COMPUTATIONAL COMPLEXITY OF MAXIMUM DISTANCE-(*k*, *l*) MATCHINGS IN GRAPHS

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In this paper, we introduce the concept of a *distance-*(k, l) *matching* of a graph, which is a subset of edges of this graph such that the number of intermediate edges in the shortest path between any two edges of this set lies between k and l. We prove that the problem MAXIMUM DISTANCE-(k, l) MATCHING, which asks whether a graph contains a distance-(k, l) matching of size exceeding a given number, is NP-complete for arbitrary given or variable k and l, and that the weighted variant of this problem is strongly NP-complete even for bipartite graphs. We also present several upper bounds on the size of a maximum distance-(k, l) matching.

Key words: matching, k-independent set, NP-completeness.

INTRODUCTION

A problem which we call MAXIMUM DISTANCE-(k, l) MATCHING is studied. It is a generalization of the classical matching problem, in which the distance between the selected objects in the matching is bounded. Special cases of this problem have applications in the areas of communication network testing [22], concurrent transmission of messages in wireless ad hoc networks [1], secure communication channels in broadcast networks [14], and many others. In these applications, the distance between the objects of a matching is restricted due to the security or interference reasons.

The standard graph-theoretic [2] and computational complexity [12] terminology is used throughout this paper.

We consider only simple finite graphs without loops or multiple edges and assume that the graphs are *connected*, i. e., for any pair of vertices there exists a path from one vertex to the other. Let $G = (V, E)$ be such a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *distance between vertices* $x, y \in V$, denoted as $dist_G(x, y)$ or simply $dist(x, y)$, is equal to the number of edges in a shortest path between x and y. In particular, $dist(x, y) = 0$ if and only if $x = y$. For a vertex $x \in V$ and an edge $e \in E$ of G, $dist(x, e) = \min\{dist(x, y) : y \in e\}$ is the *distance between vertex x and edge e*. The *distance* $dist(e, e')$ *between two distinct edges* $e, e' \in E$ is defined as $dist(e, e') = \min\{dist(x, y) : x \in e, y \in e'\}$. This means that $dist(e, e') = 0$ if and only if the edges e and e' are *adjacent*, i.e., $e \cap e' \neq \emptyset$. The edges e and e' are *independent* if they are not adjacent, i.e., $dist(e, e') \geq 1$. If S is a set of vertices (edges, respectively) and x is a vertex of G, then the *distance from x to S*, denoted by $dist(x, S)$, is defined as $dist(x, S) = \min\{dist(x, y) : y \in S\}$.

For every integer $k \geq 1$, the *k-th power graph* of graph G, denoted as G^k, is a graph with the same vertex set, and the set of edges such that two vertices are adjacent in G^k if and only if the distance between them is at most k in G, that is, $V(G^k) = V(G)$ and $E(G^k) = \{xy : dist_G(x, y) \leq k\}$. The *line graph* $L(G)$ is defined as follows: the vertices of $L(G)$ bijectively correspond to the edges of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. For a set H of graphs, graph G is called *H-free* if no induced subgraph of G is isomorphic to a graph in H. The complete graph on n vertices is denoted by K_n. For $n \geq 1$, let P_n denote the chordless path on n vertices, and $K_{1,n}$ denote the star with center of degree n. For $n \geq 3$, let C_n denote the chordless cycle on n vertices. For vertex-disjoint graphs G_1 and G_2, the *disjoint union* $G_1 \cup G_2$ denotes the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

For a positive integer k, a subset I of vertices of G is called *k-independent set* if the distance between any two distinct vertices in I is greater than k. The *k-independence number* of G, denoted $\alpha_k(G)$, is defined as the maximum size taken over all k-independent sets of G. An 1-independent set is an *independent set*, and $\alpha_1(G) = \alpha(G)$ is the *independence number* of G. A subset $D \subseteq V(G)$ is a *k-dominating set* of G if $dist_G(x, D) \leq k$ for each vertex $x \in V(G) - D$. The minimum size of a k-dominating set is called the *k-domination number* $\gamma_k(G)$. Note that if $k = 1$, then $\gamma_1(G) = \gamma(G)$, the *domination number* of G. A subset $F \subseteq E(G)$ is an *k-edge cover* of G if $dist_G(x, F) < k$ for every vertex $x \in V(G)$. The minimum size of an k-edge cover is called the *k-edge covering number* $\rho_k(G)$ of G. A *clique* is a set of pairwise adjacent vertices of G. The size of the largest clique in G is the *clique number* of G, denoted by $\omega(G)$.

A set of pairwise independent edges of graph G is called *matching*, while matching of the maximum size is called *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G and is denoted by $\alpha'(G)$. Recently, several authors have studied constrained matchings such as the ones with the bounded pairwise distance of edges.

We introduce and study the following constrained matchings. A subset M of edges of G is called a *distance-*(k, l) *matching* if the pairwise distance of edges in M is at least k and at most l in G. In other words, relation $k \leq dist(e, e') \leq l$ holds for each pair e and e' of distinct edges from M. We define *distance-*(k, l) *matching number* of G, denoted $\Sigma^{(k, l)}(G)$, as the maximum size of the distance-(k, l) matchings of G. A *maximum distance-*(k, l) *matching* is a distance-(k, l) matching of size $\Sigma^{(k, l)}(G)$.

Note that the distance-$(1, \infty)$ matching is the ordinary matching, and hence, $\Sigma^{(1, \infty)}(G) = \alpha'(G)$ for any graph G. However, not every distance-(k, l) matching is an ordinary matching. Indeed, each subset $M \subseteq E(G)$ is a distance-$(0, \infty)$ matching of G, and thus, $\Sigma^{(0, \infty)}(G) = |E(G)|$. The distance-$(k, \infty)$ matchings were first introduced under the name of *k-separated matchings* by Stockmeyer and Vazirani [22], and have recently been studied by Chang [8], Brandstädt and Mosca [4]. The distance-$(2, \infty)$ matchings have also been studied under the names of *induced matchings* (i.e., matchings which form an induced subgraph in G) by Cameron [5] and *strong matchings* by Golumbic and Laskar [13]. Golumbic and Lewenstein [14] demonstrated applications of the induced matchings in developing secure communication channels, VLSI design and network flow problems. It is interesting to note that there is an immediate connection between $\Sigma^{(2, \infty)}(G)$ (the *induced matching number*) and the irredundancy number of a graph G; see [13] for details. Finally, the important problem of finding a strong edge-coloring in a graph G [10] is to partition the edge set of G into the minimum number of induced matchings.

The distance-$(0, 1)$ matchings were first introduced and investigated by Mahdian [18] under the name of *antimatchings*. This notion also appears in the context of chordal graphs as the *neighborly set* [5]. The distance-$(1, 1)$ matchings are known in the literature as *connected matchings*. This concept was introduced by Plummer, Stiebitz and Toft [21] in connection with their study of the famous Hadwiger's Conjecture. Connected matchings have been further studied by Cameron [6]. Note also that, if G is the complete graph K_3, then $\Sigma^{(0, 0)}(G) = 3$; while if $G \neq K_3$, then $\Sigma^{(0, 0)}(G) = \Delta(G)$, where $\Delta(G)$ is the maximum degree of the graph G.

Let $k \geq 1$. It is easy to see that $dist_G(e, e') \geq k$ for distinct edges e and e' if and only if e and e' are independent in the k-th power graph $(L(G))^k$ of the line graph of G. Thus, the following property holds: for any $k \geq 1$ and graph G, the edge set M is a distance-(k, ∞) matching in G if and only if M is an independent set of vertices in $(L(G))^k$. On the other hand, by similar considerations it is easy to see that the following property holds: for any $l \geq 1$, set $M \subseteq E(G)$ is a distance-$(0, l)$ matching in G if and only if M is a clique in $(L(G))^{l+1}$. In particular, for all $k, l \geq 1$ and any graph G, we have $\Sigma^{(k, \infty)}(G) = \alpha((L(G))^k)$ and $\Sigma^{(0, l)}(G) = \omega((L(G))^{l+1})$.

Consider the following decision problem associated with the parameter $\Sigma^{(k, l)}(G)$.

MAXIMUM DISTANCE-(k, l) MATCHING: *Instance.* A graph G and a positive integer K. *Question.* Is there a distance-(k, l) matching M in G such that $|M| \geq K$? In other words, is $\Sigma^{(k, l)}(G) \geq K$?

The MAXIMUM DISTANCE-$(1, \infty)$ MATCHING problem is known to be solvable in polynomial time for general graphs [9]. On the other hand, Stockmeyer and Vazirani [22] have shown that for every $k \geq 2$, the MAXIMUM DISTANCE-(k, ∞) MATCHING problem is NP-complete even for bipartite graphs of maximum degree 4. Brandstädt and Mosca [4] have shown that for every $k \geq 1$, the MAXIMUM DISTANCE-$(2k + 1, \infty)$ MATCHING problem is NP-complete for chordal graphs, while the MAXIMUM DISTANCE-$(2k, \infty)$ MATCHING problem can be solved in polynomial time for these graphs. A number of papers [4, 5, 7, 8, 14–17, 19, 20] deal with the computational complexity of the MAXIMUM DISTANCE-$(2, \infty)$ MATCHING problem. For $k \in \{0, 1\}$, the MAXIMUM DISTANCE-$(k, 1)$ MATCHING problem is NP-complete for general graphs [18, 21], but can be solved in polynomial time for chordal graphs and for graphs with no cycle of length 4 [6, 18].

The rest of the paper is organized as follows. In Section 2, we give lower and upper bounds on $\Sigma^{(k,\,l)}(G)$ in terms of certain parameters of G. In Section 3, we present NP-completeness results for the MAXIMUM DISTANCE-(k, l) MATCHING problem and its weighted version, as well as some polynomially solvable cases.

BOUNDS ON THE DISTANCE-(k, l) MATCHING NUMBER

The problem of finding $\Sigma^{(k,\,l)}(G)$ is NP-complete for arbitrary given or variable k and l. Therefore, developing good upper bounds on this parameter is of interest. Throughout this section we assume that $k \le l \le \infty$.

Proposition 1. For $k \ge 1$, if G is a connected graph with $n \ge k + 1$ vertices, then $\Sigma^{(2k,\,l)}(G) \le \ (n-1)/k$ and $\Sigma^{(2k+1,\,l)}(G) \le \ n/(k+1)$.

The bounds in Proposition 1 are tight. To see this, consider a star $K_{1,p}$ where $1 \le p \le k-1$ and construct graph G by subdividing each edge in $K_{1,p}$ exactly k times. Graph G has $n = p(k+1) + 1$ vertices and $\Sigma^{(2k,\,l)}(G) = p$. Consequently, $\Sigma^{(2k,\,l)}(G) = \ (n-1)/k$. Further, consider a complete graph K_p, $1 \le p \le k$, and construct G by attaching a path P_{k+2} of length $k + 1$ to every vertex of K_p. This graph G has order $n = p\,(k+1) + p$ and $\Sigma^{(2k+1,\,l)}(G) = p$, therefore, $\Sigma^{(2k+1,\,l)}(G) = \ n/(k+1)$.

More bounds on $\Sigma^{(k,\,l)}(G)$ are given below.

Proposition 2. Let G be a connected graph. Then

(a) if $k \ge 2$, then $\Sigma^{(k,\,l)}(G) \le \alpha_{k-1}(G)$ and $\Sigma^{(k,\,\infty)}(G) \ge \alpha_{k+1}(G)$,

(b) if $k \ge 1$, then $\Sigma^{(2k,\,l)}(G) \le \rho_k(G)$ and $\Sigma^{(2k+1,\,l)}(G) \le \gamma_k(G)$.

COMPLEXITY RESULTS

Throughout this section, we assume that $0 \le k \le l < \infty$ and $\max\{k, l\} > 0$.

The proofs of the following four statements can be done by a polynomial transformation from the NP-complete problem CLIQUE [12].

Theorem 1. For any $k \ge 1$ and $l \ge 0$, the MAXIMUM DISTANCE-($2k$, $2k + l$) MATCHING problem is NP-complete for bipartite graphs.

Theorem 2. For any $k \ge 0$ and $l \ge 1$, the MAXIMUM DISTANCE-($2k + 1$, $2k + l + 1$) MATCHING problem is NP-complete for bipartite graphs.

Theorem 3. For any $l \ge 1$, the MAXIMUM DISTANCE-(0, l) MATCHING problem is NP-complete, and for any $l \ge 2$ it is NP-complete even for bipartite graphs.

Theorem 4. For any $k \ge 1$, the MAXIMUM DISTANCE-($2k + 1$, $2k + 1$) MATCHING problem is NP-complete.

Plummer, Stiebitz and Toft [21] proved that MAXIMUM DISTANCE-(1, 1) MATCHING is an NP-complete problem. Combining their result with Theorems 1 – 4, we obtain the following corollary: MAXIMUM DISTANCE-(k, l) MATCHING is NP-complete for arbitrary given or variable k and l.

A *weighted graph* is a pair (G, w) including graph G and edge weights represented by a non-negative integer valued function $w : E(G) \to \mathbf{Z}$. The weight $w(M)$ of a subset of edges $M \subseteq E(G)$ is defined as the sum of the weights of edges $e \in M$. The *weighted dis-*

tance-(k, l) *matching number* $\Sigma_w^{(k,l)}(G)$ of a weighted graph (G, w) is the maximum weight of a distance-(k, l) matching in (G, w).

Consider the following problem associated with the parameter $\Sigma_w^{(k,l)}(G)$.

MAXIMUM WEIGHT DISTANCE-(k, l) MATCHING: *Instance.* A weighted graph (G, w) and a positive integer K. *Question.* Is there a distance-(k, l) matching M in (G, w) such that $w(M) \geq K$? In other words, is $\Sigma_w^{(k,l)}(G) \geq K$?

Edmonds [9] gave a polynomial algorithm for the MAXIMUM WEIGHT DISTANCE-(1, ∞) MATCHING problem on general graphs. In contrast, for each $k \geq 2$, the MAXIMUM WEIGHT DISTANCE-(k, ∞) MATCHING problem is NP-complete even for special graphs [11]. We show that the complexity of the MAXIMUM WEIGHT DISTANCE-(k, l) MATCHING problem is similar.

Theorem 5. For any k and l, the MAXIMUM WEIGHT DISTANCE-(k, l) MATCHING problem is strongly NP-complete for bipartite graphs, even if the edge weights can take only two values.

Some polynomially solvable special cases for the MAXIMUM WEIGHT DISTANCE-(k, l) MATCHING problem are described below.

Proposition 3. For any odd $l \geq 1$, the MAXIMUM WEIGHT DISTANCE-(0, l) MATCHING problem can be solved in polynomial time for chordal graphs, and for arbitrary $l \geq 1$, it can be solved in polynomial time for strongly chordal graphs (see Brandstädt et al. [3] for definitions of these graphs).

Theorem 6. The MAXIMUM WEIGHT DISTANCE-(0, 1) MATCHING problem can be solved in polynomial time in the classes of (P_5, *kite*, *butterfly*)-free and ($K_3 \cup K_2$, $K_{1,3} \cup K_2$, $P_4 \cup K_2$, $C_4 \cup K_2$)-free graphs.

Kite is the graph consisting of five vertices u, v, w, x, y and edges uv, uw, vw, wx, xy. *Butterfly* is the graph obtained from kite by adding the edge wy.

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