{A}_0(t) u, u\right)_{0,\Omega} \\
&= a a_1 |u(0)|^2 + a a_1 |u(l)|^2 + a a_2 |\partial u(0)/\partial x|^2 + a a_2 |\partial u(l)/\partial x|^2 + a a_3 \left|\partial^2 u(0)/\partial x^2\right|^2 \\
&\quad + a a_3 \left|\partial^2 u(l)/\partial x^2\right|^2 + a \left\|\partial^3 u/\partial x^3\right\|^2_{0,\Omega} + c_0 \|u\|^2_{0,\Omega} \qquad \forall u \in W_2^6\left(]0, l[; (9_1)_t\right).
\end{aligned}$$

The verification of conditions I_2 and III_2 is complete; therefore, Theorem 7 implies the following assertion.

Theorem 8. *If the coefficients satisfy the conditions* $0 < a_0 \leq a(t)$, $a_i(t) \geq 0$, $a(t) \in C[0, T]$, $\partial a(t)/\partial t \in L_\infty(0, T)$, $a_i \in C[0, T]$, *and* $\partial a_i/\partial t \in L_\infty(0, T)$, $i = 1, 2, 3$, *then for arbitrary functions*

$$f \in L_2\left(]0, T[, W_2^{-3}\left(]0, l[; (9_1)_t\right)\right)$$

and $u_0 \in L_2(0, l)$, *there exists a unique weak solution* $u \in L_2(]0, T[\times]0, l[)$ *of the mixed problem* (34_1), (9_1), (35_1).

Here $W_2^{-3}(]0,l[;(9_1)_t)$ are the antidual spaces of the Hilbert spaces $W_2^3(]0,l[;(9_1)_t)$, which are obtained by the closure of the set of all functions $u(t) \in W_2^6(0,l)$ satisfying the boundary conditions (9_1) in the Hermitian norms $[u]_{(t)} = \sqrt{(A_0(t)u,u)_{0,\Omega}}$ corresponding to this problem.

Remark 9. Theorem 6.1 in [2, p. 142] does not apply to problem (34_1), (9_1), (35_1), since the square roots of the self-adjoint positive definite operators $\tilde{A}_0(t) = -a(t)\partial^6/\partial x^6 + c_0 I$ in $L_2(0,l)$ with the boundary conditions (9_1) are only integro-differential operators.

2. Let us analyze the second class of mixed problems for odd-order partial differential equations with time-dependent boundary conditions. In the cylinder $G =]0,T[\times \Omega$ of the variables t and $x = (x_1,\ldots,x_n)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $S \in C^\infty$, consider the equations

$$\frac{\partial u(t,x)}{\partial t} + \sum_{|\alpha|\le 2m+1} a_\alpha(t,x) D_x^\alpha u(t,x) = f(t,x), \qquad m = 0,1,\ldots, \tag{36}$$

with the boundary conditions (23) on S for $t \in [0,T]$ and with the initial condition (35).

Let the differential expressions $\tilde{A}(t) = \sum_{|\alpha|\le 2m+1} a_\alpha(t,x)D_x^\alpha$ with the boundary conditions (23) satisfy the following conditions.

I_3. The coefficients a_α of the equations satisfy the condition

$$D_x^\beta a_\alpha \in C(\Omega), \qquad |\beta| \le |\alpha| \le 2m+1,$$

for each $t \in [0,T]$, the differential operators $\tilde{A}(t)$ are lower semibounded in $L_2(\Omega)$ on

$$W_2^{2m+1}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m+1}\cup J_{2m+1}^-}\right) = \left\{u(t) \in W_2^{2m+1}(\Omega):\ u(t) \in (23)\right\},$$

and their adjoint operators in $L_2(\Omega)$ on $W_2^{2m+1}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m+1}\cup J_{2m+1}^-}\right)$ are given by some differential expressions $\tilde{A}^*(t) = \sum_{|\alpha|\le 2m+1} a_\alpha^*(t,x)D_x^\alpha$ with some adjoint boundary conditions $\left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}$.

The coefficients a_α^* of the adjoint equations satisfy the condition

$$D_x^\beta a_\alpha^* \in C(\Omega), \qquad |\beta| \le |\alpha| \le 2m+1,$$

for all $t \in [0,T]$, and the differential operators $\tilde{A}^*(t)$ are also lower semibounded in $L_2(\Omega)$ on

$$W_2^{2m+1}\left(\Omega; \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right) = \left\{v(t) \in W_2^{2m+1}(\Omega):\ v(t) \in \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right\}.$$

II_3. The operators $\tilde{A}_0(t) = \tilde{A}(t) + c_0 I$ satisfy the inequalities

$$\left|\left(\tilde{A}_0(t)u,w\right)_{0,\Omega}\right| \le c_{11}[u]_{(t)} \left\|\tilde{A}_0(t)w\right\|_{0,\Omega} \quad \forall u,w \in W_2^{2m+1}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m+1}\cup J_{2m+1}^-}\right) \tag{37}$$

for some $c_0 > 0$ and for each $t \in [0,T]$, where $c_{11} \ge 0$ is a constant independent of u, w, and t.

III_3. All coefficients a_α belong to $C(G)$, and their derivative $\partial a_\alpha/\partial t$ belongs to $L_\infty(G)$. The operators $\tilde{A}_0(t)$ are coercive in $L_2(\Omega)$ for any $t \in [0,T]$; i.e.,

$$\|u\|_{2m+1,\Omega} \le c_{12}\left\|\tilde{A}_0(t)u\right\|_{0,\Omega} \quad \forall u \in W_2^{2m+1}\left(\Omega; \{\Gamma_j(t)\}_{J_{2m+1}\cup J_{2m+1}^-}\right),$$

where $c_{12} \ge 0$ is a constant independent of u and t. All coefficients a_α^* belong to $C(G)$, their derivative $\partial a_\alpha^*/\partial t$ belongs to $L_\infty(G)$, and

$$\left\|\left(\partial \tilde{A}^*(t)/\partial t\right) v\right\|_{0,\Omega} \le c_{13}\left\|\tilde{A}_0^*(t)v\right\|_{0,\Omega} \quad \forall v \in W_2^{2m+1}\left(\Omega; \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right), \tag{38}$$

where $c_{13} \geq 0$ is a constant independent of v and t. All coefficients $a_{i,j}$ of boundary conditions belong to $C[0,T]$, their derivative $\partial a_{i,j}/\partial t$ belongs to $L_\infty(0,T)$, they satisfy inequality (31), and functions admit the continuations (25) and (28) from S_k^- to the entire set $\bar{\Omega}$.

Since conditions I_3–III_3 provide the validity of properties I_1—III_1 in $V = W_2^{2m+1}(\Omega)$ and hence of conditions I–III in $H = L_2(\Omega)$, from Theorems 1–4 and 6, we obtain the following assertion.

Theorem 9. *If conditions* I_3–III_3 *are satisfied, then for arbitrary functions*

$$f \in L_2\left(]0,T[, W_2^{-m}\left(\Omega; \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right)\right)$$

and $u_0 \in L_2(\Omega)$, *there exists a unique weak solution* $u \in L_2(G)$ *of the mixed problems* (36), (23), (35).

Here $W_2^{-m}\left(\Omega; \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right)$ are the antidual spaces of the Hilbert spaces $W_2^{m}\left(\Omega; \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right)$, which are obtained by the closure of the sets

$$W_2^{2m+1}\left(\Omega; \left\{\Gamma_j^*(t)\right\}_{J_{2m+1}^*}\right)$$

in Hermitian norms $\langle\cdot\rangle_{(t)}$, $t \in [0,T]$, corresponding to these problems (see formula (4) in [1]).

A special case of problem (36), (23), (35) is given by the following mixed problem for the linearized Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} - a(t)\frac{\partial^3 u}{\partial x^3} = f(t,x), \qquad 0 < t < T, \qquad 0 < x < l, \tag{36_1}$$

$$\frac{\partial^2 u(t,0)}{\partial x^2} = a_1(t)u(t,0), \qquad \frac{\partial^2 u(t,l)}{\partial x^2} = -a_2(t)u(t,l), \qquad \frac{\partial u(t,0)}{\partial x} = 0, \qquad 0 \leq t \leq T, \tag{23_1}$$

$$u(0,x) = u_0(x), \qquad 0 < x < l, \tag{35_1}$$

where $a \in C[0,T]$ is a strictly positive coefficient, $\partial a/\partial t \in L_\infty(0,T)$, $a_i(t) \in C[0,T]$ are nonnegative coefficients, $\partial a_i/\partial t \in L_\infty(0,T)$, $i = 1,2$, and

$$\begin{aligned} &\text{if} \quad \exists t_0 \in [0,T] \quad \text{such that} \quad a_1(t_0) = 0 \quad (\text{respectively}, \quad a_2(t_0) = 0), \\ &\text{then} \quad a_1 \equiv 0 \quad (\text{respectively}, \quad a_2 \equiv 0). \end{aligned} \tag{39}$$

The differential operators $-\tilde{A}(t) = a(t)\partial^3/\partial x^3$ with the boundary conditions (23_1) are dissipative in $L_2(0,l)$. Their adjoint operators in $L_2(0,l)$ are given by the differential expressions

$$-\tilde{A}^*(t) = -a(t)\partial^3/\partial x^3$$

with the boundary conditions

$$\frac{\partial^2 v(t,0)}{\partial x^2} = -a_1(t)v(t,0), \qquad \frac{\partial^2 v(t,l)}{\partial x^2} = a_2(t)v(t,l), \qquad \frac{\partial v(t,l)}{\partial x} = 0, \qquad 0 \leq t \leq T, \tag{40}$$

which are also dissipative in $L_2(0,l)$. Inequality (37) with $c_{11} = \sqrt{2}$ and $m = 1$ is valid for $c_0 = 1$ and for any $t \in [0,T]$, since

$$\left(\mathrm{Re}\,\left(\tilde{A}_0(t)u, u\right)_{0,\Omega}\right)^{1/2} = \left(aa_1|u(0)|^2 + aa_2|u(l)|^2 + (a/2)|\partial u(l)/\partial x|^2 + \|u\|_{0,\Omega}^2\right)^{1/2},$$

$$\begin{aligned} \left\|\tilde{A}_0(t)w\right\|_{0,\Omega}^2 &= 2aa_1|w(0)|^2 + 2aa_2|w(l)|^2 + a|\partial w(l)/\partial x|^2 \\ &+ a^2\left\|\partial^3 w/\partial x^3\right\|_{0,\Omega}^2 + \|w\|_{0,\Omega}^2. \end{aligned}$$

Inequalities (38) are also valid. By virtue of condition (39), we have (31). Conditions (25) and (28) on the continuation of functions from the boundary on the entire domain are necessarily valid for a segment [3]. The verification of conditions I_3–III_3 is complete; therefore, Theorem 9 implies the following assertion.

Theorem 10. *If* $0 < a_0 \le a(t)$, $a_1(t) \ge 0$, $a_2(t) \ge 0$, $a(t) \in C[0,T]$, $\partial a(t)/\partial t \in L_\infty(0,T)$, $a_i \in C[0,T]$, $\partial a_i/\partial t \in L_\infty(0,T)$, $i = 1, 2$, *and condition* (39) *is satisfied, then for each function* $f \in L_2(]0,T[,\ W_2^{-1}(]0,l[;(40)_t))$ *and* $u_0 \in L_2(0,l)$, *there exists a unique weak solution* u *belonging to* $L_2(]0,T[\times]0,l[)$ *of the mixed problem* (36_1), (23_1), (35_1).

Here $W_2^{-1}(]0,l[;(40)_t)$ are the antidual spaces of the Hilbert spaces $W_2^1(]0,l[;(40)_t)$, which are obtained by the closure of the set of all functions $u(t) \in W_2^3(0,l)$ satisfying the boundary conditions (40) in the norms

$$\left(\mathrm{Re}\left(\tilde{A}_0^*(t)v, v\right)_{0,\Omega}\right)^{1/2} = \left(aa_1|v(0)|^2 + aa_2|v(l)|^2 + (a/2)|\partial v(0)/\partial x|^2 + \|v\|_{0,\Omega}^2\right)^{1/2}$$

corresponding to this problem.

Remark 10. The corresponding *a priori* estimates in Remark 4 are valid for all weak solutions of the mixed problems (34), (9), (35) and (36), (23), (35). If one can somehow derive stronger *a priori* estimates, for example, from the estimate in Remark 3 for strong solutions, or even stronger estimates, then one can supplement Eqs. (34) and (36) with lower-order terms, and the fact that the resulting mixed problems are well posed can be justified by the well-known method of continuation with respect to a parameter with the use of Theorems 7 and 9.

In conclusion, we note that all above-considered abstract (in particular, operator-valued) functions of the variable t are assumed to be strongly measurable in the Lebesgue measure dt on $[0,T]$ in H.

The generalizing Theorems 1 and 3 were announced in [4], and Theorem 6 was announced in [5].

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