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SHORT COMMUNICATIONS

Smoothness of Strong Solutions of Complete Hyperbolic Second-Order Differential Equations with Variable Domains of Operator Coefficients

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Smoothness properties of strong solutions of hyperbolic second-order differential equations with variable domains of operator coefficients in the case of a two-term leading part were analyzed in the paper [1, 2]. The paper [3] deals with the correct solvability of hyperbolic second-order operator-differential equations with variable domains and with a three-term leading part. In the present paper, we consider smoothness properties of strong solutions of hyperbolic second-order operator-differential equations with variable domains in the case of a three-term leading part. Boundary value problems for singular hyperbolic partial differential equations are examples of such operator-differential equations [3].

1. STATEMENT OF THE PROBLEM

Let H be a Hilbert space with inner product $(\cdot\,,\cdot)$ and norm $|\cdot|$. On a bounded interval $]0,T[$ of the real line $\mathbf{R}$, we consider the Cauchy problem for the differential equation

$$\mathscr{L}u \equiv d^2u/dt^2 + B(t)du/dt + A(t)u = f, \qquad t \in\,]0,T[\,, \tag{1}$$

with the initial conditions

$$l_0 u \equiv u|_{t=0} = \varphi, \qquad l_1 u \equiv (du/dt)|_{t=0} = \psi. \tag{2}$$

Here u and f are functions of t ranging in H; $A(t)$ and $B(t)$, $t \in \Theta$, are linear bounded operators in H with domains $D(A(t))$ and $D(B(t))$, respectively, depending on t; Θ is some set of full measure in $[0,T]$; φ and ψ are elements of H.

We assume that the operators $A(t)$, $t \in \Theta$, satisfy the following conditions.

A_1. The operators $A(t)$, $t \in \Theta$, are self-adjoint in H and satisfy the inequalities

$$(A(t)u,u) \ge c_1(t)|u|^2 \qquad \forall u \in D(A(t)),$$

where the constants $c_1(t) > 0$ are independent of u.

A_2. The inverses $A^{-1}(t) \in L_\infty(\,]0,T[\,,\mathscr{L}(H))$, $t \in \Theta$, have a strong regular derivative [4] $dA^{-1}(t)/dt \in L_\infty(\,]0,T[\,,\mathscr{L}(H))$ on $]0,T[\,$, which satisfies the inequality

$$-\left(\left(dA^{-1}(t)/dt\right)g,g\right) \le c_2\left(A^{-1}(t)g,g\right) \quad \forall g \in H,$$

where the constant $c_2 \ge 0$ is independent of g and t.

We assume that the operators $B(t)$, $t \in \Theta$, satisfy the following conditions.

B_1. One has $D(A(t)) \subset D(B(t))$, $t \in \Theta$, $B(t)A^{-1/2}(t) \in L_\infty(\,]0,T[\,,\mathscr{L}(H))$, where $A^{-1/2}(t)$ is the inverse of the square root $A^{1/2}(t)$, $t \in \Theta$, and $-\operatorname{Re}(B(t)u,u) \le c_3|u|^2$ for all $u \in D(A(t))$, where $c_3 \ge 0$ is a constant independent of u and t.

B_2. We have $B(t)\left(dA^{-1}(t)/dt\right) \in L_\infty(\,]0,T[\,,\mathscr{L}(H))$ and

$$\left|\left(B(t)\left(dA^{-1}(t)/dt\right)g,v\right)\right| \le c_4|g|\left|A^{-1/2}(t)v\right| \qquad \forall g,v \in H,$$

$$-\operatorname{Re}(B(t)u,A(t)u) \le c_5\left|A^{1/2}(t)u\right|^2 \qquad \forall u \in D(A(t)),$$

where the constants $c_4, c_5 \ge 0$ are independent of g, v, u, and t.

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Under the above-mentioned conditions, we investigate the smoothness of strong solutions of the Cauchy problem (1), (2) by the abstract method suggested in [1].

2. EXISTENCE AND UNIQUENESS THEOREMS FOR STRONG SOLUTIONS

Let us first introduce spaces and define strong solutions of the Cauchy problem (1), (2). By [5], the fractional powers $A^{\alpha}(t)$, $0 < \alpha < 1$, with domains $D(A^{\alpha}(t))$ in which $D(A(t))$ is dense are well defined for each $t \in \Theta$. By equipping the vector spaces $D\left(A^{q/2}(t)\right) \neq \{\varnothing\}$ with the Hermitian norms $|v|_{q(t)} = \left|A^{q/2}(t)v\right|$, we obtain a family $W^q(t)$, $t \in \Theta$, of Hilbert spaces with $W^0(t) = H$.

We define the space of strong solutions as the Banach space E that is the completion of the set

$$D(L) = \left\{u \in L_2(\,]0,T[\,,H) : u(t) \in D(A(t)),\ t \in \Theta;\ d^2u/dt^2,\ B(t)(du/dt),\ A(t)u \in L_2(\,]0,T[\,,H)\right\}$$

in the norm

$$\|u\|_E = \left[\sup_{0<t<T}\left(|du(t)/dt|^2 + \left|A^{1/2}(t)u(t)\right|^2\right)\right]^{1/2}.$$

Let $D(A(0)) \neq \{\varnothing\}$. As the space of right-hand sides f and initial data φ and ψ, we choose the Hilbert space $F = L_2(\,]0,T[\,,H) \times W^1(0) \times H$. The Cauchy problem (1), (2) corresponds to an unbounded linear operator $L \equiv \{\mathscr{L}, l_0, l_1\} : E \supset D(L) \to F$ with domain $D(L)$. Suppose that L is closable in these Banach spaces, i.e., if $u_n \to 0$ in E and $Lu_n = \{\mathscr{L}u_n, l_0u_n, l_1u_n\} \to \mathscr{F} = \{f, \varphi, \psi\}$ in F as $n \to \infty$, then $\mathscr{F} = 0$. Note that, for this condition to hold, it suffices to require that if $u_n \to 0$ in E and $\mathscr{L}u_n \to f$ in $L_2(\,]0,T[\,,H)$ as $n \to \infty$, then $f = 0$. Let $\bar{L}$ be the closure of L and $D\left(\bar{L}\right)$ the corresponding domain. Solutions of the operator equation $\bar{L}u = \mathscr{F}$, $\mathscr{F} = \{f, \varphi, \psi\} \in F$, are referred to as *strong solutions* of the Cauchy problem (1), (2).

Let us now derive an *a priori* estimate for strong solutions of the Cauchy problem (1), (2).

Theorem 1. *Suppose that $D(L)$ is dense in $L_2(\,]0,T[\,,H)$ and the operator L has a closure $\bar{L}$. If conditions A_1, A_2, and B_1 are satisfied, then there exists a constant $c_0 > 0$ independent of u such that*

$$\|u\|_E \le c_0 \left\|\bar{L}u\right\|_F \qquad \forall u \in D\left(\bar{L}\right). \tag{3}$$

Proof. The proof is similar to that of Theorem 1 in [1, 3].

It follows from Theorem 1 that if a solution of the Cauchy problem (1), (2) exists, then it is unique and continuous with respect to f, φ, and ψ.

The solvability of the Cauchy problem (1), (2) is established in the following assertion.

Theorem 2. *Let $D(L)$ be dense in $L_2(\,]0,T[\,,H)$, and let L admit a closure $\bar{L}$. If conditions A_1, A_2, B_1, and B_2 are satisfied and the operator family $dA^{-1}(t)/dt$ has a strong regular derivative $d^2A^{-1}(t)/dt^2 \in L_\infty(\,]0,T[\,,\mathscr{L}(H))$ satisfying the inequalities*

$$\left|\left(\left(d^2A^{-1}(t)/dt^2\right)g, v\right)\right| \le c_6|g|\left|A^{-1/2}(t)v\right| \qquad \forall g, v \in H, \qquad c_6 \ge 0,$$

then, for each $\mathscr{F} = \{f, \varphi, \psi\} \in F$, there exists a unique strong solution $u \in E$ of the Cauchy problem (1), (2); this solution admits the estimate

$$\|u\|_E \le c_0\left(\int_0^T |f(t)|^2 dt + |\varphi|^2_{1(0)} + |\psi|^2\right)^{1/2}.$$

The proof of the theorem is similar to that of Theorem 2 in [1, 3].

Remark 1. Using the same method for Theorem 1 (possibly, with a larger constant c_0) and the Schauder–Ladyzhenskaya method of continuation with respect to a parameter for Theorem 2, we can generalize these results to equations

$$d^2u/dt^2 + B(t)du/dt + A(t)u + \tilde{B}(t)du/dt + \tilde{A}(t)u = f \tag{4}$$

with lower-order terms, where $\tilde{B}(t), \tilde{A}(t)A^{-1/2}(t) \in L_\infty(\,]0,T[\,,\mathscr{L}(H))$.

3. SMOOTHNESS OF STRONG SOLUTIONS

It readily follows from the *a priori* estimate (3) for strong solutions that each strong solution of the Cauchy problem (1), (2) [respectively, (4), (2)] coincides with some function $u \in C^{(1)}([0,T],H)$ almost everywhere on $]0,T[$. Let us establish conditions providing that this function u has the second-order derivative with respect to t square-integrable in the scale of Hilbert spaces $W^q(t)$ almost everywhere on $]0,T[$.

Theorem 3. *Suppose that the assumptions of Theorem 2 are valid,* $D\left(A^{(q+1)/2}(t)\right) \neq \{\emptyset\}$, $t \in \Theta \cup 0$, *for some* $q \geq 1$, *and there exist strong regular derivatives*

$$d^i A^{-q/2}(t)/dt^i \in L_\infty(\,]0,T[\,,\mathscr{L}(H)), \qquad i=1,2,$$

such that

$$\begin{aligned} dA^{-q/2}(t)/dt &\in L_\infty\left(\,]0,T[\,,\mathscr{L}\left(H,W^q(t)\right)\right) \\ &\quad \cap L_\infty\left(\,]0,T[\,,\mathscr{L}\left(W^1(t),W^1(t)\right)\right), \\ B(t)\left(dA^{-q/2}(t)/dt\right),\ d^2A^{-q/2}(t)/dt^2 &\in L_\infty\left(\,]0,T[\,,\mathscr{L}\left(W^1(t),W^q(t)\right)\right), \\ B(t) &\in L_\infty\left(\,]0,T[\,,\mathscr{L}\left(W^{q+1}(t),W^q(t)\right)\right) \\ &\quad \cap L_\infty\left(\,]0,T[\,,\mathscr{L}\left(W^q(t),W^{q-1}(t)\right)\right), \\ B(t)A^{-q/2}(t)\left(dA^{-1}(t)/dt\right) &\in L_\infty\left(\,]0,T[\,,\mathscr{L}\left(H,W^q(t)\right)\right), \\ -\operatorname{Re}\left(A^{q/2}(t)B(t)A^{-q/2}(t)v,v\right) &\leq c_7|v|^2 \qquad \forall v \in D(A(t)), \qquad c_7 \geq 0; \\ -\operatorname{Re}\left(A^{q/2}(t)B(t)A^{-q/2}(t)v,A(t)v\right) &\leq c_8\left|A^{1/2}(t)v\right|^2 \qquad \forall v \in D(A(t)), \qquad c_8 \geq 0; \\ \left|\left(A^{q/2}(t)B(t)A^{-q/2}(t)\left(dA^{-1}(t)/dt\right)g,v\right)\right| &\leq c_9|g|\left|A^{-1/2}(t)v\right| \qquad \forall g,v \in H, \qquad c_9 \geq 0. \end{aligned}$$

If $f \in L_2(\,]0,T[\,,W^q(t))$, $\varphi \in W^{q+1}(0)$, *and* $\psi \in W^q(0)$, *then the Cauchy problem* (1), (2) *has a unique strong solution* $u \in E$ *such that* $u(t) \in D\left(A^{(q+1)/2}(t)\right)$ *for almost all* $t \in]0,T[$; $u \in L_\infty(\,]0,T[\,,W^{q+1}(t))$, $du/dt \in L_\infty(\,]0,T[\,,W^q(t))$, *and* $d^2u/dt^2 \in L_2(\,]0,T[\,,W^{q-1}(t))$.

Proof. Consider the auxiliary Cauchy problem

$$\begin{aligned} \tilde{\mathscr{L}}\, w \equiv d^2w/dt^2 + A^{q/2}(t)B(t)A^{-q/2}(t)(dw/dt) + A(t)w + A^{q/2}(t)B(t)\left(dA^{-q/2}(t)/dt\right)w& \\ + 2A^{q/2}(t)\left(dA^{-q/2}(t)/dt\right)(dw/dt) + A^{q/2}(t)\left(d^2A^{-q/2}(t)/dt^2\right)w = \tilde{f}&, \end{aligned} \tag{5}$$

$$l_0 w \equiv w|_{t=0} = \tilde{\varphi}, \qquad l_1 w \equiv (dw/dt)|_{t=0} = \tilde{\psi}, \tag{6}$$

where $\tilde{f} = A^{q/2}(t)f$, $\tilde{\varphi} = A^{q/2}(0)\varphi$, and $\tilde{\psi} = A^{q/2}(0)\psi - A^{q/2}(0)\left(dA^{-q/2}(0)/dt\right)A^{q/2}(0)\varphi$. Since $\tilde{f} \in L_2(]0,T[,H)$, $\tilde{\varphi} \in W^1(0)$, $\tilde{\psi} \in H$, and the operator coefficients $A(t)$, $A^{q/2}(t)B(t)A^{-q/2}(t)$, $\tilde{A}(t) = A^{q/2}(t)B(t)\left(dA^{-q/2}(t)/dt\right) + A^{q/2}(t)\left(d^2A^{-q/2}(t)/dt^2\right)$, and $\tilde{B}(t) = 2A^{q/2}(t)\left(dA^{-q/2}(t)/dt\right)$ satisfy all assumptions of Theorem 2 and Remark 1, it follows that there exists a unique strong solution $w \in E$ of problem (5), (6). By the definition of a strong solution, w is a solution of the operator equation $\bar{\tilde{L}}w \equiv \left\{\bar{\tilde{\mathscr{L}}}\, w, l_0 w, l_1 w\right\} = \tilde{\mathscr{F}}$, $\tilde{\mathscr{F}} = \left\{\tilde{f},\tilde{\varphi},\tilde{\psi}\right\} \in F$, where $\bar{\tilde{\mathscr{L}}}$ is the closure of the operator $\tilde{\mathscr{L}} : E \supset D(L) \to L_2(]0,T[,H)$.

We can readily prove the following assertion.

Lemma 1. *Let* $\mathscr{E}$ *and* $\mathscr{H}$ *be Banach spaces. If* $T_1 : \mathscr{E} \to \mathscr{H}$ *is a linear operator admitting a closure* $\overline{T}_1$, $S_1 : \mathscr{H} \to \mathscr{H}$ *is a linear bounded operator, and the product* $\overline{S_1 \cdot T_1}$ *admits a closure* $S_1 \cdot T_1 : \mathscr{E} \to \mathscr{H}$, *then* $S_1 \cdot \overline{T}_1 \subset \overline{S_1 \cdot T_1}$.

Using Lemma 1 with $\mathscr{E} = E$, $\mathscr{H} = L_2(]0,T[,H)$, $T_1 = \tilde{\mathscr{L}}$, and $S_1 = A^{-q/2}(t)$, we obtain the inclusion

$$A^{-q/2}(t)\bar{\tilde{\mathscr{L}}} \subset \overline{A^{-q/2}(t)\tilde{\mathscr{L}}}. \qquad (7)$$

We can readily see that

$$A^{-q/2}(t)\tilde{\mathscr{L}}\, w = \mathscr{L} A^{-q/2}(t)w \qquad \forall w \in D(L). \qquad (8)$$

The following assertion is quite obvious.

Lemma 2. *Let $\mathscr{E}$ and $\mathscr{H}$ be two Banach spaces. If $S_1 : \mathscr{E} \to \mathscr{E}$ is a linear bounded operator and $T_1 : \mathscr{E} \to \mathscr{H}$ is a linear operator admitting a closure $\overline{T}_1$, then their product $T_1 \cdot S_1 : \mathscr{E} \to \mathscr{H}$ admits a closure $\overline{T_1 \cdot S_1}$, and $\overline{T_1 \cdot S_1} \subset \overline{T}_1 \cdot S_1$.*

Using Lemma 2 with $\mathscr{E} = E$, $\mathscr{H} = L_2(]0,T[,H)$, $T_1 = \mathscr{L}$, and $S_1 = A^{-q/2}(t)$, we obtain the inclusion

$$\overline{\mathscr{L} A^{-q/2}(t)} \subset \bar{\mathscr{L}} A^{-q/2}(t). \qquad (9)$$

From relations (7)–(9), we obtain $\bar{\mathscr{L}}\, A^{-q/2}(t)w = f$, and, as straightforward verification shows, $l_0\left(A^{-q/2}(t)w\right) = \varphi$ and $l_1\left(A^{-q/2}(t)w\right) = \varphi$. This, together with the uniqueness theorem for strong solutions, implies that $u = A^{-q/2}(t)w$. Since $w \in E$, it follows from the last relation that $u(t) \in D\left(A^{(q+1)/2}(t)\right)$ for almost all $t \in]0,T[$, $A^{(q+1)/2}(t)u$, $A^{q/2}(t)(du/dt) \in L_\infty(]0,T[,H)$, which, in turn, together with (1), gives $A^{(q-1)/2}(t)\left(d^2u/dt^2\right) \in L_2(]0,T[,H)$. The proof of Theorem 3 is complete.

Corollary. Let the assumptions of Theorem 3 be valid, and moreover, let

$$\tilde{A}(t) \in L_\infty\left(]0,T[,\mathscr{L}\left(W^{q+1}(t), W^q(t)\right)\right), \qquad \tilde{B}(t) \in L_\infty\left(]0,T[,\mathscr{L}\left(W^q(t), W^q(t)\right)\right).$$

Then the assertion of Theorem 3 holds for the Cauchy problem (4), (2).

The proof is similar to that of Theorem 3.

Remark 2. The condition $q \geq 1$ is used only to prove that $\tilde{\mathscr{L}}$ is closable. In applications, this property usually holds for $q \geq 0$ and can readily be proved in a straightforward way in each specific problem.

REFERENCES

1. Lomovtsev, F.E., *Differents. Uravn.*, 1992, vol. 28, no. 5, pp. 877–886.
2. Lomovtsev, F.E., *Differents. Uravn.*, 1994, vol. 30, no. 8, pp. 1412–1425.
3. Lomovtsev, F.E., *Differents. Uravn.*, 2000, vol. 36, no. 4, pp. 542–548.
4. Lomovtsev, F.E., *Differents. Uravn.*, 1995, vol. 31, no. 7, pp. 1132–1141.
5. Krein, S.G., *Lineinye differentsial'nye uravneniya v banakhovom prostranstve* (Linear Differential Equations in a Banach Space), Moscow, 1967.