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PARTIAL DIFFERENTIAL EQUATIONS

On a Stable Approximation to Boundary Value Problems for Evolution Operator-Differential Equations with Variable Domains

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In the present paper, we introduce an approximation to linear operator-differential equations that provides well-posed solvability of boundary value problems for these equations. For the approximate solution of such problems, we suggest a method stable with respect to the input data of these problems and, in particular, with respect to the operator coefficients of the equations in the nonweakened energy inequalities. This method, which will be called the *internal approximation method*, can even be used to analyze the well-posedness and obtain approximate solutions of boundary value problems for variable-order partial differential equations.

1. INTRODUCTION

We illustrate the internal approximation method by the Cauchy problem for the first-order evolution differential equation

$$du(t)/dt + A(t)u(t) = f(t), \qquad t \in \,]0,T[\,, \tag{1}$$

with linear unbounded operators $A(t)$ acting in a Banach space E. We distinguish between two types of approximation to well-posed boundary value problems, internal and external. An approximation to an unbounded operator by bounded operators is referred to as an external approximation. By definition, an approximation to an operator-differential equation (a partial differential equation with or without boundary conditions) for which all or part of the operators (differential operators with or without the corresponding boundary conditions) are approximated by bounded operators is referred to as an *external approximation* to this operator-differential equation (partial differential equation). Such an approximation was used in [1, pp. 241–247] for finding smooth solutions of operator-differential equations and in [2] for finding nonsmooth solutions (see Remark 1). The conventional terminology is somewhat contradictory. In [1], solutions of operator-differential equations, referred to as strong, are, in a sense, smooth ("classical") solutions of those equations. But in the present paper and in [2, 3], solutions of Eq. (1), which are also said to be strong, are nonsmooth (generalized) solutions. The well-known papers by Lions, Kato, Tanabe, and others deal with weak (generalized) solutions of operator-differential equations. A specific feature of the external approximation to well-posed boundary value problems for operator-differential equations is that their nonsmooth solutions are approximated by smooth solutions only in energy inequalities weakened in comparison with the energy inequalities for the original boundary value problems.

In the present paper, nonsmooth solutions of Eq. (1) are locally and globally approximated by *a priori* smooth solutions of the first-order evolution differential equation (in explicit form, unlike [3])

$$d\tilde{u}(t)/dt + B(t)\tilde{u}(t) = \tilde{f}(t), \qquad t \in \,]0,T[\,, \tag{2}$$

with linear unbounded operators $B(t)$ acting in the same Banach space E. Such an approximation will be called *internal*. An approximation to an unbounded operator by unbounded operators will be referred to as an *internal* approximation. In accordance with this definition, an approximation to an operator-differential equation (a partial differential equation with or without boundary conditions) for which all operators (differential operators with or without the corresponding boundary

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conditions) of that equation or part of them are approximated by unbounded operators is referred to as an internal approximation to that operator-differential equation (partial differential equation). An internal approximation permits one to approximate nonsmooth solutions by smooth ones in nonweakened energy norms of the original boundary value problems. This is important in forthcoming proofs of the well-posed solvability of the Cauchy problem for Eq. (1), since the original Eq. (1) may fail to have smooth solutions however smooth the original data of the Cauchy problem, and the smoothing auxiliary operators $B_\varepsilon^{-1}(t)$ can be discontinuous with respect to t. The internal approximation can be treated as a special version of the method, known in the theory of ill-posed problems [4], where the original boundary value problem is replaced by a close boundary value problem. Such an approximation permits one to solve boundary value problems that at first glance seem ill-posed but, on the basis of the character of their stable approximation, should rather be classified as well-posed problems.

Remark 1. Unlike the theory of partial differential equations, in the modern theory of operator and operator-differential equations, the notions of their smooth ("classical") and nonsmooth (generalized) solutions have a conditional and relative character. Smooth solutions of operator and operator-differential equations are treated as solutions that belong to the domains of these equations. In applications, the sets of the latter rarely coincide with the sets of functions that are (accordingly many times) continuously differentiable in the classical sense with respect to time and space variables in some domain of the Euclidean space. For example, the solutions of the abstract operator equation $\bar{L}u = \mathscr{F}$ (the rigorous definition of the operator $\bar{L}$ will be given in Section 3), which are actually not smooth with respect to Eq. (1), can be treated as smooth solutions, since they belong to the domain $D\left(\bar{L}\right)$, and can be used for approximating the less smooth weak solutions of Eq. (1).

2. STATEMENT OF THE PROBLEM

In the Hilbert space H with inner product $(\cdot\,,\cdot)$ and norm $|\cdot|$, we consider the Cauchy problem

$$\mathscr{L}(t)u \equiv du/dt + A(t)u = f, \qquad t \in \,]0, T] \subset \mathbb{R}, \tag{3}$$

$$lu \equiv u|_{t=0} = \varphi, \qquad \varphi \in H, \tag{4}$$

where u and f are functions of t ranging in H, the $A(t)$ are given linear unbounded closed operators in H for almost all t with domain $D(A(t))$ depending on t.

We assume that the operators $A(t)$ satisfy the following conditions.

I. The operators $A(t)$ and their adjoint operators $A^*(t)$ in H with domains $D\left(A^*(t)\right)$ satisfy the inequalities

$$[u]^2_{(t)} \equiv \operatorname{Re}\,(A(t)u + c_0u, u) \geq c_1|u|^2 \qquad \forall u \in D(A(t)), \tag{5}$$

$$\operatorname{Re}\,(A^*(t)v + c_0v, v) \geq c_1|v|^2 \qquad \forall v \in D\left(A^*(t)\right), \tag{6}$$

where $c_0 \geq 0$ and $c_1 > 0$ are constants independent of u, v, and t.

To approximate operators $A(t)$ locally with respect to t by more smooth ones, we introduce auxiliary operators $B(t)$.

II. The interval $[0, T[$ is divided into disjoint subintervals $I_r = [t_r, t_{r+1}[\,$, $r = 0, \ldots, R$, $t_0 = 0$, $t_{R+1} = T$, such that on each of them, there exist linear dissipative operators $-B(t)$ in H with domain $D(B(t))$ depending on t, these operators have inverses $B^{-1}(t) \in \mathscr{B}\left(I_r, \mathfrak{L}(H)\right)$ such that $A(t)B^{-1}(t) \in L_\infty\left(I_r, \mathfrak{L}(H)\right)$, their strong derivatives with respect to t [1] satisfy the inclusion $dB^{-1}(t)/dt \in \mathscr{B}\left(I_r, \mathfrak{L}(H)\right)$, and there exist limits

$$-\lim_{\varepsilon\to 0}\int\limits_{I_r} \operatorname{Re}\,\left(A(t)B_\varepsilon^{-1}(t)\left(B_\varepsilon^{-1}(t)\right)^* u, u\right) dt \leq c_2 \int\limits_{I_r} |u|^2 dt \qquad \forall u \in L_2\left(I_r, H_t^+\right), \tag{7}$$

$$\lim_{\varepsilon\to 0}\int\limits_{I_r} \left(\left(dB_\varepsilon^{-1}(t)/dt\right)\left(B_\varepsilon^{-1}(t)\right)^* u, u\right) dt = 0 \qquad \forall u \in L_2\left(I_r, H_t^+\right), \tag{8}$$

where $\mathscr{B}(I_r, \mathfrak{L}(H))$ is the set of operators bounded with respect to $t \in I_r$ and in the norm of $\mathfrak{L}(H)$, the Hilbert spaces H_t^+ are the completions of the sets $D(A(t))$ in the Hermitian norms $[\cdot]_{(t)}$ (5), $B_\varepsilon^{-1}(t) = (I + \varepsilon B(t))^{-1}$, $\varepsilon > 0$, is a family of smoothing operators, $dB_\varepsilon^{-1}(t)/dt$ is their strong derivative with respect to t, $(B_\varepsilon^{-1}(t))^* = (I + \varepsilon B^*(t))^{-1}$, $\varepsilon > 0$, is the adjoint family of smoothing operators, the $B^*(t)$ are the adjoint operators of $B(t)$ in H, and $c_2 \geq 0$ is a constant independent of u and t. The number of intervals I_r can be infinite.

By [3], on each interval I_r, Conditions I and II are sufficient for the existence and uniqueness of local strong solutions of the Cauchy problem for Eq. (3) on $[t_r, t_{r+1}[$. But to "sew" global strong solutions of the Cauchy problem (3), (4) from those local strong conditions, we need matching conditions at nonremovable points t_r, $r = 1, \ldots, R$, of nonsmoothness and discontinuity of the operators $B^{-1}(t)$. Let the operators $B(t)$ admit one-sided left continuations $B(t_r - 0)$ from I_{r-1} to points t_r such that the continued operators also satisfy Condition II on $[t_{r-1}, t_r]$, $r = 1, \ldots, R$. Throughout the following, the existence of such continuations is provided by the differentiability of the operators $B^{-1}(t)$ with respect to t, see Condition II. For each $g \in H$, we set $\tilde{u}(t_r) \in D(B(t_r - 0))$ and $g \equiv B(t_r - 0)\tilde{u}(t_r)$ provided that $u(t) = B^{-1}g \to \tilde{u}(t_r)$ [and, obviously, $B(t)u(t) = g \to g$] in H as $t \to t_r$, $t < t_r$. The operators $B(t_r + 0) = B(t_r)$ and $A(t_r + 0) = A(t_r)$, $r = 1, \ldots, R$, are defined by Condition II and Condition III, item (b) below, respectively.

We assume that the following matching conditions are satisfied.

III. If the partition I_r, $r = 0, \ldots, R$, of the interval $[0, T[$ consists of two or more intervals, then the following conditions are valid for two arbitrary adjacent intervals I_{r-1} and I_r:

(a) one of the inequalities

$$|B(t_r + 0)u| \leq c_3 |B(t_r - 0)u| \qquad \forall u \in D(B(t_r - 0)), \tag{9_1}$$

$$|A_0(t_r + 0)u| \leq c_4 |B(t_r - 0)u| \qquad \forall u \in D(B(t_r - 0)) \tag{9_2}$$

is valid at their common point t_r, where $A_0(t) = A(t) + c_0 I$ and $c_3, c_4 > 0$ are constants independent of u and t_r;

(b) if inequality (9_1) is valid, then, on the right interval I_r, the operator

$$B^{-1}(t) \in \mathscr{B}(I_r, \mathfrak{L}(H)) \cap L_\infty\left(I_r, \mathfrak{L}\left(\hat{H}_t^-\right)\right)$$

has the strong derivative $dB^{-1}(t)/dt \in \mathscr{B}(I_r, \mathfrak{L}(H))$ such that

$$B(t)\left(dB^{-1}(t)/dt\right) \in L_\infty\left(I_r, \mathfrak{L}\left(\hat{H}_t^+, H\right)\right).$$

Here the Hilbert spaces $\hat{H}_t^+$ are analogs of the spaces H_t^+ constructed on the basis of $B(t)$ instead of $A(t)$, and the $\hat{H}_t^-$ are the anti-dual spaces of the Hilbert spaces $\hat{H}_t^+$; if inequality (9_2) holds, then all these smoothness conditions are valid for the inverse operators $A_0^{-1}(t)$ of $A_0(t)$ instead of $B^{-1}(t)$ in H_t^+ and H_t^- instead of $\hat{H}_t^+$ and $\hat{H}_t^-$, respectively.

Unlike [2, 3], where for proving the existence of strong solutions of the Cauchy problem, it was proved that the range of the problem is dense in the corresponding space, in the present paper, we prove their existence by "sewing" them from local strong solutions with the use of the internal approximation. Let us prove the strong stability of these strong solutions, that is, their continuous dependence on the operator coefficient, the right-hand side of the equation, and the initial data. We suggest a method for the approximate solution of the Cauchy problem, which is strongly stable in nonweakened energy inequalities. In the final part, we give a new example of mixed problems for partial differential equations of variable order with respect to the space derivative, whose well-posed solvability follows neither from [2], since operators $B_\varepsilon^{-1}(t)$ differentiable with respect to t do not exist on the entire $[0, T[$, nor from [3], since the operators $A_0^{-1}(t)$ are not necessarily differentiable with respect to t on any I_r. These mixed problems are mainly characterized by the alternation of odd orders of differentiation with respect to x in the course of time t.

Remark 2. Inequality (9_1) and the corresponding part of assumptions in Condition III(b) are new matching conditions as compared with [3]. Inequality (9_1) is more restrictive than inequality (9_2) in [3], since the second inequality follows from (9_1) and Condition II. But in the case of (9_1), all smoothness requirements for the operators $A(t)$ are transferred to the operator $B(t)$, which permits one to construct new well-posed mixed problems for partial differential equations of the form (21)–(23) (see below). In the case of self-adjoint operators $A(t)$ and $B(t)$, Theorem 3 in [3] permits one to replace (without violating the correctness) inequalities (9_1) and (9_2) by the weaker requirements $\left|B^{1/2}(t_r+0)\,u\right| \le c_3^{1/2}\left|B^{1/2}(t_r-0)\,u\right|$ for all $u \in D\left(B^{1/2}(t_r-0)\right)$ and $\left|A_0^{1/2}(t_r+0)\,u\right| \le c_4^{1/2}\left|B^{1/2}(t_r-0)\,u\right|$ for all $u \in D\left(B^{1/2}(t_r-0)\right)$, respectively. Moreover, in Condition III(b), the operators $B(t)$ and $A_0(t)$ can be replaced by their square roots $B^{1/2}(t)$ and $A_0^{1/2}(t)$ with domains $D\left(B^{1/2}(t)\right)$ and $D\left(A_0^{1/2}(t)\right)$, respectively.

3. EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS

First, we introduce some spaces and state the definition of strong solutions of the Cauchy problem (3), (4). We set $\mathscr{H}^+ = L_2\left(]0,T[, H_t^+\right)$, $\mathscr{H} = L_2(]0,T[, H)$, and $\mathscr{H}^- = L_2\left(]0,T[, H_t^-\right)$, where the H_t^- are the antidual spaces of H_t^+. The space of strong solutions of the Cauchy problem (3), (4) is the Banach space E that is the completion of the set $D(L) = \{u \in \mathscr{H} : \ u(t) \in D(A(t))$ for almost all t; $du/dt, A(t)u \in \mathscr{H}\}$ in the norm

$$\|u\|_E = \left(\sup_{0<t<T} |u(t)|^2 + \int_0^T [u(t)]_{(t)}^2 dt \right)^{1/2}.$$

The space of right-hand sides of Eq. (3) and initial data in (4) is the Hilbert space $F = \mathscr{H}^- \times H$ that is the completion of the set $\mathscr{H} \times H$ of all elements $\mathscr{F} = \{f, \varphi\}$ in the Hermitian norm

$$\|\mathscr{F}\|_F = \left(\int_0^T [f(t)]_{(-t)}^2 dt + |\varphi|^2 \right)^{1/2},$$

where the $[\cdot]_{(-t)}$ are the Hermitian norms in the Hilbert spaces H_t^-. The Cauchy problem (3), (4) corresponds to a linear unbounded operator $L \equiv \{\mathscr{L}(t), l\} : E \supset D(L) \to F$ with dense domain $D(L)$. By using the closability criterion for linear operators in Banach spaces, one can justify the following assertion in a standard way [2].

Lemma 1. *Let the operators $A(t)$ be closed in H and satisfy inequalities* (5). *If $D(L)$ is dense in $\mathscr{H}^+$, then $L : E \supset D(L) \to F$ is closable.*

Recall that a linear operator acting in Banach spaces is said to be closable if the closure of its graph is again the graph of some linear operator, referred to as the *closure* of the original operator. We construct the closure $\bar{L} = \left\{\bar{\mathscr{L}}(t), l\right\} : E \supset D\left(\bar{L}\right) \to F$ of the operator L and assume that its domain $D\left(\bar{L}\right)$ contains all functions $u \in E$ for which there exist a sequence $u_n \in D(L)$ and an element $\mathscr{F} = \{f, \varphi\} \in F$ such that $\|u_n - u\|_E \to 0$ and $\|Lu_n - \mathscr{F}\|_F \to 0$ as $n \to \infty$. Furthermore, we set $\bar{L}u = \lim_{n\to\infty} Lu_n = \mathscr{F}$. Solutions of the operator equation $\bar{L}u = \mathscr{F}$ are referred to as *strong solutions* of the Cauchy problem (3), (4).

Now we derive an energy inequality for strong solutions of the Cauchy problem (3), (4). The following assertion can be proved in a standard way [2, 3].

Theorem 1. *Let the operators $A(t)$ be closed and satisfy inequalities* (5), *and let $D(L)$ be dense in $\mathscr{H}^+$. Then*

$$\|u\|_E \le c_5 \left\|\bar{L}u\right\|_F \qquad \forall u \in D\left(\bar{L}\right), \qquad c_5 = \exp\left(c_0 T\right). \tag{10}$$

Theorem 1 implies the uniqueness of strong solutions of the Cauchy problem (3), (4) and their continuous dependence on the original data f and φ.

The following assertion justifies the solvability of the Cauchy problem in question.

Theorem 2. *Let Conditions* I–III *be satisfied, and let* $D(L)$ *be dense in* $\mathscr{H}^+$. *Then for any* $f \in \mathscr{H}^-$ *and* $\varphi \in H$, *there exists a unique strong solution* $u \in E$ *of the Cauchy problem* (3), (4), *which satisfies the inequality* $\|u\|_E \le c_5 \|\mathscr{F}\|_F$, $\mathscr{F} = \{f, \varphi\}$.

Proof. By inequality (5), from Theorem 1 in [3], we obtain the inequality

$$\sup_{t_0 \le t \le t_1} |u(t)|^2 + \int_{t_0}^{t_1} [u(t)]_{(t)}^2 dt \le e^{2c_0(t_1 - t_0)} \left(\int_{t_0}^{t_1} \left[\bar{\mathscr{L}}(t)u\right]_{(-t)}^2 dt + |lu|^2 \right) \qquad (O_{t_0,t_1})$$

for each $u \in D\left(\bar{L}_0\right)$, where the domain $D\left(\bar{L}_0\right)$ is obtained from $D\left(\bar{L}\right)$ by the replacement of $[0,T[$ by $[t_0,t_1[$. Relations (5)–(8), together with Theorem 2 in [3], imply that for each $f_0 = f \in L_2\left(I_0, H_t^-\right)$ and for each $\varphi \in H$, there exists a unique strong solution $u_0 \in E_0 \cap C\left([t_0,t_1], H\right)$ of the Cauchy problem for Eq. (3) on $[t_0,t_1[$, that is, a solution of the operator equation $\bar{L}_0 u = \mathscr{F}_0$, $\mathscr{F}_0 = \{f_0, \varphi\} \in F_0$, where the norms in the Banach space E_0 and the Hilbert space F_0 are determined by the left- and right-hand sides, respectively, of inequality (O_{t_0,t_1}) and $\bar{L}_0$ is the closure of the operator $L_0 \equiv \{\mathscr{L}(t), l\} : E_0 \supset D\left(L_0\right) \to F_0$. By the definition of strong solutions, hence we obtain the existence of a sequence $u_0^{(n)} \in D\left(L_0\right)$ such that $u_0^{(n)} \to u_0$ in E_0 and $L_0 u_0^{(n)} \to \mathscr{F}_0$ in F_0 as $n \to \infty$. An analysis of Theorems 1 and 2 in [3] shows also that $u_0^{(n)} \in L_2\left(I_0, D(B(t))\right)$.

By Theorem 1 in [3], we have (O_{t_1,t_2}) for all $u \in D\left(\bar{L}_1\right)$, where the domain $D\left(\bar{L}_1\right)$ is obtained from $D\left(\bar{L}\right)$ by the replacement of $[0,T[$ by the interval $[t_1,t_2[$. By Theorem 2 in [3], for any $f_1 = f \in L_2\left(I_1, H_t^-\right)$ and $\varphi_1 = u_0\left(t_1\right) \in H$, there exists a unique strong solution $u_1 \in E_1 \cap C\left([t_1,t_2], H\right)$ of the Cauchy problem for Eq. (3) on $[t_1,t_2[$, that is, a solution of the operator equation $\bar{L}_1 u = \mathscr{F}_1$, $\bar{L}_1 = \left\{\bar{\mathscr{L}}(t), \cdot\,|_{t=t_1}\right\}$, $\mathscr{F}_1 = \{f_1, \varphi_1\} \in F_1$, where the norms in the Banach space E_1 and the Hilbert space F_1 are determined by the left- and right-hand sides, respectively, of inequality (O_{t_1,t_2}).

We sum inequalities (O_{t_0,t_1}) and (O_{t_1,t_2}), use the estimate

$$h\left(t_1\right) + \sup_{t_0 \le t \le t_2} h(t) \le \sup_{t_0 \le t \le t_1} h(t) + \sup_{t_1 \le t \le t_2} h(t),$$

which is obviously valid for functions h continuous at the point t_1, collect similar terms, estimate the remainder in the last term on the right-hand side in inequality (O_{t_1,t_2}) by the right-hand side of inequality (O_{t_0,t_1}), and find that, for any $f \in L_2\left(]t_0,t_2[, H_t^-\right)$ and $\varphi \in H$, there exists a unique function $u_{0,1} \in E_{0,1} \cap C\left([t_0,t_2], H\right)$ equal to u_r on I_r, $r = 0,1$, and satisfying the operator equations $\bar{L}_0 u = \mathscr{F}_0$ and $\bar{L}_1 u = \mathscr{F}_1$ and the inequality (O_{t_0,t_2}). By the definition of strong solutions, the function $u_{0,1}$ is a strong solution of the Cauchy problem for Eq. (3) on $[t_0,t_2[$, that is, a solution of the operator equation $\bar{L}_{0,1} u = \mathscr{F}_{0,1}$, $\mathscr{F}_{0,1} = \{f, \varphi\} \in F_{0,1}$, where $\bar{L}_{0,1} = \left\{\bar{\mathscr{L}}(t), l\right\}$, provided that there exists a sequence $u_{0,1}^{(n)} \in D\left(L_{0,1}\right)$ such that $u_{0,1}^{(n)} \to u_{0,1}$ in $E_{0,1}$ and $L_{0,1} u_{0,1}^{(n)} \to \mathscr{F}_{0,1}$ in $F_{0,1}$ as $n \to \infty$. The norms in the Banach space $E_{0,1}$ and the Hilbert space $F_{0,1}$ are given by the left- and right-hand sides of inequality (O_{t_0,t_2}), and the domain $D\left(L_{0,1}\right)$ is obtained from $D(L)$ by the replacement of $[0,T[$ by the interval $[t_0,t_2[$.

If inequality (9_1) is valid at the point t_1, then we obtain $\varphi_1^{(n)} = u_0^{(n)}\left(t_1\right) \in D\left(B\left(t_1 + 0\right)\right)$. By Theorem 2 in [3], for f_1 and $\varphi_1^{(n)}$, there exist strong solutions $u_1^{(n)} \in D\left(\bar{L}_1\right)$ of the Cauchy problem for Eq. (3) on $[t_1,t_2[$, i.e., $\bar{L}_1 u_1^{(n)} = \{f_1, \varphi_1^{(n)}\}$. By construction, $\varphi_1^{(n)} \to \varphi_1$ in H; hence $u_1^{(n)} \to u_1$ in E_1 as $n \to \infty$ by inequality (O_{t_1,t_2}).

Consider the operator equations $\bar{\hat{L}}_r u = \hat{\mathscr{F}}$, $\hat{\mathscr{F}} \in \hat{F}_r$, $r = 0,1,\ldots$, where the operators $\bar{\hat{L}}_r \equiv \{\bar{\hat{\mathscr{L}}}(t),\ \cdot\,|_{t=t_r}\}$ with domains $D\left(\bar{\hat{L}}_r\right)$ are the closures of the linear operators $\hat{L}_r \equiv \{\hat{\mathscr{L}}(t),\ \cdot\,|_{t=t_r}\}$, $\hat{\mathscr{L}}(t) = d/dt + B(t)$ mapping the Banach spaces $\hat{E}_r$ into the Hilbert spaces $\hat{F}_r$ with domains

$$D\left(\hat{L}_r\right) = \left\{u \in L_2\left(I_r, H\right) :\ u(t) \in D(B(t)),\ t \in I_r;\ du/dt, B(t)u \in L_2\left(I_r, H\right)\right\}.$$

The spaces $\hat{E}_r$ and $\hat{F}_r$ and the domain $D\left(\hat{L}_r\right)$ are obtained from E_r, F_r, and $D\left(L_r\right)$, respectively, by the replacement of the operator $A(t)$ by the operator $B(t)$. Since, obviously, Conditions I and II with the operators $B(t)$ instead of the operators $A(t)$ are valid, it follows that for each $\hat{\mathscr{F}} \in \hat{F}_r$, there exists a unique strong solution $\hat{u} \in \hat{E}_r$ of the auxiliary Cauchy problem for Eq. (2) on $[t_r, t_{r+1}[$, i.e., there exists a linear bounded inverse operator $\left(\bar{\hat{L}}_r\right)^{-1} \in \mathfrak{L}\left(\hat{F}_r, \hat{E}_r\right)$. Let

$$\hat{u}_1^{(n)} = \left(\bar{\hat{L}}_1\right)^{-1}\left\{0, \varphi_1^{(n)}\right\}.$$

We use the following lemma on increasing the local smoothness of strong solution (see Theorem 3 in [3]).

Lemma 2. *Let Conditions* I *and* II *be satisfied, and let* $D\left(\hat{L}_r\right)$ *be a dense set in* $\hat{\mathscr{H}}_r^+$. *Suppose that for some* $q > 0$, *the inverse operators*

$$B^{-q}(t) \in \mathscr{B}\left(I_r, \mathfrak{L}(H)\right) \cap L_\infty\left(I_r, \mathfrak{L}\left(\hat{H}_t^+\right)\right) \cap L_\infty\left(I_r, \mathfrak{L}\left(\hat{H}_t^-\right)\right)$$

have a strong derivative $dB^{-q}(t)/dt \in \mathscr{B}\left(I_r, \mathfrak{L}(H)\right)$ *such that*

$$B^q(t)\left(dB^{-q}(t)/dt\right) \in L_\infty\left(I_r, \mathfrak{L}\left(\hat{H}_t^+, H\right)\right),$$

and there exists a $q_0 \ge 0$ *and a constant* $c_6 > 0$ *independent of* u *and* t *and such that*

$$\langle u \rangle_{(t)}^2 \ge c_6 \langle |u| \rangle_{q_0(t)}^2$$

for all $u \in D(B(t))$ *on* I_r, *where the norm* $\langle \cdot \rangle_{(t)}$ *is obtained from the norm* $[\cdot]_{(t)}$ *by the replacement of* $A(t)$ *by the operators* $B(t)$. *If the domains* $D\left(B^{q+q_0}(t)\right)$ *are dense in* H *and*

$$B^q(t) g \in \hat{\mathscr{H}}_r^- = L_2\left(I_r, \hat{H}_t^-\right),$$

then, for any $g \in \hat{\mathscr{H}}_r^{q+q_0-1}$ *and* $\psi \in \hat{W}^q\left(t_r\right)$, *the Cauchy problem for Eq.* (2) *on* $[t_r, t_{r+1}[$ *has a unique solution* $\hat{u} \in \hat{E}_r$ *such that* $d^k\hat{u}/dt^k \in \hat{\mathscr{H}}_r^{q+q_0-k}$, $k = 0, 1$.

Here $\hat{\mathscr{H}}_r^s = L_2\left(I_r, \hat{W}^s(t)\right)$, $\hat{\mathscr{H}}_r^0 = \hat{\mathscr{H}}_r$, and the Hilbert spaces $\hat{W}^s(t)$ are the domains $D\left(B^s(t)\right)$ of the powers $B^s(t)$ equipped with the Hermitian norms $\langle |u| \rangle_{s(t)} = |B^s(t)u|$.

By Lemma 2 used on I_1 for $q = 1$ and $q_0 = 0$ and by Condition III(b), for any $g \in L_2\left(I_1, D(B(t))\right)$ and $\psi \in D\left(B\left(t_1 + 0\right)\right)$, there exists a unique strong solution $\hat{u} \in D\left(\hat{L}_1\right)$ of the auxiliary Cauchy problem for Eq. (2) on $[t_1, t_2[$. Note that if $q = 1$, then the condition $B^{-1}(t) \in L_\infty\left(I_1, \mathfrak{L}\left(\hat{H}_t^+\right)\right)$ of Lemma 2 is necessarily valid; therefore, $\hat{u}_1^{(n)} \in D\left(\hat{L}_1\right)$. Then there exist $\check{u}_1^{(n)} \in D\left(\bar{L}_1\right)$ such that

$$\bar{L}_1 u_1^{(n)} = L_1 \hat{u}_1^{(n)} + \bar{L}_1 \check{u}_1^{(n)}, \tag{11}$$

since, obviously, $D\left(\hat{L}_1\right) \subset D\left(L_1\right)$,

$$L_1 \hat{u}_1^{(n)} = \hat{L}_1 \hat{u}_1^{(n)} + \left(L_1 - \hat{L}_1\right)\hat{u}_1^{(n)} = \left\{0, \varphi_1^{(n)}\right\} + \left\{(A(t) - B(t))\hat{u}_1^{(n)}, 0\right\} = \left\{(A(t) - B(t))\hat{u}_1^{(n)}, \varphi_1^{(n)}\right\},$$

and consequently,

$$\bar{L}_1 \check{u}_1^{(n)} = \left\{f_1, \varphi_1^{(n)}\right\} - \left\{(A(t) - B(t))\hat{u}_1^{(n)}, \varphi_1^{(n)}\right\} = \left\{f_1 + (B(t) - A(t))\hat{u}_1^{(n)}, 0\right\}. \tag{12}$$

By the definition of strong solutions, there exists a sequence $\breve{u}_1^{(n,m)} \in D(L_1)$ such that $\breve{u}_1^{(n,m)} \to \breve{u}_1^{(n)}$ in E_1 and $L_1\breve{u}_1^{(n,m)} \to \bar{L}_1\breve{u}_1^{(n)}$ in F_1 as $m \to \infty$. An analysis of the proofs of Theorems 1 and 2 in [3] shows that indeed $\breve{u}_1^{(n,m)} \in D\left(\hat{L}_1\right)$. If we multiply this sequence by the cut-off functions

$$q_m^+(t) = \begin{cases} 0 & \text{if } \ 0 \le t \le t_1 \\ m(t-t_1) & \text{if } \ t_1 \le t \le t_1 + 1/m \\ 1 & \text{if } \ t_1 + 1/m \le t \le t_2, \end{cases}$$

then we obtain the inclusion $q_m^+\breve{u}_1^{(n,m)} \in D\left(\hat{L}_1\right)$. The inequalities

$$\begin{aligned} &\int_{t_1}^{t_2} \left[\bar{\mathscr{L}}(t)\left(q_m^+\breve{u}_1^{(n,m)} - \breve{u}_1^{(n)}\right)\right]^2_{(-t)} dt \\ &\qquad \le 4\int_{t_1}^{t_2} \left[\mathscr{L}(t)\breve{u}_1^{(n,m)} - \bar{\mathscr{L}}(t)\breve{u}_1^{(n)}\right]^2_{(-t)} dt + 4\int_{t_1}^{t_2} \left(q_m^+ - 1\right)^2 \left[\bar{\mathscr{L}}(t)\breve{u}_1^{(n)}\right]^2_{(-t)} dt \\ &\qquad\quad + 4m\int_{t_1}^{t_1+1/m} \left|\breve{u}_1^{(n,m)} - \breve{u}_1^{(n)}\right|^2 dt + 4m\int_{t_1}^{t_1+1/m} \left|\breve{u}_1^{(n)}\right|^2 dt \end{aligned}$$

are valid for all sufficiently large m, which imply that $L_1 q_m^+\breve{u}_1^{(n,m)} \to \bar{L}_1\breve{u}_1^{(n)}$ in F_1 as $m \to \infty$, since, by (12), $\breve{u}_1^{(n)}(t) \to 0$ in H as $t \to t_1 + 0$. By using the diagonal process, from the sequence $u_1^{(n,m)} = \hat{u}_1^{(n)} + q_m^+\breve{u}_1^{(n,m)} \in D\left(\hat{L}_1\right)$, we extract a subsequence $u_1^{(n,m_n)} = \hat{u}_1^{(n)} + q_{m_n}^+\breve{u}_1^{(n,m_n)}$ such that $u_1^{(n,m_n)} \to u_1$ in E_1 by construction and $L_1u_1^{(n,m_n)} \to \mathscr{F}_1$ in F_1 as $n \to \infty$ by (11).

Obviously, the sequence $u_{0,1}^{(n)}$ equal to $u_0^{(n)}$ on $[t_0, t_1]$ and $u_1^{(n,m_n)}$ on $[t_1, t_2]$ belongs to $D(L_{0,1})$ for all sufficiently large n. The relations

$$\int_{t_0}^{t_2} \left[\mathscr{L}(t)u_{0,1}^{(n)} - f\right]^2_{(-t)} dt = \int_{t_0}^{t_1} \left[\mathscr{L}(t)u_0^{(n)} - f_0\right]^2_{(-t)} dt + \int_{t_1}^{t_2} \left[\mathscr{L}(t)u_1^{(n,m_n)} - f_1\right]^2_{(-t)} dt$$

and the inequality (O_{t_0,t_2}) imply that $u_{0,1}^{(n)} \to u_{0,1}$ in $E_{0,1}$ and $L_{0,1}u_{0,1}^{(n)} \to \mathscr{F}_{0,1}$ in $F_{0,1}$ as $n \to \infty$, since, by construction, $\mathscr{L}(t)u_0^{(n)} \to f_0$ in $L_2\left(I_0, H_t^-\right)$ and $\mathscr{L}(t)u_1^{(n,m_n)} \to f_1$ in $L_2\left(I_1, H_t^-\right)$ as $n \to \infty$. Therefore, the function $u_{0,1}$ is indeed a strong solution of the Cauchy problem for Eq. (3) on $[t_0, t_2[$.

If inequality (9_2) were valid at the point t_1, then, by repeating the above-performed constructions on I_1 with the operators $A(t)$ instead of $B(t)$, we would find that the corresponding function $u_{0,1}$ is a strong solution of the Cauchy problem for Eq. (3) on $[t_0, t_2[$.

By Theorems 1 and 2 in [3], there exists a unique strong solution $u_2 \in E_2$ of the Cauchy problem for Eq. (3) on $[t_2, t_3[$ for $f_2 = f \in L_2\left(I_2, H_t^-\right)$, and $\varphi_2 = u_1(t_2) \in H$. To show that the function $u_{0,2}$ equal to u_r on I_r, $r = 0, 1, 2$, is a strong solution of the Cauchy problem for Eq. (3) on $[t_0, t_3[$, one should indicate a sequence $u_{0,2}^{(n)} \in D(L_{0,2})$ such that $u_{0,2}^{(n)} \to u_{0,2}$ in $E_{0,2}$ and $L_{0,2}u_{0,2}^{(n)} \to \mathscr{F}_{0,2} = \{f, \varphi\}$ in $F_{0,2}$, $L_{0,2} = \{\mathscr{L}(t), l\}$, as $n \to \infty$. The norms in the Banach space $E_{0,2}$ and in the Hilbert space $F_{0,2}$ are given by the left- and right-hand sides, respectively, of inequality (O_{t_0,t_3}), and the domain $D(L_{0,2})$ is obtained from $D(L)$ by the replacement of $[0, T[$ by the interval $[t_0, t_3[$.

By repeating all above-performed considerations for the intervals I_1 and I_2 instead of the intervals I_0 and I_1, we obtain a sequence $u_{1,2}^{(n)} \in D(L_{1,2})$ such that $u_{1,2}^{(n)} \to u_{1,2}$ in $E_{1,2}$ and $L_{1,2}u_{1,2}^{(n)} \to \mathscr{F}_{1,2} = \{f, \varphi_1\}$ in $F_{1,2}$ as $n \to \infty$, where $L_{1,2} = \{\mathscr{L}(t), \cdot\,|_{t=t_1}\}$, $E_{1,2}$ and $F_{1,2}$ are defined by analogy with $E_{0,1}$ and $F_{0,1}$ and the domain $D(L_{1,2})$ is obtained from $D(L)$ by the replacement of $[0, T[$

by the interval $[t_1, t_3[$. The function $u_{1,2}$ is equal to u_r on I_r, $r = 1, 2$. Then the sequence $u_{0,2}^{(n)} = q_n^- u_{0,1}^{(n)} + q_n^+ u_{1,2}^{(n)}$ belongs to $D(L_{0,2})$ for all sufficiently large n, where the cutoff functions occurring in the partition $q_n^-(t) + q_n^+(t) = 1$ of unity for all $t \in [t_0, t_3]$ are

$$q_n^-(t) = \begin{cases} 1 & \text{if} \quad t_0 \le t \le \bar{t} = t_1 + (t_2 - t_1)/2 \\ 1 - n(t - \bar{t}) & \text{if} \quad \bar{t} \le t \le \bar{t} + 1/n \\ 0 & \text{if} \quad \bar{t} + 1/n \le t \le t_3. \end{cases}$$

The inequalities

$$\int_{t_0}^{t_3} \left[\mathscr{L}(t)u_{0,2}^{(n)} - f\right]_{(-t)}^2 dt \le 4\int_{t_0}^{t_2} \left[\mathscr{L}(t)u_{0,1}^{(n)} - f\right]_{(-t)}^2 dt + 4\int_{t_1}^{t_3} \left[\mathscr{L}(t)u_{1,2}^{(n)} - f\right]_{(-t)}^2 dt$$
$$+ 4n \int_{\bar{t}}^{\bar{t}+1/n} \left(\left|u_{0,1}^{(n)} - u_{0,2}\right|^2 + \left|u_{1,2}^{(n)} - u_{0,2}\right|^2\right) dt$$

are valid for all sufficiently large n, which imply that $\mathscr{L}(t)u_{0,2}^{(n)} \to f$ in $L_2\left(]t_0, t_3[\,, H_t^-\right)$ as $n \to \infty$, since $\mathscr{L}(t)u_{0,1}^{(n)} \to f$ in $L_2\left(]t_0, t_2[\,, H_t^-\right)$, $\mathscr{L}(t)u_{1,2}^{(n)} \to f$ in $L_2\left(]t_1, t_3[\,, H_t^-\right)$, and, by construction, $u_{0,1}^{(n)} \to u_{0,2}$ in $E_{0,1}$ and $u_{1,2}^{(n)} \to u_{0,2}$ in $E_{1,2}$ as $n \to \infty$. Then it follows from inequality (O_{t_0,t_3}) that $u_{0,2}^{(n)} \to u_{0,2}$ in $E_{0,2}$ as $n \to \infty$, and so on.

As a result, in a finite number R of "sewing" steps, for all $f \in \mathscr{H}^-$ and $\varphi \in H$, we obtain unique strong solutions $u_{0,R} \in E_{0,R} \cap C([t_0, t_{R+1}], H)$ of the Cauchy problem for Eq. (3) on $[t_0, t_{R+1}[$. These solutions are equal to u_r on I_r, $r = 0, \ldots, R$, and satisfy the inequalities

$$\sup_{0 \le t \le t_{R+1}} |u_{0,R}(t)|^2 + \int_0^{t_{R+1}} [u_{0,R}(t)]_{(t)}^2 dt \le e^{2c_0 t_{R+1}} \left(\int_0^{t_{R+1}} \left[\bar{\mathscr{L}}(t)u_{0,R}\right]_{(-t)}^2 dt + |lu_{0,R}|^2\right). \tag{13}$$

By using this iterative "sewing" process and the closeness of the right-hand side of inequalities (13) to the squared right-hand side of inequality (10), in the case of finitely many intervals I_R, one can claim that the limit function $\lim_{R\to\infty} u_{0,R} = u$ exists in E, satisfies inequality (10), and is a strong solution of the Cauchy problem (3), (4), since u admits an approximation with arbitrary accuracy in the norms of the spaces E and F by smooth solutions of Eq. (2). The proof of Theorem 2 is complete.

Remark 3. If $A_1(t) \in L_\infty\left(]0, T[, \mathfrak{L}\left(H_t^+, H\right)\right)$, then, by using the method of continuation with respect to a parameter, one can generalize the assertion of Theorem 2 (possibly, with a larger constant c_5) to equations $\mathscr{L}(t)u + A_1(t)u = f(t)$, $t \in]0, T[$, with lower-order part.

4. A METHOD FOR THE APPROXIMATE SOLUTION

Let us describe the method for the approximate solution of boundary value problems for operator-differential equations which was used in the proof of Theorem 2 using the Cauchy problem (3), (4) as an example and derive method accuracy estimates stable with respect to the operator coefficient, the right-hand side, and the initial data. Earlier, it was more common practice to argue oppositely: known analytic solution methods were used to prove abstract theorems. For example, in their first works on the abstract operator-differential equation (3), Hille and Yosida implemented a method for the solution of ordinary stationary differential equations in the case of constant $A(t) = A$ with the help of semigroup theory. Kato's first works on the evolution equation (3) with variable operators $A(t)$ and with constant domain $D(A(t)) = D(A)$ represented an abstract generalization of the polygon method for ordinary differential equations. In almost all later works by Sobolevskii, Kato, Tanabe, and others dealing with the evolution equation (3) with variable $A(t)$ and $D(A(t))$, its solution was reduced to the solution of an abstract integral equation by the successive approximation method. Our method is essentially different in that boundary value problems for operator-differential equations are not reduced to any boundary value problem for ordinary differential equations or to integral equations, but their solutions are stably approximated by solutions of the same of close boundary value problems.

Suppose that we need to find solutions $u_r \in E_r$ of the operator equations

$$\bar{L}_r u_r = \mathscr{F}_r, \qquad \mathscr{F}_r = \{f_r, \varphi_r\} \in F_r, \tag{14}$$

where the $\bar{L}_r \equiv \left\{\bar{\mathscr{L}}(t), \cdot\,|_{t=t_r}\right\}$ are the closures of the operators

$$L_r \equiv \{\mathscr{L}(t), \cdot\,|_{t=t_r}\} : E_r \supset D(L_r) \to F_r = L_2\left(I_r, H_t^-\right) \times H$$

occurring in the proof of Theorem 2. Under the assumptions of Theorem 2, exact strong solutions of the Cauchy problem (3), (4) can be found by the successive exhaustion of the intervals I_r, $r = 0, \ldots, R$. Suppose that, instead of the exact initial data $\{A(t), f_r, \varphi_r\}$, we have approximate initial data $\{B(t), \hat{f}_r, \hat{\varphi}_r\}$, i.e., solutions of Eq. (14) are approximated by solutions of the equations

$$\bar{\hat{L}}_r \hat{u}_r = \hat{\mathscr{F}}_r, \qquad \hat{\mathscr{F}}_r = \left\{\hat{f}_r, \hat{\varphi}_r\right\} \in \hat{F}_r, \tag{15}$$

where $\bar{\hat{L}}_r \equiv \{\bar{\hat{\mathscr{L}}}(t), \cdot\,|_{t=t_r}\}$ are the closures of the operators

$$\hat{L}_r \equiv \left\{\hat{\mathscr{L}}(t), \cdot\,|_{t=t_r}\right\} : \hat{E}_r \supset D\left(\hat{L}_r\right) \to \hat{F}_r = L_2\left(I_r, \hat{H}_t^-\right) \times H$$

occurring in the proof of Theorem 2. In general, in the case of nonself-adjoint operators $A(t)$ and $B(t)$, we have $E_r \not\subset \hat{E}_r$, $\hat{E}_r \not\subset E_r$, $F_r \not\subset \hat{F}_r$, $\hat{F}_r \not\subset F_r$, and only $D\left(\hat{L}_r\right) \subset D\left(L_r\right)$.

Since the sets $L_2\left(I_r, D(B(t))\right)$ are dense in $\mathscr{H}_r^- = L_2\left(I_r, H_t^-\right)$, it follows that for any $\delta_f > 0$ and $f_r \in \mathscr{H}_r^-$, there exists an $\hat{f}_r \in L_2\left(I_r, D(B(t))\right)$ such that

$$\int\limits_{I_r} \left[f_r - \hat{f}_r\right]_{(-t)}^2 dt \le \delta_f^2.$$

Since $D\left(B\left(t_r\right)\right)$ is dense in H, it follows that for any $\delta_\varphi > 0$ and $\varphi_r \in H$, there exists a $\hat{\varphi}_r \in D\left(B\left(t_r\right)\right)$ such that $|\varphi_r - \hat{\varphi}_r| \le \delta_\varphi$. By Lemma 2, the solutions $\hat{u}_r = \left(\bar{\hat{L}}_r\right)^{-1}\{\hat{f}_r, \hat{\varphi}_r\}$ of Eqs. (15) belong to $D\left(\hat{L}_r\right) \subset D\left(L_r\right)$. Then the errors $w_r = u_r - \hat{u}_r$ of the approximations $\hat{u}_r$ are solutions of the equations

$$\bar{L}_r w_r = \left\{f_r - \hat{f}_r + (B(t) - A(t))\hat{u}_r, \varphi_r - \hat{\varphi}_r\right\} \in F_r, \tag{16}$$

since

$$\bar{L}_r w_r = \bar{L}_r u_r - L_r \hat{u}_r = \{f_r, \varphi_r\} - \left\{\hat{f}_r + (A(t) - B(t))\hat{u}_r, \hat{\varphi}_r\right\}.$$

By Theorem 2 in [3], Eqs. (16) have unique strong solutions $w_r \in E_r$ satisfying the corresponding energy inequalities (10). We have thereby justified the following assertion.

Theorem 3. *Under the assumptions of Theorem* 2, *each strong solution $u \in E$ of the Cauchy problem* (3), (4) *is locally and stably approximated by smooth solutions $\hat{u}_r \in D\left(\hat{L}_r\right)$, $r = 0, \ldots, R$, of Eqs.* (15) *with accuracy*

$$\|u_r - \hat{u}_r\|_{E_r}^2 \le e^{2c_0(t_{r+1} - t_r)}\left\{\int\limits_{I_r}\left(\left[f_r - \hat{f}_r\right]_{(-t)} + [(A(t) - B(t))\hat{u}_r]_{(-t)}\right)^2 dt + |\varphi_r - \hat{\varphi}_r|^2\right\}, \tag{17}$$

$$\operatorname*{ess\,sup}_{t \in I_r} |u_r - \hat{u}_r|^2 \le e^{2c_0(t_{r+1} - t_r)}\left\{\int\limits_{I_r}\left(\left[f_r - \hat{f}_r\right]_{(-t)}^2 + [(A(t) - B(t))\hat{u}_r]_{(-t)}^2\right) dt + |\varphi_r - \hat{\varphi}_r|^2\right\}, \tag{18}$$

$$\int_{I_r} |u_r - \hat{u}_r|^2 \, dt \le e^{2c_0(t_{r+1}-t_r)} \Bigg\{ \int_{I_r} (t_{r+1} - t) \left(\left[f_r - \hat{f}_r \right]^2_{(-t)} + [(A(t) - B(t))\hat{u}_r]^2_{(-t)} \right) dt \qquad (19)$$

$$+ (t_{r+1} - t_r) \, |\varphi_r - \hat{\varphi}_r|^2 \Bigg\},$$

where $\| \cdot \|_{E_r}$ *is the norm in* E_r*. For any* $\mathscr{F} = \{f, \varphi\} \in F$*, the function* $\hat{u} \in L_\infty(]0, T[, H)$ *equal to* $\hat{u}_r$ *on* I_r*,* $r = 0, \ldots, R$*, is some approximate solution of the Cauchy problem* (3), (4)*, which is stable with respect to the operator coefficient, the right-hand side, and the initial data with the error* $\|u - \hat{u}\|_E \to 0$ *as* δ_f*,* $\delta_\varphi \to 0$ *provided that*

$$B(t)u \to A(t)u \quad \forall u \in L_2(]0, T[, D(B(t))) \quad in \quad \mathscr{H}_r^-. \qquad (20)$$

In particular, $\hat{u}$ *tend to the exact solution* u *in* $L_\infty(]0, T[, H)$ *at the rate* $O\left(\sqrt{h_R}\right)$ *with respect to the operator coefficient and the right-hand side of Eq.* (3) *provided that* $\hat{f} \to f$ *in* $L_\infty\left(]0, T[, H_t^-\right)$*, where* $\hat{f} = \hat{f}_r$ *on* I_r*,* $r = 0, \ldots, R$*, and* $h_R = \max_{0 \le r \le R} |t_{r+1} - t_r|$*. Moreover, the approximate solutions* $\hat{u} \in C(I_r, H)$*,* $r = 0, \ldots, R$*, can be continuous at each point* t_r *at which condition* (9_1) *or* (9_2) *is satisfied and can have a jump discontinuity at the remaining points* t_r*.*

The proof of the estimates (17)–(19) is similar to the derivation of inequality (10). By the definition of strong solutions, the pair $\{u, \mathscr{F}\}$ can be approximated with arbitrary accuracy in $E \times F$ by the pair $\{v, \mathscr{F}_v\}$, where $v \in D(L)$ and $Lv = \mathscr{F}_v = \{f_v, \varphi_v\} \in F$. Therefore, the assertion of Theorem 3 on the convergence follows from the inequality

$$\|u - \hat{u}\|_E \le \|u - v\|_E + \|v - \hat{v}\|_E + \|\hat{v} - \hat{u}\|_E,$$

where $\hat{v} = \hat{v}_r \in D\left(\hat{L}_r\right)$, $\hat{v}_r$ are the solutions of Eq. (15) corresponding to $\mathscr{F}_v$ and the last two norms tend to zero by virtue of (20) and since $\hat{\mathscr{F}} \to \mathscr{F}_v$ in F, $\hat{\mathscr{F}} = \hat{\mathscr{F}}_r$ on I_r, $r = 0, \ldots, R$. Further, for $\hat{\varphi}_r = \varphi_r$, there exists a constant $c_7 > 0$ independent of t and R such that

$$\operatorname*{ess\,sup}_{0<t<T} |u - \hat{u}| \le c_7 \sqrt{h_R}$$

for all R.

Remark 4. The proof of similar estimates for the approximation error in [5, p. 62; 6, p. 954] is difficult in the case of variable domains $D(A(t))$ and $D(B(t))$ and in the case of nonself-adjoint operators $A(t)$ and $B(t)$ even if their domains $D(A)$ and $D(B)$ are constant. If, as in [5, 6], A is a self-adjoint operator and is independent on t in I_r, then Lemma 2 applied to Eq. (16) with A instead of B and with $q = q_0 = 1/2$, together with the Banach closed graph theorem, implies the following estimates for the error of the local approximation to smooth solutions of Eq. (3) (with $c_0 = 0$) :

$$\operatorname*{ess\,sup}_{t \in I_r} \left| A^{1/2} (u_r - \hat{u}_r) \right|^2 + \int_{I_r} \left(\left| \frac{d(u_r - \hat{u}_r)}{dt} \right|^2 + |A(u_r - \hat{u}_r)|^2 \right) dt$$

$$\le c_7 \int_{I_r} \left(\left| f_r - \hat{f}_r \right| + |(A - B)\hat{u}_r| \right)^2 dt + \left| A^{1/2} (\varphi_r - \hat{\varphi}_r) \right|^2, \qquad c_7 = 1.$$

These estimates can also be derived directly. If the $A(t)$ are variable and $D(A)$ is constant, then the same estimates are also valid but with some constant $c_7 \ge 1$. However, in the case of variable domains $D(A(t))$, such *a priori* estimates for smooth solutions of Eq. (3) require, in addition, the restrictive condition of the monotonicity of the domains $D\left(A_0^{1/2}(t)\right)$ with respect to t at least in some neighborhood of the points t_r (of nondecrease in the present paper and nonincrease in [7, p. 63]).

For example, in problems (21)–(23) (see below) of the present paper, the sets $D\left(A_0^{1/2}(t)\right)$ are not monotone on $]0,T[$ and not nondecreasing in neighborhoods of all t_r. Therefore, in the present paper, only *a priori* estimates for nonsmooth solutions of the form (10) and (17)–(19) are valid.

5. EXAMPLE

We pose boundary value problems whose well-posedness does not follow from [3]. In the bounded domain $G = [0,T[\times]0,l[$ of the variables t and x with the partition of the interval $]0,T[$ into disjoint subintervals I_r, $r = 0,1,2,3,4$, we consider partial differential equations with real-valued coefficients

$$\partial u(t,x)/\partial t + \mathscr{A}_r(t)u(t,x) = f(t,x), \qquad t \in I_r, \qquad r = 0,1,2,3,4, \tag{21}$$

where $\mathscr{A}_0(t)u = \sum_{k=0}^{m_0}(-1)^k D^k a_k(t,x) D^k u$ on I_0, $D^k = \partial^k/\partial x^k$, $D^i a_k(t,x) \in L_\infty\left(I_0 \times]0,l[\right)$, $i \le m_0$, $a_{m_0}(t,x) > 0$, with the Dirichlet conditions

$$D^i u\big|_{x=0} = D^i u\big|_{x=l} = 0, \qquad 0 \le i \le m_0 - 1, \qquad t \in I_0, \qquad m_0 \ge 3; \tag{22_0}$$

$\mathscr{A}_1(t)u = (-1)^{m_1-1} b_1(t) D^{2m_1-1}$ on I_1, $b_1(t) \in L_\infty(I_1)$, with the conditions

$$D^i u\big|_{x=0} = D^i u\big|_{x=l} = D^{m_1-1}u\big|_{x=0} = 0, \quad 0 \le i \le m_1 - 2, \quad t \in I_1, \quad 3 \le m_1 \le m_0; \tag{22_1}$$

$\mathscr{A}_2(t)u = c_2(t)D^5 u$ on I_2, $c_2(t) \in L_\infty(I_2)$, with the conditions

$$D^i u\big|_{x=0} = D^i u\big|_{x=l} = D^2 u\big|_{x=0} = 0, \qquad i = 0,1, \qquad t \in I_2; \tag{22_2}$$

$\mathscr{A}_3(t)u = -d_3(t)D^2 u$ on I_3, $d_3(t) \in L_\infty(I_3)$, $d_3(t) > 0$, with the Neumann conditions

$$Du|_{x=0} = Du|_{x=l} = 0, \qquad t \in I_3; \tag{22_3}$$

and $\mathscr{A}_4(t)u = \sum_{k=0}^{m_4(t)}(-1)^k e_k(t) D^{2k} u$ on I_4 with the conditions

$$D^{2i+1}u\big|_{x=0} = D^{2i+1}u\big|_{x=l} = 0, \qquad 0 \le i \le m_4(t) - 1, \qquad t \in I_4, \tag{22_4}$$

where $e_k(t) \in L_\infty(I_4)$, $e_{m_4(t)} > 0$, and $m_4(t) \ge 0$ is an integer-valued function bounded by a number m_4. Let us add the initial condition

$$u(0,x) = \varphi(x), \qquad x \in]0,l[. \tag{23}$$

Theorem 4. *For any $f \in \mathscr{H}^-(G)$ and $\varphi \in L_2(0,l)$, the mixed problems* (21)–(23) *have a unique strong solution $u \in \mathscr{E}(G) \cap C\left([0,T], L_2(0,l)\right)$ satisfying the inequality $\|u\|_{\mathscr{E}} \le c_5 \|\mathscr{F}\|_\Phi$, $\mathscr{F} = \{f,\varphi\}$.*

Proof. Here the Banach space $\mathscr{E}(G)$ is the completion of the set

$$\mathscr{D}(G) = \left\{u \in W_2^{1,2\max\{m_0,m_4\}}(G):\ u \in (22_r),\ r = 0,1,2,3,4\right\}$$

in the norm

$$\|u\|_{\mathscr{E}} = \left\{\sup_{0<t<T} \|u(t,x)\|_0^2 + \operatorname{Re}\int_0^T \left(A_0(t)u(t,x), u(t,x)\right)_0 dt\right\}^{1/2},$$

where $A_0(t) = A(t) + c_0 I$, the $A(t)$ are the operators in $L_2(0,l)$ induced for $t \in I_r$ by the expressions $\mathscr{A}_r(t)$ with conditions (22_r), $r = 0,1,2,3,4$, and $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ are the inner product and the norm, respectively, in $L_2(0,l)$. The Hilbert space $\Phi(G) = \mathscr{H}^-(G) \times L_2(0,l)$ is the completion of the set $L_2(G) \times L_2(0,l)$ in the Hermitian norm $\|\mathscr{F}\|_\Phi = \left\{\|f\|_-^2 + \|\varphi\|_0^2\right\}^{1/2}$, $\mathscr{F} = \{f,\varphi\}$, where the Hilbert space $\mathscr{H}^-(G)$ with the norm $\|\cdot\|_-$ is the anti-dual space of the Hilbert space $\mathscr{H}^+(G)$ with Hermitian norm

$$\|v\|_+ = \left\{ \operatorname{Re} \int_0^T \left(A_0(t)v, v \right)_0 dt \right\}^{1/2}.$$

Obviously, Condition I is satisfied. Conditions II and III are valid if, as local smoothing operators $B(t)$ in $L_2(0,l)$, one takes the expression

$$\mathscr{B}_0(t) = (-1)^{m_0} D^{2m_0}$$

with condition (22_0) on I_0,

$$\mathscr{B}_1(t) = (-1)^{m_1-1} D^{2m_1-1} + I$$

with (22_1) on I_1,

$$\mathscr{B}_2(t) = D^5 + I$$

with (22_2) on I_2, and

$$\mathscr{B}_3(t) = \mathscr{B}_4(t) = (-1)^{m_4} D^{2m_4} + I$$

with (22_4) for $m_4(t) = m_4$ on $I_3 \cup I_4$; moreover, condition (9_2) rather than (9_1) is valid at t_3. One can show that $D(L) = \mathscr{D}(G)$ is a dense set in $\mathscr{H}^+(G)$. Obviously, property (8) takes place, since the $B(t)$ are locally independent of t on each I_r. Theorem 4 follows from Theorem 2.

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