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PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problems for Complete Quasi-Hyperbolic Differential Equations with Variable Domains of Smooth Operator Coefficients: II

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The present paper is a continuation of [1]. We continue the numbering of sections, assertions, remarks, and formulas in [1].

4. THE EXISTENCE OF STRONG SOLUTIONS

The solvability of the boundary value problems (1), (2) in the strong sense for all $f \in \hat{F}^{-(m-1)}$ is justified in the following assertion.

Theorem 2. *Let the assumptions of Theorem* 1 *in* [1] *and Conditions* II *and* IV *be satisfied, and let* $d^j A_0^{-1}(t)/dt^j \in \mathscr{B}\left([0,T], \mathscr{L}\left(H, W^{m-j}(t)\right)\right)$, $2 \le j \le m-1$. *Then for each* $\lambda_m \in \tilde{\Lambda}_m$, *where* $\tilde{\Lambda}_1 = \hat{\Lambda}_1$ *for* $m = 1$ *and* $\tilde{\Lambda}_m = [\tilde{\lambda}_m, +\infty[$, $\tilde{\lambda}_m \ge \hat{\lambda}_m$, *for* $m > 1$, *and for each* $f \in \hat{F}^{-(m-1)}$, *there exists a unique strong solution* $u \in E^m$ *of the boundary value problems* (1), (2), *and*

$$|||u|||_m \le c_0(m)\langle\|f\|\rangle_{-(m-1)}, \qquad m = 1, 2, \ldots \tag{36}$$

Proof. Since the *a priori* estimates (28) imply that the ranges $R\left(\overline{L}_m\left(\lambda_m\right)\right)$ of the operators $\overline{L}_m\left(\lambda_m\right)$ are closed in $\hat{F}^{-(m-1)}$, it follows that, to justify the solvability of the boundary value problems (1), (2) in the strong sense for all $f \in \hat{F}^{-(m-1)}$, it suffices to show that the ranges $R\left(L_m\left(\lambda_m\right)\right)$ of the operators $L_m\left(\lambda_m\right)$ are dense in $\hat{F}^{-(m-1)}$. Since the spaces $\hat{F}^{-(m-1)}$ are reflexive, it suffices to show that $v = 0$ whenever

$$\int_0^T \left(L_m\left(\lambda_m\right) u, v\right) dt = 0 \qquad \forall u \in D\left(L_m\right), \qquad m = 1, 2, \ldots, \tag{37}$$

for some function $v \in \hat{E}^{m-1}$.

Identities (37) admit $(m-1)$-fold integration by parts:

$$\int_0^T \left(\frac{d^{m+1}u}{dt^{m+1}}, \frac{d^{m-1}v}{dt^{m-1}}\right) dt + \sum_{k=1}^{m-1} (-1)^k \int_0^T \left(\left[A_{2k+1}(t)\frac{d}{dt} + A_{2k}(t)\right] \frac{d^k u}{dt^k}, \frac{d^k v}{dt^k}\right) dt$$
$$+ \int_0^T \left(\left[A_1(t)\frac{d}{dt} + \lambda_m A_0(t)\right] u, v\right) dt = 0.$$

By passing to the limit, we generalize the last identities to all $u \in \mathscr{H}$ such that

$$d^{m+1}u/dt^{m+1}, A_s(t) d^{[(s+1)/2]}u/dt^{[(s+1)/2]} \in \mathscr{H}, \qquad s = 0, \ldots, 2m-1,$$
$$d^k u/dt^k \in \mathscr{H}^{m-k}, \qquad k = 1, \ldots, m,$$

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and u satisfies the boundary conditions (2); we set $u = A_0^{-1}(t)h$ for all $h \in \mathscr{H}$ such that $d^k h/dt^k \in \mathscr{H}$, $k = 1, \ldots, m+1$, and h satisfies the boundary conditions (2) and then obtain the relations

$$\begin{aligned}\int_0^T \left(\frac{d^{m+1}h}{dt^{m+1}}, A_0^{-1}(t)\frac{d^{m-1}}{dt^{m-1}}\left[e^{c(T-t)}J(t)w\right]\right) dt \\ = -\int_0^T e^{c(T-t)} \left(A_1(t)\frac{dA_0^{-1}(t)h}{dt} + \lambda_m h, J(t)w\right) dt \\ - \sum_{i=1}^{m+1} C_{m+1}^i \int_0^T \left(\frac{d^i A_0^{-1}(t)}{dt^i}\frac{d^{m+1-i}h}{dt^{m+1-i}}, \frac{d^{m-1}}{dt^{m-1}}\left[e^{c(T-t)}J(t)w\right]\right) dt \\ - \sum_{k=1}^{m-1}(-1)^k \int_0^T \left(\left[A_{2k+1}(t)\frac{d}{dt} + A_{2k}(t)\right]\frac{d^k A_0^{-1}(t)h}{dt^k}, \frac{d^k}{dt^k}\left[e^{c(T-t}J(t)w\right]\right) dt,\end{aligned} \tag{38}$$

where w is the solution of the Cauchy problem $J(t)w = e^{c(t-T)}v$, $w(0) = 0$ in $\mathscr{H}$. The relations $d^{i+1}w/dt^{i+1} = (T-t)^{-1}d^i\left(e^{c(t-T)}v\right)/dt^i - (m-1-i)(T-t)^{-1}d^i w/dt^i$, $0 \le i \le m-1$, imply that at least

$$\begin{aligned} w \in \mathscr{W}^m = \{w \in \mathscr{H}:\ d^k w/dt^k \in \mathscr{H},\ 1 \le k \le m;\ \left(d^i w/dt^i\right)\big|_{t=0} = \left(d^j w/dt^j\right)\big|_{t=T} = 0, \\ 0 \le i \le m-1,\ 0 \le j \le m-2\}.\end{aligned}$$

It follows from (38) that the functions $A_0^{-1}(t)(T-t)d^m w/dt^m$ have derivatives in $\mathscr{H}$ and vanish for $t = T$. We integrate relation (38) by parts once more, extend the results to all $h \in \mathscr{W}^m$ by passing to the limit, set $h = w$, take the doubled real part, and obtain

$$\begin{aligned} -2\,\mathrm{Re}\int_0^T \left(\frac{d^m w}{dt^m}, \frac{d}{dt}\left[A_0^{-1}(t)e^{c(T-t)}(T-t)\frac{d^m w}{dt^m}\right]\right) dt \\ = 2\,\mathrm{Re}\sum_{i=1}^{m-1} C_{m-1}^i \int_0^T \left(\frac{d^m w}{dt^m}, \frac{d}{dt}\left[A_0^{-1}(t)\frac{d^i e^{c(T-t)}}{dt^i}\frac{d^{m-1-i}J(t)w}{dt^{m-1-i}}\right]\right) dt \\ - 2\,\mathrm{Re}\sum_{i=1}^{m+1} C_{m+1}^i \int_0^T \left(\frac{d^i A_0^{-1}(t)}{dt^i}\frac{d^{m+1-i}w}{dt^{m+1-i}}, \frac{d^{m-1}}{dt^{m-1}}\left[e^{c(T-t)}J(t)w\right]\right) dt \\ - 2\,\mathrm{Re}\sum_{k=1}^{m-1}(-1)^k \int_0^T \left(\left[A_{2k+1}(t)\frac{d}{dt} + A_{2k}(t)\right]\frac{d^k A_0^{-1}(t)w}{dt^k}, \frac{d^k}{dt^k}\left[e^{c(T-t)}J(t)w\right]\right) dt \\ - 2\,\mathrm{Re}\int_0^T e^{c(T-t)} \left(A_1(t)\frac{dA_0^{-1}(t)w}{dt} + \lambda_m w, J(t)w\right) dt.\end{aligned} \tag{39}$$

Here integration by parts is based on the following assertion [2].

Lemma 8. *Let E_1, F_1, and G_1 be Banach spaces, let $T_1 : E_1 \to F_1$ be a bounded linear operator, and let $S_1 : F_1 \to G_1$ be a closed linear operator with dense domain. If the domain of the product $S_1 \cdot T_1$ is dense in E_1, then the adjoint operator $(S_1 \cdot T_1)^*$ is equal to the weak closure of the product $T_1^* \cdot S_1^*$ of the adjoint operators T_1^* and S_1^*.*

By applying Lemma 8 in $E_1 = F_1 = \mathscr{H}$ and $G_1 = \mathscr{H} \times H$ to the operators

$$T_1 u = A_0^{-1}(t) e^{c(T-t)}(T-t)u, \qquad S_1 g = \{dg/dt, g(0)\}$$

with domain $D(S_1) = \{g \in \mathscr{H} : \ dg/dt \in \mathscr{H}, \ g(T) = 0\}$ in the first two terms of the expressions

$$-\int_0^T \left(\frac{d^m w}{dt^m}, \frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \frac{d^m w}{dt^m} \right] \right) dt \\ - Te^{cT} \left(\frac{d^m w}{dt^m}, A_0^{-1}(t) \frac{d^m w}{dt^m} \right)\Bigg|_{t=0} + Te^{cT} \left| A_0^{-1/2}(t) \frac{d^m w}{dt^m} \right|^2 \Bigg|_{t=0}, \tag{40}$$

we find that the expressions (40) are equal to

$$\int_0^T \left(\frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \frac{d^m w}{dt^m} \right], \frac{d^m w}{dt^m} \right) dt \\ - \int_0^T \left(\left(\frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \right] \right) \frac{d^m w}{dt^m}, \frac{d^m w}{dt^m} \right) dt \\ + Te^{cT} \left| A_0^{-1/2}(t) \frac{d^m w}{dt^m} \right|^2 \Bigg|_{t=0}, \tag{41}$$

since $T_1^* = T_1$, $S_1^*(\{p, p(0)\}) = -dp/dt$, $D(S_1^*) = \{\{p, p(0)\} \in G_1 : \ dp/dt \in \mathscr{H}\}$, and

$$\overline{T_1^* \cdot S_1^*} \frac{d^m w}{dt^m} = -\frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \frac{d^m w}{dt^m} \right] + \left(\frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \right] \right) \frac{d^m w}{dt^m},$$

where $\overline{T_1^* \cdot S_1^*}$ is the closure of the product $T_1^* \cdot S_1^*$. The fact that $d^m w/dt^m$ belongs to the domain $\left((S_1 \cdot T_1)^*\right)$ of the operator $(S_1 \cdot T_1)^*$ follows from relations (38), which can be represented in the form

$$\int_0^T \left(\frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \frac{d^m h}{dt^m} \right], \frac{d^m w}{dt^m} \right) dt \\ = \int_0^T \left(\left(\frac{d}{dt}\left[A_0^{-1}(t) e^{c(T-t)}(T-t) \right] \right) \frac{d^m h}{dt^m}, \frac{d^m w}{dt^m} \right) dt \\ + \sum_{i=1}^{m-1} C_{m-1}^i \int_0^T \left(\frac{d^m h}{dt^m}, \frac{d}{dt}\left[A_0^{-1}(t) \frac{d^i e^{c(T-t)}}{dt^i} \frac{d^{m-1-i} J(t) w}{dt^{m-1-i}} \right] \right) dt \\ - \sum_{i=1}^{m+1} C_{m+1}^i \int_0^T \left(\frac{d^i A_0^{-1}(t)}{dt^i} \frac{d^{m+1-i} h}{dt^{m+1-i}}, \frac{d^{m-1}}{dt^{m-1}}\left[e^{c(T-t)} J(t) w \right] \right) dt \\ - \sum_{k=1}^{m-1} (-1)^k \int_0^T \left(\left[A_{2k+1}(t) \frac{d}{dt} + A_{2k}(t) \right] \frac{d^k A_0^{-1}(t) h}{dt^k}, \frac{d^k}{dt^k}\left[e^{c(T-t)} J(t) w \right] \right) dt \\ - \int_0^T e^{c(T-t)} \left(A_1(t) \frac{d A_0^{-1}(t) h}{dt} + \lambda_m h, J(t) w \right) dt.$$

Therefore, by (39)–(41), we have

$$
\begin{aligned}
&\int_0^T e^{c(T-t)}\left|A_0^{-1/2}(t)\frac{d^m w}{dt^m}\right|^2 dt + Te^{cT}\left|A_0^{-1/2}(t)\frac{d^m w}{dt^m}\right|^2\Bigg|_{t=0} \\
&= 2\,\mathrm{Re}\sum_{i=2}^{m-1} C_{m-1}^i \int_0^T \left(\frac{d^m w}{dt^m}, \frac{d}{dt}\left[A_0^{-1}(t)\frac{d^i e^{c(T-t)}}{dt^i}\frac{d^{m-1-i}J(t)w}{dt^{m-1-i}}\right]\right) dt \\
&\quad - (2m-2)c\,\mathrm{Re}\int_0^T \left(\frac{d^m w}{dt^m}, \frac{d}{dt}\left[A_0^{-1}(t)e^{c(T-t)}\frac{d^{m-2}J(t)w}{dt^{m-2}}\right] - A_0^{-1}(t)e^{c(T-t)}(T-t)\frac{d^m w}{dt^m}\right) dt \\
&\quad - 2\,\mathrm{Re}\sum_{i=2}^{m+1} C_{m+1}^i \int_0^T \left(\frac{d^i A_0^{-1}(t)}{dt^i}\frac{d^{m+1-i}w}{dt^{m+1-i}}, \frac{d^{m-1}}{dt^{m-1}}\left[e^{c(T-t)}J(t)w\right]\right) dt \qquad (42) \\
&\quad - (2m+2)\,\mathrm{Re}\int_0^T \left(\frac{dA_0^{-1}(t)}{dt}\frac{d^m w}{dt^m}, \frac{d^{m-1}}{dt^{m-1}}\left[e^{c(T-t)}J(t)w\right] - e^{c(T-t)}(T-t)\frac{d^m w}{dt^m}\right) dt \\
&\quad - (2m+1)\int_0^T e^{c(T-t)}(T-t)\left(\frac{dA_0^{-1}(t)}{dt}\frac{d^m w}{dt^m}, \frac{d^m w}{dt^m}\right) dt - \Phi^{(2)}(w,w) - \Phi^{(1)}(w,w) \\
&\quad - (2m-1)c\int_0^T e^{c(T-t)}(T-t)\left|A_0^{-1/2}(t)\frac{d^m w}{dt^m}\right|^2 dt - \lambda_m\int_0^T [c(T-t)+2m-1]e^{c(T-t)}|w|^2 dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi^{(l)}(w,w) &= 2\,\mathrm{Re}\sum_{s=l}^{2m-l}(-1)^{[s/2]} \\
&\quad \times \Bigg\{\left[\frac{s+1}{2}\right]\int_0^T e^{c(T-t)}(T-t)\left(A_s(t)\frac{dA_0^{-1}(t)}{dt}\frac{d^{[(s-1)/2]}w}{dt^{[(s-1)/2]}}, \frac{d^{[(s+2)/2]}w}{dt^{[(s+2)/2]}}\right) dt \\
&\quad + \left[\frac{s+1}{2}\right]\int_0^T \Bigg(A_s(t)\frac{dA_0^{-1}(t)}{dt}\frac{d^{[(s-1)/2]}w}{dt^{[(s-1)/2]}}, \Bigg(\frac{d^{[s/2]}}{dt^{[s/2]}}\left[e^{c(T-t)}J(t)w\right] \\
&\quad - e^{c(T-t)}(T-t)\frac{d^{[(s+2)/2]}w}{dt^{[(s+2)/2]}}\Bigg)\Bigg) dt \\
&\quad + \sum_{j=2}^{[(s+1)/2]} C_{[(s+1)/2]}^j \int_0^T \left(A_s(t)\frac{d^j A_0^{-1}(t)}{dt^j}\frac{d^{[(s+1)/2]-j}w}{dt^{[(s+1)/2]-j}}, \frac{d^{[s/2]}}{dt^{[s/2]}}\left[e^{c(T-t)}J(t)w\right]\right) dt \qquad (43) \\
&\quad + \sum_{i=1}^{[s/2]} C_{[s/2]}^i \int_0^T \left(A_s(t)A_0^{-1}(t)\frac{d^{[(s+1)/2]}w}{dt^{[(s+1)/2]}}, \frac{d^i e^{c(T-t)}}{dt^i}\frac{d^{[s/2]-i}J(t)w}{dt^{[s/2]-i}}\right) dt \\
&\quad + \sum_{j=0}^{1}\int_0^T e^{c(T-t)}\left((1-j)\left(m-1-\left[\frac{s}{2}\right]\right) + j(T-t)\right) \\
&\quad \times \left(A_s(t)A_0^{-1}(t)\frac{d^{[(s+1)/2]}w}{dt^{[(s+1)/2]}}, \frac{d^{[s/2]+j}w}{dt^{[s/2]+j}}\right) dt\Bigg\}, \qquad l = 2; 1.
\end{aligned}
$$

Moreover, the summation in (43) ranges over even values of s for $l = 2$ and over odd values of s for $l = 1$. For the operators $A_{2k}(t)$, $k > 0$, and $A_{2k+1}(t)$, $k \geq 0$, that do not satisfy inequality (8) for $j = 1$, in the first integrals of these forms, we integrate by parts once:

$$\begin{aligned}\int_0^T & e^{c(T-t)}(T-t)\left(A_s(t)\frac{dA_0^{-1}(t)}{dt}\frac{d^{[(s-1)/2]}w}{dt^{[(s-1)/2]}}, \frac{d^{[(s+2)/2]}w}{dt^{[(s+2)/2]}}\right)dt \\ &= -\sum_{j=0}^{1}\int_0^T \frac{d^j e^{c(T-t)}(T-t)}{dt^j}\left(A_s(t)\frac{dA_0^{-1}(t)}{dt}\frac{d^{[(s+1)/2]-j}w}{dt^{[(s+1)/2]-j}}, \frac{d^{[s/2]}w}{dt^{[s/2]}}\right)dt \\ &\quad -\sum_{i=1}^{2}\int_0^T e^{c(T-t)}(T-t)\left(\frac{d^{2-i}A_s(t)}{dt^{2-i}}\frac{d^i A_0^{-1}(t)}{dt^i}\frac{d^{[(s-1)/2]}w}{dt^{[(s-1)/2]}}, \frac{d^{[s/2]}w}{dt^{[s/2]}}\right)dt;\end{aligned} \tag{44}$$

in (44), we use the symmetry of the operators $A_s(t)\left(dA_0^{-1}(t)/dt\right)$ in H for odd s.

If $m = 1$, then the right-hand side of (42) can be estimated above by the expression

$$\begin{aligned}&\left(c_0^{(2)} + 3c_0^{(1)} + \tilde{c}_1^{(1)} + 2\tilde{c}_1 - c\right)\int_0^T e^{c(T-t)}(T-t)\left|A_0^{-1/2}(t)\frac{dw}{dt}\right|^2 dt \\ &\quad + \left[\left(c_0^{(2)} + \tilde{c}_1^{(1)} - c\lambda_1\right)(T-t) - \lambda_1\right]\int_0^T e^{c(T-t)}|w|^2 dt;\end{aligned} \tag{45}$$

owing to inequalities (4) in the third integral, (3) in the fifth integral, (9) in the twelfth integral, (7) in the fourteenth and fifteenth integrals, (13) in the fifteenth integral, and either inequality (8) with $j = 1$ applied to the eleventh integral on the right-hand side of (42) or inequalities (8) with $j = 2$, (9), and (10) applied to (44) with $s = 1$. We see that the quantity (45) is nonpositive for all $c \geq c_4 = c_0^{(2)} + 3c_0^{(1)} + \tilde{c}_1^{(1)} + 2\tilde{c}_1$ and all $\lambda_1 \geq 1$.

If $m > 1$, then the right-hand side of (42) can be estimated above by the expressions

$$\begin{aligned}\int_0^T &(T-t)\left(\sum_{i=1}^{m}\sum_{j=0}^{m-1} c_{i,j}\left|\frac{d^i w}{dt^i}\right|_{-i,t}\left|\frac{d^j w}{dt^j}\right|_{-j,t} - c\lambda_m|w|^2\right)dt \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{m-1}\tilde{c}_{i,j}\left\|\frac{d^i w}{dt^i}\right\|_{-i}\left\|\frac{d^j w}{dt^j}\right\|_{-j} - (2m-1)\lambda_m\|w\|_0^2 \\ &\quad + \left[(2m+1)c_0^{(1)} - (2m-1)c\right]\int_0^T e^{c(T-t)}(T-t)\left|\frac{d^m w}{dt^w}\right|_{-m,t}^2 dt;\end{aligned} \tag{46}$$

owing to inequalities (14) with $\alpha = 1/2$ and $\beta = 1/(2m)$ applied in the first, second, and fourth integrals, (4) in the third integral, (3) in the fifth integral, (9) in the seventh and twelfth integrals, (8) in the eighth and thirteenth integrals, (7) in the ninth, tenth, fourteenth, and fifteenth integrals, (12) in the tenth integral, (13) in the fifteenth integral, and either inequality (8) with $j = 1$ in the sixth and eleventh integrals on the right-hand sides in (42) or inequalities (8) with $j = 2$ and (9)–(11) used in (44). We assume that, all in all, there are seventeen integrals on the right-hand side in (42). The constants $c_{i,j}, \tilde{c}_{i,j} \geq 0$ depend only on c, m, T, and the constants occurring in Conditions I–IV. If we use the δ-inequality and the estimates (18) and (19) in (46), then we find first $c_{2m+11} \geq (2m+1)c_0^{(1)}/(2m-1)$ and then $\tilde{\lambda}_m \geq \hat{\lambda}_m$ such that, for $c = c_{2m+11}$ and all $\lambda_m \geq \tilde{\lambda}_m$, the expressions (46) do not exceed $(1 - c_{2m+12})\left\|d^m w/dt^m\right\|_{-m}^2$, $c_{2m+12} > 0$.

To complete the proof, it remains to estimate the left-hand sides of (42) via $\|d^m w/dt^m\|_{-m}^2$ for $c = c_4$ and $\lambda_1 \geq 1$ if $m = 1$ and for $c = c_{2m+11}$ and $\lambda_m \geq \tilde{\lambda}_m$ if $m > 1$, estimate the right-hand sides via $(1 - c_{2m+12}) \|d^m w/dt^m\|_{-m}^2$, collect similar terms, and find that $w = 0$ and hence $v = 0$. Inequalities (36) follow from (28).

Remark 2. In the same way, the assertion of Theorem 1 [possibly with larger values of $c_0(m)$ and $\hat{\lambda}_m$] and the assertion of Theorem 2 (with the use of the method of continuation with respect to a parameter and possibly with a larger value of $\tilde{\lambda}_m$) can be generalized to equations with lower-order terms:

$$L_m(\lambda_m) u + \sum_{k=0}^{2m-1} B_k(t) \frac{d^k u}{dt^k} = f, \qquad t \in]0, T[, \qquad m = 1, 2, \ldots,$$

where the linear unbounded operators

$$B_k(t) \in \mathscr{B}\left([0, T], \mathscr{L}\left(W^{2m-k-1}(t), H\right)\right), \qquad 0 \leq k \leq 2m - 1,$$

with domain $D(B_k(t)) \supset D(A_0(t))$ depending on t have strong derivatives

$$d^j B_k(t)/dt^j \in \mathscr{B}\left([0, T], \mathscr{L}\left(W^{2m-k+j-1}(t), H\right)\right),$$

$1 \leq j \leq k - m$, $0 \leq k \leq 2m - 1$, in H for all $t \in [0, T]$ such that

$$\left|\left(\frac{d^j B_k(t)}{dt^j} u, v\right)\right| \leq \begin{cases} c_{2m+13}^{(0)} |u|_{m-k,t} |v|_{m-1,t} & \text{if } \ 0 \leq k \leq m \\ c_{2m+13}^{(j)} |u| |v|_{2m-k+j-1,t} & \text{if } \ 1 \leq j \leq k - m, \ m < k \leq 2m - 1, \end{cases}$$

for all $u, v \in D(A_0(t))$; here the $c_{2m+13}^{(j)} \geq 0$ are constants independent of u, v, and t.

5. EXAMPLES OF PROBLEMS

For each $m = 1, 2, \ldots$, we construct new well-posed boundary value problems for complete "hyperbolic" partial differential equations whose order with respect to $x \in \mathbb{R}^n$ depends smoothly on t. These problems illustrate sufficient existence and uniqueness conditions for strong solutions of the boundary value problems (1), (2) in the framework of Theorems 1 and 2. First, we describe the operator coefficients of the differential equations (1).

Let the differential operators

$$A_0(t) = (I - \Delta_x)^{p(t)}, \qquad p(t) > n/2, \qquad p(t) \in C[0, T], \qquad x \in \mathbb{R}^n, \tag{47}$$

with t-dependent domains

$$D(A_0(t)) = \left\{u(x) \in L_2(\mathbb{R}^n) : \ (I - \Delta_x)^{p(t)} u(x) \in L_2(\mathbb{R}^n)\right\} \tag{48}$$

and the differential operators

$$\begin{gathered} A_s(t) = (-1)^{[s/2]} (I - \Delta_x)^{p(t)(1 - s/(2m) - \varepsilon_s)}, \\ 0 < \varepsilon_{2k} < 1/(2m), \qquad k = 1, \ldots, m - 1; \qquad \varepsilon_{2k+1} = 0, \qquad k = 0, \ldots, m - 1, \end{gathered} \tag{49}$$

with t-dependent domains

$$\begin{gathered} D(A_s(t)) = \left\{u(x) \in L_2(\mathbb{R}^n) : \ (I - \Delta_x)^{p(t)(1 - s/(2m) - \varepsilon_s)} u(x) \in L_2(\mathbb{R}^n)\right\}, \\ s = 1, \ldots, 2m - 1, \end{gathered} \tag{50}$$

act in the Hilbert space $H = L_2(\mathbb{R}^n)$, $n \geq 1$, for each $t \in [0, T]$, where Δ_x is the Laplace operator with respect to $x = (x_1, \ldots, x_n)$.

Here the fractional partial derivatives are defined via the direct and inverse Fourier–Plancherel transforms in $L_2(\mathbb{R}^n)$ [3, p. 108]:

$$F[g](\xi) = \lim_{R\to\infty} \int\limits_{|x|<R} g(x)e^{i(\xi,x)}dx, \qquad F^{-1}[g](x) = \frac{1}{(2\pi)^n} \lim_{R\to\infty} \int\limits_{|x|<R} g(\xi)e^{-i(x,\xi)}d\xi.$$

Namely,

$$A_s(t)u(x) = (-1)^{[s/2]}F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)(1-s/(2m)-\varepsilon_s)} F[u](\xi)\right](x),$$
$$D\left(A_s(t)\right) = \left\{u(x) \in L_2\left(\mathbb{R}^n\right): \ \left(1+|\xi|^2\right)^{p(t)(1-s/(2m)-\varepsilon_s)} F[u](\xi) \in L_2\left(\mathbb{R}^n\right)\right\}, \qquad s > 0$$

[3, pp. 135–136; 4, p. 38 of the Russian translation]. The operators $A_0(t)$ with domains $D\left(A_0(t)\right)$ [see Eq. (57) below with $\alpha = 2m$] are defined in a similar way. The boundary value problems obtained from problems (1), (2) with the differential operator coefficients (47)–(50) will be denoted by (1′), (2′).

Now, following Section 3, for the operators (47)–(50), we construct spaces of strong solutions of the boundary value problems (1′), (2′), namely, the Hilbert spaces $\mathscr{E}^m = \mathscr{E}^m\left([0,T]\times\mathbb{R}^n\right)$ defined as the completions of the sets $\mathscr{D}\left(L_m\right)$ corresponding to the operators (47)–(50) with respect to the Hermitian norms

$$|||u|||_{\mathscr{E}^m} = \left(\int\limits_0^T\int\limits_{\mathbb{R}^n}\left(\left|\partial^m u(t,x)/\partial t^m\right|^2 + \left|(I-\Delta_x)^{p(t)/2}u(t,x)\right|^2\right)dx\,dt\right)^{1/2};$$

we also construct the spaces of right-hand sides of Eq. (1′), that is, the Banach spaces $\hat{\mathscr{F}}^{-(m-1)} = \hat{\mathscr{F}}^{-(m-1)}\left(]0,T[\times\mathbb{R}^n\right)$ defined as the completions of the set $L_2\left(]0,T[\times\mathbb{R}^n\right)$ with respect to the norms

$$\langle\|f\|\rangle_{\hat{\mathscr{F}}^{-(m-1)}} = \sup_v \left\{\left|\int\limits_0^T\int\limits_{\mathbb{R}^n} f(t,x)\overline{v(t,x)}\,dx\,dt\right| \Big/ \langle\|v\|\rangle_{m-1}\right\},$$
$$v \in \hat{\mathscr{E}}^{m-1} = \hat{\mathscr{E}}^{m-1}\left([0,T]\times\mathbb{R}^n\right),$$

where the $\hat{\mathscr{E}}^{m-1}$ are the completions of the sets

$$\hat{\mathscr{D}}^m = \Big\{v \in L_2\left(]0,T[\times\mathbb{R}^n\right): \ \partial^k v/\partial t^k \in D\left(A_0^{(m-k)/(2m)}(t)\right), \ t\in[0,T];$$
$$A_0^{(m-k)/(2m)}(t)\partial^k v/\partial t^k \in L_2\left(]0,T[\times\mathbb{R}^n\right), \ k=0,\dots,m;$$
$$\left.\partial^i v/\partial t^i\right|_{t=0} = \left.\partial^i v/\partial t^i\right|_{t=T} = 0, \ i=0,\dots,m-1\Big\}$$

with respect to the Hermitian norms

$$\langle\|v\|\rangle_{m-1} = \left(\sum_{k=0}^{m-1}\int\limits_0^T\int\limits_{\mathbb{R}^n}(T-t)^{-2}\left|A_0^{(m-1-k)/(2m)}(t)\frac{\partial^k v}{\partial t^k}\right|^2dx\,dt\right)^{1/2}.$$

Theorem 3. *If $n/2 < p(t) \in C^{(m+1)}[0,T]$ and $p'(t) \le 0$, $t\in[0,T]$, then there exist constants $\tilde{c}_0(m) > 0$ and sets $\tilde{\Lambda}_1 = [1,+\infty[$ for $m = 1$ and $\tilde{\Lambda}_m = [\tilde{\lambda}_m,+\infty[$ for $m > 1$ such that for each $\lambda_m \in \tilde{\Lambda}_m$ and $f \in \hat{\mathscr{F}}^{-(m-1)}$, there exists a unique strong solution $u \in \mathscr{E}^m$ of the boundary value problems* (1′), (2′) *and*

$$|||u|||_{\mathscr{E}^m} \le \tilde{c}_0(m)\langle\|f\|\rangle_{\hat{\mathscr{F}}^{-(m-1)}}, \qquad m = 1,2,\dots \tag{51}$$

Proof. Let us show that all assumptions of Theorem 2 are valid for the boundary value problems (1′), (2′).

I′. The operators $A_0(t)$ are self-adjoint in $L_2(\mathbb{R}^n)$ for all t, since they are symmetric on $D(A_0(t))$ and have bounded inverses $A_0^{-1}(t)g = F^{-1}\left[(1+|\xi|^2)^{-p(t)}\right] * g \in L_2(\mathbb{R}^n)$ on $L_2(\mathbb{R}^n)$ for all $g \in L_2(\mathbb{R}^n)$ [3, pp. 182–185] provided that $p(t) > n/2$ for all t, which follows from the convolution estimate [3, p. 62] $\|h * g\|_{L_2} \le \|h\|_{L_1}\|g\|_{L_2}$. The symmetry of the operators $A_0(t)$ in $L_2(\mathbb{R}^n)$ follows from the relations

$$\begin{aligned}\langle A_0(t)u, v\rangle &= \left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)} F[u]\right], v\right\rangle = (2\pi)^{-n}\left\langle \left(1+|\xi|^2\right)^{p(t)} F[u], F[v]\right\rangle \\ &= (2\pi)^{-n}\left\langle F[u], \left(1+|\xi|^2\right)^{p(t)} F[v]\right\rangle = \left\langle u, F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)} F[v]\right]\right\rangle \\ &= \langle u, A_0(t)v\rangle\end{aligned}$$

for all $u, v \in D(A_0(t)) = \left\{u \in L_2(\mathbb{R}^n) : \ (1+|\xi|^2)^{p(t)} F[u](\xi) \in L_2(\mathbb{R}^n)\right\}$, $t \in [0, T]$, which can be obtained with the use of the Parseval relation [3, p. 108]

$$\langle h, g\rangle = (2\pi)^{-n}\langle F[h], F[g]\rangle \qquad \forall h, g \in L_2(\mathbb{R}^n) \tag{52}$$

for the inner product $\langle\cdot,\cdot\rangle$ in $L_2(\mathbb{R}^n)$ with the norm $\|\cdot\|$ and the inversion formulas [3, p. 102]

$$F^{-1}\cdot F[g] = F\cdot F^{-1}[g] = g \qquad \forall g \in L_2(\mathbb{R}^n). \tag{53}$$

In particular, from the relations obtained above for $v = u$, we have

$$\begin{aligned}\langle A_0(t)u, u\rangle &= (2\pi)^{-n}\left\|\left(1+|\xi|^2\right)^{p(t)/2} F[u]\right\|^2 \ge (2\pi)^{-n}\|F[u]\|^2 \\ &= \|u\|^2 \qquad \forall u \in D(A_0(t)), \qquad \forall t,\end{aligned}$$

which implies that the operators $A_0(t)$ are positive definite in $L_2(\mathbb{R}^n)$ for $c_0(t) = 1$.

By taking account of the continuity of F^{-1} in $L_2(\mathbb{R}^n)$ and by differentiating with respect to t, we find that there exists a strong derivative

$$\left(dA_0^{-1}(t)/dt\right) g = -p'(t)\left(F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)} \ln\left(1+|\xi|^2\right)\right] * g\right),$$

which is bounded in $L_2(\mathbb{R}^n)$ for all t and all $g \in L_2(\mathbb{R}^n)$ by virtue of the inequalities

$$\begin{aligned}\left\|\frac{dA_0^{-1}(t)}{dt} g\right\|^2 &= (p'(t))^2\left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)}\ln\left(1+|\xi|^2\right)\right] * g\right\|^2 \\ &= \frac{(p'(t))^2}{(2\pi)^n}\left\|\left(1+|\xi|^2\right)^{-p(t)}\ln\left(1+|\xi|^2\right) F[g]\right\|^2 \\ &\le (2\pi)^{-n}(\varrho e)^{-2}(p'(t))^2\left\|\left(1+|\xi|^2\right)^{-p(t)+\varrho} F[g]\right\|^2 \\ &\le (2\pi)^{-n}(\varrho e)^{-2}(p'(t))^2\|F[g]\|^2 = (\varrho e)^{-2}(p'(t))^2\|g\|^2;\end{aligned}$$

here we have used relations (52) and (53) and the estimates

$$\ln^i z \le (i/(\varrho e))^i z^\varrho \quad \forall z \ge 1, \quad \forall \varrho > 0, \quad i = 1, 2, \ldots, \tag{54}$$

valid for $i = 1$ and for any sufficiently small exponent $\varrho > 0$. In a similar way, we obtain the inequalities $\left\|A_0^{-1}(t)g\right\|^2 \le \|g\|^2$ for all $g \in L_2(\mathbb{R}^n)$ and all t.

Since $p'(t) \le 0$ for all t, it follows from (52) and (53) and the convolution transformation formula [3, p. 105]

$$F[h * g] = F[h]\cdot F[g] \quad \forall h \in L_1(\mathbb{R}^n), \quad \forall g \in L_2(\mathbb{R}^n) \tag{55}$$

that inequality (3) is valid for $c_0^{(1)} = 0$ for any t:

$$-\left\langle \left(dA_0^{-1}(t)/dt\right) g, g\right\rangle = p'(t) \left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)} \ln\left(1+|\xi|^2\right)\right] * g, g\right\rangle$$
$$= (2\pi)^{-n} p'(t) \left\langle \left(1+|\xi|^2\right)^{-p(t)} \ln\left(1+|\xi|^2\right) F[g], F[g]\right\rangle \le 0 \quad \forall g \in L_2\left(\mathbb{R}^n\right).$$

When deriving equalities and inequalities in what follows, we often use properties (52), (53), and (55) without mentioning this explicitly.

II′. By using the continuity of F^{-1} in $L_2(\mathbb{R}^n)$ and by differentiating with respect to t, we obtain the second strong derivative

$$\frac{d^2 A_0^{-1}(t)}{dt^2} g = -p''(t) F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)} \ln\left(1+|\xi|^2\right)\right] * g$$
$$+ \left(p'(t)\right)^2 F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)} \ln^2\left(1+|\xi|^2\right)\right] * g$$

for all $g \in L_2(\mathbb{R}^n)$, and then we successively compute the strong derivatives $d^j A_0^{-1}(t)/dt^j$, $3 \le j \le m+1$, since $p(t) \in C^{(m+1)}[0,T]$. The fact that they are bounded in $L_2(\mathbb{R}^n)$ can be proved by analogy with the boundedness of $dA_0^{-1}(t)/dt$. If

$$0 < \varrho \le \min_{[0,T]} p(t)/(2m) \le (j-1) \min_{[0,T]} p(t)/(2m), \qquad j = 2, \ldots, m+1,$$

in (54) for all t, g, and $v \in L_2(\mathbb{R}^n)$, then these derivatives satisfy inequalities (4):

$$\left|\left\langle \left(d^j A_0^{-1}(t)/dt^j\right) g, v\right\rangle\right|$$
$$\le \sum_{i=1}^{j} c_i(t) \left|\left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)} \ln^i\left(1+|\xi|^2\right)\right] * g, v\right\rangle\right|$$
$$\le \frac{1}{(2\pi)^n} \sum_{i=1}^{j} \left(\frac{i}{\varrho e}\right)^i c_i(t) \left\|\left(1+|\xi|^2\right)^{-p(t)/2+\varrho} F[g]\right\| \left\|\left(1+|\xi|^2\right)^{-p(t)/2} F[v]\right\|$$
$$\le \frac{1}{(2\pi)^n} \sum_{i=1}^{j} \left(\frac{i}{\varrho e}\right)^i c_i(t) \left\|\left(1+|\xi|^2\right)^{-p(t)(m+1-j)/(2m)} F[g]\right\| \left\|\left(1+|\xi|^2\right)^{-p(t)/2} F[v]\right\|$$
$$= \sum_{i=1}^{j} \left(\frac{i}{\varrho e}\right)^i c_i(t) \left\|A_0^{-(m-j+1)/(2m)}(t) g\right\| \left\|A_0^{-1/2}(t) v\right\|$$
$$= \sum_{i=1}^{j} \left(\frac{i}{\varrho e}\right)^i c_i(t) |g|_{-(m+1-j),t} |v|_{-m,t},$$

where the functions $c_i(t)$ can be expressed via $p(t)$ and its derivatives of orders $\le j-i+1$. Here we have used the representation of negative fractional powers of the operators $A_0(t)$ in the form

$$A_0^{-\alpha/(2m)}(t) g = F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)\alpha/(2m)}\right] * g, \qquad \alpha > 0. \tag{56}$$

III′. The operators $A_s(t)$, $s > 0$, satisfy the estimate (5), since

$$|\langle A_s(t) u, v\rangle| = \left|\left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)(1-s/(2m)-\varepsilon_s)} F[u]\right], v\right\rangle\right|$$
$$= (2\pi)^{-n} \left|\left\langle \left(1+|\xi|^2\right)^{p(t)(1-s/(2m)-\varepsilon_s)} F[u], F[v]\right\rangle\right|$$
$$= (2\pi)^{-n} \left|\left\langle \left(1+|\xi|^2\right)^{p(t)(1/2-[(s+1)/2]/(2m)-\varepsilon_s)} F[u], \left(1+|\xi|^2\right)^{p(t)(1/2-[s/2]/(2m))} F[v]\right\rangle\right|$$
$$\le \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)(1/2-[(s+1)/2]/(2m))} F[u]\right]\right\|$$
$$\times \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)(1/2-[s/2]/(2m))} F[v]\right]\right\| = |u|_{m-[(s+1)/2],t} |v|_{m-[s/2],t}$$

for all t and u and for all $v \in L_2(\mathbb{R}^n)$. Here we have used the following representations of positive fractional powers of the operators $A_0(t)$:

$$A_0^{\alpha/(2m)}(t)u = F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)\alpha/(2m)} F[u]\right], \qquad \alpha > 0. \tag{57}$$

Let us show that the operators $A_s(t)$ are strongly differentiable with respect to t in $L_2(\mathbb{R}^n)$ on the sets $D\left(A_0^{1-s/(2m)-\varepsilon_s+\eta}(t)\right)$ for each small $\eta > 0$ and compute their strong derivative $dA_s(t)/dt$ with t-dependent domains $D(A_s(t))$, $s > 0$ (see Definition 1). For each $t_0 \in [0,T]$ and

$$u(t_0) \in D\left(A_0^{1-s/(2m)-\varepsilon_s+\eta}(t_0)\right) \subset D(A_s(t_0)),$$

there exists a $g_0 \in L_2(\mathbb{R}^n)$ such that $u(t_0) = A_0^{-(1-s/(2m)-\varepsilon_s+\eta)}(t_0)\, g_0$. Then there exists

$$u(t) = A_0^{-(1-s/(2m)-\varepsilon_s)}(t) A_0^{-\eta}(t_0)\, g_0 \in D(A_s(t)), \qquad t \neq t_0;$$

moreover, by (56) and by the continuity of F^{-1} in $L_2(\mathbb{R}^n)$ and of $p(t)$ with respect to t, we have

$$\begin{aligned}\lim_{t\to t_0} u(t) &= F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t_0)(1-s/(2m)-\varepsilon_s)}\right] * A_0^{-\eta}(t_0)\, g_0 \\ &= A_0^{-(1-s/(2m)-\varepsilon_s+\eta)}(t_0)\, g_0 = u(t_0)\end{aligned}$$

in $L_2(\mathbb{R}^n)$. By using the continuity of F^{-1} in $L_2(\mathbb{R}^n)$, we also show that the derivative

$$\begin{aligned}u'(t_0) = &-(1-s/(2m)-\varepsilon_s)\, p'(t_0) \\ &\times \left(F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t_0)(1-s/(2m)-\varepsilon_s)} \ln\left(1+|\xi|^2\right)\right] * A_0^{-\eta}(t_0)\, g_0\right) \in D(A_s(t_0))\end{aligned}$$

exists provided that $0 < \varrho \le \eta \min_{[0,T]} p(t)$ in (54), where $\eta > 0$. Moreover, the derivative

$$h'(t_0) = (A_s u)'_t(t_0) = (-1)^{[s/2]}\left(A_0^{-\eta}(t_0)\, g_0\right)'_t(t_0) = 0$$

exists. This, together with (57) and Definition 1, implies that for $s > 0$ in $L_2(\mathbb{R}^n)$, there exists a strong derivative

$$\begin{aligned}(dA_s(t_0)/dt)\, u(t_0) &= h'(t_0) - A_s(t_0)\, u'(t_0) \\ &= -(-1)^{[s/2]} F^{-1}\left[\left(1+|\xi|^2\right)^{p(t_0)(1-s/(2m)-\varepsilon_s)} F[u'(t_0)]\right] \\ &= (-1)^{[s/2]}(1-s/(2m)-\varepsilon_s)\, p'(t_0)\, F^{-1}\left[\ln\left(1+|\xi|^2\right) F\left[A_0^{-\eta}(t_0)\, g_0\right]\right] \\ &= (-1)^{[s/2]}(1-s/(2m)-\varepsilon_s)\, p'(t_0) \\ &\quad \times F^{-1}\left[\left(1+|\xi|^2\right)^{p(t_0)(1-s/(2m)-\varepsilon_s)} \ln\left(1+|\xi|^2\right) F[u(t_0)]\right].\end{aligned} \tag{58}$$

Since $p(t) \in C^{(m+1)}[0,T]$, it follows that, in a similar way, one can compute higher-order strong derivatives $d^iA_s(t)/dt^i$ such that

$$\begin{gathered}\left(d^iA_s(t)/dt^i\right) A_0^{-(1-s/(2m)+\tau/(2m))}(t) \in \mathscr{B}([0,T], \mathscr{L}(L_2(\mathbb{R}^n))), \\ \tau > 0, \qquad i = 1,\dots,[s/2], \qquad s = 1,\dots,2m-1.\end{gathered}$$

The symmetry of $A_s(t)$ in $L_2(\mathbb{R}^n)$ and the validity of inequalities (6) for $i = 0$ and $s = 2k+1$, $k = 0,\dots,m-1$, follow from the fact that $A_0(t)$ is self-adjoint in $L_2(\mathbb{R}^n)$. Since $p'(t) \le 0$ for all t, it follows from (58) that the validity of (6) for $i = 1$ and $s = 2k$, $k = 1, \ldots, m-1$, is a corollary to the relation

$$\begin{aligned}
&(-1)^k \langle (dA_{2k}(t)/dt)\, u, u\rangle \\
&\quad = (1 - k/m - \varepsilon_{2k})\, p'(t) \left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{p(t)(1-k/m-\varepsilon_{2k})} \ln\left(1+|\xi|^2\right) F[u]\right], u\right\rangle \\
&\quad = (2\pi)^{-n} (1 - k/m - \varepsilon_{2k})\, p'(t) \left\langle \left(1+|\xi|^2\right)^{p(t)(1-k/m-\varepsilon_{2k})} \ln\left(1+|\xi|^2\right) F[u], F[u]\right\rangle \\
&\quad \le 0 \qquad \forall u \in D\left(A_0(t)\right), \qquad \forall t.
\end{aligned}$$

IV′. By (56) and (57), all operators $A_s(t)$, $s > 0$, satisfy inequalities (7) for all t:

$$\begin{aligned}
\left|\left\langle A_s(t) A_0^{-1}(t) g, v\right\rangle\right| &= \left|\left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} F[g]\right], v\right\rangle\right| \\
&= (2\pi)^{-n} \left|\left\langle \left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} F[g], F[v]\right\rangle\right| \\
&\le (2\pi)^{-n} \left\|\left(1+|\xi|^2\right)^{-p(t)[(s+1)/2]/(2m)} F[g]\right\| \left\|\left(1+|\xi|^2\right)^{-p(t)[s/2]/(2m)} F[v]\right\| \\
&= \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)[(s+1)/2]/(2m)}\right] * g\right\| \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)[s/2]/(2m)}\right] * v\right\| \\
&= |g|_{-[(s+1)/2],t} |v|_{-[s/2],t} \quad \forall g, v \in L_2\left(\mathbb{R}^n\right).
\end{aligned}$$

The boundedness of the operators $A_s(t)\left(dA_0^{-1}(t)/dt\right)$, $s > 0$, in $L_2\left(\mathbb{R}^n\right)$ for all t follows from the relations

$$\begin{aligned}
\left\|A_s(t)\left(dA_0^{-1}(t)/dt\right) g\right\|^2 &= \left(p'(t)\right)^2 \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} \ln\left(1+|\xi|^2\right) F[g]\right]\right\|^2 \\
&= \frac{\left(p'(t)\right)^2}{(2\pi)^n} \left\|\left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} \ln\left(1+|\xi|^2\right) F[g]\right\|^2 \\
&\le \frac{\left(p'(t)\right)^2}{(\varrho e)^2 (2\pi)^n} \left\|\left(1+|\xi|^2\right)^{\varrho - p(t)s/(2m)} F[g]\right\|^2 \\
&\le (2\pi)^{-n} (\varrho e)^{-2} \left(p'(t)\right)^2 \|F[g]\|^2 \\
&= (\varrho e)^{-2} \left(p'(t)\right)^2 \|g\|^2 \quad \forall g \in L_2\left(\mathbb{R}^n\right)
\end{aligned}$$

provided that $0 < \varrho \le \min_{[0,T]} p(t)/(2m)$ in (54). In a similar way, one can show that

$$A_s(t)\left(d^j A_0^{-1}(t)/dt^j\right) \in \mathscr{B}\left([0,T], \mathscr{L}\left(L_2\left(\mathbb{R}^n\right)\right)\right),$$

$j = 2, \ldots, [(s+1)/2]$, $s = 1, \ldots, 2m-1$, $s > 0$.

By using (56), one can show that if $A_s(t)$, $s > 0$, then inequality (8) with $j = 2$ is valid for $0 < \varrho \le \min_{[0,T]} p(t)/(2m)$ in (54), since

$$\begin{aligned}
&\left|\left\langle A_s(t)\left(d^2 A_0^{-1}(t)/dt^2\right) g, v\right\rangle\right| \\
&= (2\pi)^{-n} \left|\left\langle \left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} \ln\left(1+|\xi|^2\right) \left\{p''(t) - \left(p'(t)\right)^2 \ln\left(1+|\xi|^2\right)\right\} F[g], F[v]\right\rangle\right| \\
&\le (2\pi)^{-n} \sum_{i=1}^{2} b_i(t) \left\|\left(1+|\xi|^2\right)^{-p(t)([(s+1)/2]-1)/(2m)+\varrho} F[g]\right\| \left\|\left(1+|\xi|^2\right)^{-p(t)([s/2]+1)/(2m)} F[v]\right\| \\
&\le (2\pi)^{-n} \sum_{i=1}^{2} b_i(t) \left\|\left(1+|\xi|^2\right)^{-p(t)([(s+1)/2]-2)/(2m)} F[g]\right\| \left\|\left(1+|\xi|^2\right)^{-p(t)([s/2]+1)/(2m)} F[v]\right\| \\
&= \sum_{i=1}^{2} b_i(t) \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)([(s+1)/2]-2)/(2m)}\right] * g\right\| \left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)([s/2]+1)/(2m)}\right] * v\right\| \\
&= \sum_{i=1}^{2} b_i(t) |g|_{-[(s+1)/2]+2,t} |v|_{-[s/2]-1,t} \qquad \forall g, v \in L_2\left(\mathbb{R}^n\right)
\end{aligned}$$

for all t, where $b_i(t) = (i/(\varrho e))^i c_i(t)$, $c_1(t) = |p''(t)|$, and $c_2(t) = (p'(t))^2$. In a similar way, one can justify inequality (8) with $3 \le j \le [(s+1)/2]$, $s > 0$. Inequality (8) with $s = 2k+1 > 0$ fails for $j = 1$, although by using the inequality $\varepsilon_{2k} > 0$, one can show that (8) holds for $s = 2k > 0$ with $j = 1$, but formula (8) with any $s > 0$ is valid for $j = 2$ with another right-hand side in Condition IV. By (58), inequality (10) is valid for all $A_s(t)$, $s > 0$, t, and $g, v \in L_2(\mathbb{R}^n)$:

$$\begin{aligned}
&\left|\left\langle (dA_s(t)/dt)\left(dA_0^{-1}(t)/dt\right) g, v\right\rangle\right| \\
&\quad = (1 - s/(2m) - \varepsilon_s)\left(p'(t)\right)^2 \left|\left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} \ln^2\left(1+|\xi|^2\right) F[g]\right], v\right\rangle\right| \\
&\quad = (2\pi)^{-n}(1 - s/(2m) - \varepsilon_s)\left(p'(t)\right)^2 \left|\left\langle \left(1+|\xi|^2\right)^{-p(t)(s/(2m)+\varepsilon_s)} \ln^2\left(1+|\xi|^2\right) F[g], F[v]\right\rangle\right| \\
&\quad = (2\pi)^{-n}(1 - s/(2m) - \varepsilon_s)\left(p'(t)\right)^2 \\
&\qquad \times \left|\left\langle \left(1+|\xi|^2\right)^{-(p(t)/(2m))([(s-1)/2]+1+2m\varepsilon_s)} \ln^2\left(1+|\xi|^2\right) F[g], \left(1+|\xi|^2\right)^{-(p(t)/(2m))[s/2]} F[v]\right\rangle\right| \\
&\quad \le (2\pi)^{-n}(1 - s/(2m) - \varepsilon_s)\left(2p'(t)\right)^2 (\varrho e)^{-2} \left\|\left(1+|\xi|^2\right)^{-p(t)[(s-1)/2]/(2m)} F[g]\right\| \\
&\qquad \times \left\|\left(1+|\xi|^2\right)^{-p(t)[s/2]/(2m)} F[v]\right\| \\
&\quad = (1 - s/(2m) - \varepsilon_s)\left(2p'(t)\right)^2 (\varrho e)^{-2} |g|_{-[(s-1)/2],t} |v|_{-[s/2],t}
\end{aligned}$$

provided that $0 < \varrho \le \min_{[0,T]} p(t)/(2m)$ in (54). In a similar way, one can verify inequality (9).

By using (57), we verify inequalities (11) for all t:

$$\begin{aligned}
&(-1)^k \operatorname{Re}\left\langle A_{2k}(t)\left(dA_0^{-1}(t)/dt\right) g, g\right\rangle \\
&\qquad = -p'(t)\left\langle F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)(k/m+\varepsilon_{2k})} \ln\left(1+|\xi|^2\right) F[g]\right], g\right\rangle \\
&\qquad = -\frac{p'(t)}{(2\pi)^n}\left\|\left(1+|\xi|^2\right)^{-(p(t)/2)(k/m+\varepsilon_{2k})} \sqrt{\ln\left(1+|\xi|^2\right)}\, F[g]\right\|^2 \\
&\qquad \le \frac{|p'(t)|}{\varrho e(2\pi)^n}\left\|\left(1+|\xi|^2\right)^{-(p(t)/2)(k/m+\varepsilon_{2k})+\varrho/2} F[g]\right\|^2 \\
&\qquad \le \frac{|p'(t)|}{\varrho e(2\pi)^n}\left\|\left(1+|\xi|^2\right)^{-p(t)k/(2m)} F[g]\right\|^2 = \frac{|p'(t)|}{\varrho e}\left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)k/(2m)} F[g]\right]\right\|^2 \\
&\qquad = \frac{|p'(t)|}{\varrho e}|g|^2_{-k,t} \qquad \forall g \in L_2(\mathbb{R}^n)
\end{aligned}$$

provided that $0 < \varrho \le \varepsilon_{2k}\min_{[0,T]} p(t)$ in (54), where $\varepsilon_{2k} > 0$, $k > 0$. One can readily see that the product $A_{2k+1}(t)\left(dA_0^{-1}(t)/dt\right)$ of symmetric and obviously commuting operators $A_{2k+1}(t)$ and $dA_0^{-1}(t)/dt$ in $L_2(\mathbb{R}^n)$ is a symmetric operator. The validity of inequality (12) follows from the fact that $A_{2k}(t)$, $k > 0$, and $A_0(t)$ commute, and the validity of (13) follows from the fact that $A_{2k+1}(t)$ and $A_0(t)$ commute and from (6) with $s = 2k+1$, $k \ge 0$.

Let us verify the remaining assumptions of Theorem 2. We have

$$A_0^{1-1/(2m)}(t)\left(dA_0^{-1}(t)/dt\right) \in \mathscr{B}\left([0,T], \mathscr{L}\left(L_2(\mathbb{R}^n)\right)\right)$$

for $m > 1$, since

$$\begin{aligned}
&\left\|A_0^{1-1/(2m)}(t)\left(dA_0^{-1}(t)/dt\right) g\right\|^2 = \left(p'(t)\right)^2\left\|F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)/(2m)} \ln\left(1+|\xi|^2\right) F[g]\right]\right\|^2 \\
&\qquad \le \frac{\left(p'(t)\right)^2}{(\varrho e)^2(2\pi)^n}\left\|\left(1+|\xi|^2\right)^{-p(t)/(2m)+\varrho} F[g]\right\|^2 \le \frac{\left(p'(t)\right)^2}{(\varrho e)^2(2\pi)^n}\|F[g]\|^2 \\
&\qquad = \frac{\left(p'(t)\right)^2}{(\varrho e)^2}\|g\|^2 \qquad \forall g \in L_2(\mathbb{R}^n)
\end{aligned}$$

for all t provided that $0 < \varrho \le \min_{[0,T]} p(t)/(2m)$ in (54). In a similar way, one can show that

$$A_0^{-1/(2m)}(t) \left(dA_0^{-1}(t)/dt\right) A_0(t) \in \mathscr{B}\left([0,T], \mathscr{L}\left(L_2\left(\mathbb{R}^n\right)\right)\right)$$

for $m > 1$. Obviously, the requirement

$$A_0^{(m-j)/(2m)}(t) \left(d^j A_0^{-1}(t)/dt^j\right) \in \mathscr{B}\left([0,T], \mathscr{L}\left(L_2\left(\mathbb{R}^n\right)\right)\right), \quad 2 \le j \le m-1,$$

in Theorem 2 follows from the inclusions

$$A_s(t) \left(d^j A_0^{-1}(t)/dt^j\right) \in \mathscr{B}\left([0,T], \mathscr{L}\left(L_2\left(\mathbb{R}^n\right)\right)\right), \quad 2 \le j \le [(s+1)/2], \quad s > 0,$$

proved above in IV′.

The set $C_0^\infty\left([0,T] \times \mathbb{R}^n\right)$ of infinitely differentiable functions compactly supported in $]0,T[\times \mathbb{R}^n$ lies in $\hat{\mathscr{D}}^m$ [4, p. 18 of the Russian translation] and is dense in $L_2\left(]0,T[\times \mathbb{R}^n\right)$; therefore, the $\hat{\mathscr{D}}^m$ are dense in $L_2\left(]0,T[\times \mathbb{R}^n\right)$. The fact that the operators $L_m(\lambda_m)$ corresponding to the boundary value problems (1′), (2′) are closed follows from Lemma 7 provided that $A_0^{1-j/(2m)}(t)\left(d^j A_0^{-1}(t)/dt^j\right) \in \mathscr{B}\left([0,T], \mathscr{L}\left(L_2\left(\mathbb{R}^n\right)\right)\right)$, $2 \le j \le m-1$. This condition has been justified above for $j = 1$. The desired inclusion for $j = 2$ and for all t follows from the inequalities

$$\begin{aligned}
&\left\| A_0^{1-1/m}(t) \frac{d^2 A_0^{-1}(t)}{dt^2} g \right\|^2 \\
&\quad = \left\| F^{-1}\left[\left(1+|\xi|^2\right)^{-p(t)/m} \ln\left(1+|\xi|^2\right)\left\{p''(t) - \left(p'(t)\right)^2 \ln\left(1+|\xi|^2\right)\right\} F[g]\right]\right\|^2 \\
&\quad \le \frac{2}{(2\pi)^n} \sum_{i=1}^{2} c_i^2(t) \left\| \left(1+|\xi|^2\right)^{-p(t)/m} \ln^i\left(1+|\xi|^2\right) F[g]\right\|^2 \\
&\quad \le \frac{2}{(2\pi)^n} \sum_{i=1}^{2} \left(\frac{i}{\varrho e}\right)^{2i} c_i^2(t) \left\| \left(1+|\xi|^2\right)^{-p(t)/m+\varrho} F[g]\right\|^2 \\
&\quad \le \frac{2}{(2\pi)^n} \sum_{i=1}^{2} \left(\frac{i}{\varrho e}\right)^{2i} c_i^2(t) \|F[g]\|^2 = 2\sum_{i=1}^{2} \left(\frac{i}{\varrho e}\right)^{2i} c_i^2(t) \|g\|^2 \qquad \forall g \in L_2\left(\mathbb{R}^n\right)
\end{aligned}$$

provided that $0 < \varrho \le \min_{[0,T]} p(t)/m$ in (54). The desired inclusions for the remaining values $3 \le j \le m-1$ can be proved in a similar way. The energy inequalities (51) correspond to inequalities (36).

All assumptions of Theorem 2 are valid. The proof of Theorem 3 is complete.

Remark 3. The analysis of the proofs shows that all assumptions of Theorem 2 remain valid for the operators $a_s(t)A_s(t)$ obtained by multiplying the operators $A_s(t)$ of the boundary value problems (1′), (2′) (see Section 5) by arbitrary functions $a_s(t) \in C^{([s/2])}[0,T]$, $s > 0$. By virtue of Remark 2, the well-posedness of the boundary value problems (1′), (2′) is preserved if the left-hand sides of Eqs. (1′) are supplemented by the lower-order terms $\sum_{k=0}^{2m-1} \sum_{|\alpha(t)| \le p_k(t)} b_{\alpha(t),k}(t,x) D_x^{\alpha(t)} D_t^k$, where

$$\alpha(t) = \left(\alpha_1(t), \ldots, \alpha_n(t)\right), \qquad |\alpha(t)| = \alpha_1(t) + \cdots + \alpha_n(t), \qquad \alpha_i(t) \ge 0,$$

$$p_k(t) \le 2p(t)(1-(k+1)/(2m)), \qquad b_{\alpha(t),k}(t,x) \in C^{(0,|\alpha(t)|-p(t)(1-k/m))}\left([0,T] \times \mathbb{R}^n\right)$$

for $0 \le k \le m$ and $b_{\alpha(t),k}(t,x) \in C^{(k-m,|\alpha(t)|)}\left([0,T] \times \mathbb{R}^n\right)$ for $m < k \le 2m-1$.

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