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Equilibrium shapes of a ferrofluid drop

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A numerical solution strategy for calculating equilibrium free surfaces of a ferrofluid drop under the action of uniform magnetic fields is proposed. Based on this strategy, drop shapes of nonlinear magnetisable fluids are obtained numerically in a wide range of field intensities and compared with existing theoretical results.

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1 Introduction

The problem for calculating equilibrium free surfaces of a ferrofluid drop is described by a coupled system of the Maxwell's equations and the magnetically augmented Young-Laplace equation [4]. The Maxwell's equations are formulated inside of the bounded fluid domain Ω_1 with an a-priori unknown boundary Γ and outside in the air domain $\Omega_2 = \mathbb{R}^3 \setminus \Omega_1$. The boundary position Γ will be determined as a solution of the Young-Laplace equation. Both the magnetic field and drop surface have to be found simultaneously. One possible way to handle numerically the coupled problem is to split it into two subproblems: solving the Maxwell's equations for the given drop shape and solving the Young-Laplace equation for the given field distribution on the interface Γ. Iterations between two subproblems will result in the equilibrium shape. Discretisation of the Maxwell's equations is realised by the coupled boundary-element/finite-element method and for the Young-Laplace equation by the finite-difference method.

2 Mathematical formulation

Under the assumption of the axial symmetry for the drop shape we formulate the mathematical model in cylindrical coordinates (r, z). Taking radius R_0 of the initial circular drop shape as a characteristic length and intensity of the applied field H_0 as a characteristic field strength we write the two-dimensional Maxwell's equations complemented by the boundary conditions in dimensionless form

$$-\nabla \cdot (\mu(|\nabla u_1|)\nabla u_1) = 0 \quad \text{in} \quad \Omega_1, \quad -\nabla \cdot (\nabla u_2) = 0 \quad \text{in} \quad \Omega_2; \tag{1}$$

$$u_1 = u_2, \quad \mu(|\nabla u_1|)\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \qquad \text{on} \quad \Gamma, \tag{2}$$

$$u_2 = z, \quad (r, z) \to \infty; \tag{3}$$

$$\mu(t) = 1 + 3\chi\frac{(\coth\gamma t - 1/(\gamma t))}{\gamma t}, \quad \gamma = \frac{3\chi H_0}{M_s}.$$

Here u_1 and u_2 denote the scalar potentials in Ω_1 and Ω_2; χ is the initial susceptibility, M_s the magnetic saturation.

We describe the equilibrium shape of the ferrofluid drop by parametric functions $r = r(s)$, $z = z(s)$, where s denotes the arc length of the equilibrium line Γ. Using the approach in [3] the Young-Laplace equation can be written as

$$z'' = r'F, \quad r'' = -z'F, \quad 0 < s < 1; \qquad F = f - \frac{z'}{r} + C; \tag{4}$$

$$r(0) = 0,\ z'(0) = 0, \quad r'(1) = 0,\ z(1) = 0; \tag{5}$$

$$f = -WL\left[\frac{2}{3\chi}\ln\frac{\sinh(\gamma H)}{\gamma H} + \left(\left(\coth\gamma H - \frac{1}{\gamma H}\right)\frac{H_n}{H}\right)^2\right], \quad W = \frac{\mu_0 M_s^2 V^{1/3}}{2\sigma}.$$

The point $s = 0$ lies on the z axis and the point $s = 1$ on the plane $z = 0$. For details of calculating constants C and L see [3]. Here $H = |\nabla u_1|$ is the magnetic field intensity, $H_n = \partial u_1/\partial n$, μ_0 denotes the magnetic permeability, $V = 4\pi R_0^3/3$ the drop volume, σ the surface tension coefficient.

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3 Discretisation of the problem

To discretise equations (1)-(3) we use the idea of the coupled collocation boundary-element method (BEM) and the Galerkin finite-element method (FEM), analysed in [5]. We apply the BEM in the unbounded domain Ω_2 to fulfill exactly the boundary condition for the magnetic field at the infinity (3). The FEM is used in the bounded domain Ω_1, where nonlinearities of the Maxwell's equations occur. We reformulate equations (1)-(3) as the nonlocal boundary value problem for u_1 in $\bar{\Omega}_1$ and $\partial u_1/\partial n$ on Γ. Piecewise linear and piecewise constant functions are used for the approximation of u_1 and $\partial u_1/\partial n$, respectively. The fluid-air interface Γ is fixed during the process of solving equations (1)-(3) with a piecewise linear approximation of the interface in the FEM discretisation and a cubic spline approximation in the BEM discretisation being used. The resulting nonlinear discrete system is solved by a fixed-point iteration method with application of the Gaussian elimination method for the linearised systems.

We use a finite-difference scheme of the second order approximation to discretise equations (4)-(5), for details see [3]. Grid adaptation based on the information about the surface curvature is applied [3]. A two-layer iterative scheme is constructed with tridiagonal matrices of the linearised systems.

4 Numerical results

The current work extents results of [2], where only linear magnetisable fluids were considered. The linear magnetisation is a reasonable assumption in the region of weak fields [4], whereas nonlinear magnetisation is a necessary requirement for the problem modeling in a wide range of field intensities.

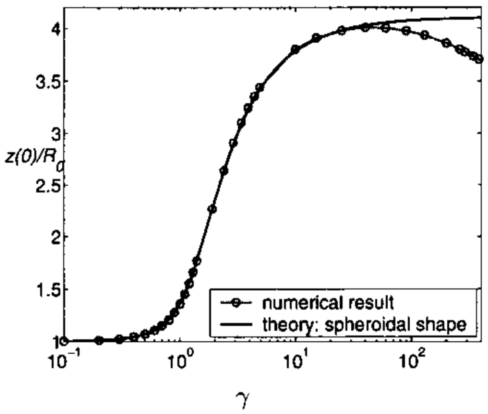

Fig. 1 The dependence of the drop elongation versus the dimensionless magnetic field. Calculations were made for the ferrofluid with parameters $\chi = 1.9$, $W = 62$.

If we assume that the drop shape is spheroid then we can determine the drop elongation for every field intensity by applying a theoretical approach, so-called the virial method [1]. The corresponding dependence is drawn by the solid line. The curves with markers present numerical results. From the picture we see that for $\gamma \in [0, 50]$ theoretical and numerical results nearly coincide. It follows that for the considered ferrofluid the drop shape is rather close to spheroid in a wide range of field intensities. For $\gamma > 50$ the numerical results show a qualitative difference with the theory: a larger applied field produces less elongated shape. Such a behavior of numerical results should be further studied.

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