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SHORT COMMUNICATIONS

Mixed Problem for the String Vibration Equation with a Time-Dependent Oblique Derivative in the Boundary Condition

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Abstract—We obtain formulas for the classical solution of the mixed problem for the equation of vibrations of a half-bounded string for the case in which the boundary condition contains a directional (oblique) derivative with time-dependent direction. We find the limit values of the solution as the direction tends to a characteristic of the equation.

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1. For a half-bounded string, we consider the mixed problem

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < \infty, \qquad t > 0, \tag{1}$$

$$u\big|_{t=0} = \varphi(x), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = \psi(x), \qquad 0 \le x < \infty, \tag{2}$$

$$\left(\alpha(t)\frac{\partial u}{\partial t} + \beta(t)\frac{\partial u}{\partial x} + \gamma(t)u\right)\bigg|_{x=0} = g(t). \tag{3}$$

The solvability of the mixed problem (1)–(3) with $\alpha(t) \equiv 0$ and variable coefficients $\beta(t)$ and $\gamma(t)$ under some conditions follows from [1, 2], where the unique solvability of the Cauchy problem for hyperbolic operator-differential equations with time-dependent domain of the operator coefficient was proved. Problem (1)–(3) was not considered for $\alpha(t) \not\equiv 0$. Since for $\alpha(t) \neq 0$, the derivative $\alpha(t)\partial/\partial t + \beta(t)\partial/\partial x$ is not directed along the normal to the line $x = 0$ and since the solvability properties of the problem change if the expression $a\alpha(t) - \beta(t)$ vanishes, i.e., if the derivative $\alpha(t)\partial/\partial t + \beta(t)\partial/\partial x$ is directed along the characteristic $x - at = 0$ of Eq. (1), we use an analogy with the elliptic case [3, p. 403] and refer to problem (1)–(3) as the *oblique derivative problem*.

2. Consider problem (1)–(3) for $a\alpha(t) \neq \beta(t)$.

Theorem 1. *Suppose that* $\varphi \in C^2[0, +\infty)$, $\psi \in C^1[0, +\infty)$, $\alpha, \beta, \gamma, g \in C^1[0, \infty)$, *the coordination conditions*

$$J_1 \equiv \alpha(0)\psi(0) + \beta(0)\varphi'(0) + \gamma(0)\varphi(0) = g(0), \tag{4}$$

$$J_2 \equiv a^2\alpha(0)\varphi''(0) + \beta(0)\psi'(0) + \gamma(0)\psi(0) + \alpha'(0)\psi(0) + \beta'(0)\varphi'(0) + \gamma'(0)\varphi(0) = g'(0) \tag{5}$$

hold for the initial and boundary data, and $a\alpha(t) \neq \beta(t)$ *in the boundary condition* (3). *Then there exists a unique classical solution of problem* (1)–(3), *which can be represented by the formulas*

$$u_+(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(\xi)\,d\xi, \qquad x - at \ge 0, \tag{6}$$

$$u_-(x,t) = \frac{\varphi(at+x) - \varphi(at-x)}{2} + \frac{1}{2a}\int\limits_{at-x}^{at+x} \psi(\xi)\,d\xi$$
$$- \int\limits_0^{t-x/a} \frac{\beta(\tau)[a\varphi'(\tau)+\psi(\tau)]}{a\alpha(\tau)-\beta(\tau)} \exp\left\{ \int\limits_{t-x/a}^{\tau} \frac{a\gamma(\xi)\,d\xi}{a\alpha(\xi)-\beta(\xi)} \right\} d\tau$$
$$+ \int\limits_0^{t-x/a} \frac{ag(\tau)}{a\alpha(\tau)-\beta(\tau)} \exp\left\{ \int\limits_{t-x/a}^{\tau} \frac{a\gamma(\xi)\,d\xi}{a\alpha(\xi)-\beta(\xi)} \right\} d\tau$$
$$+ \varphi(0)\exp\left\{ \int\limits_{t-x/a}^{\tau} \frac{a\gamma(\xi)\,d\xi}{a\alpha(\xi)-\beta(\xi)} \right\}, \qquad x - at \le 0. \tag{7}$$

Proof. It is known that the general solution of Eq. (1) has the form

$$u(x,t) = f_1(x+at) + f_2(x-at), \tag{8}$$

where $f_1, f_2 \in C^2$ are arbitrary functions. First, we substitute the expression (8) into the initial conditions (2) and obtain formula (6); then, by using the boundary condition (3) for $a\alpha(t) \ne \beta(t)$, we arrive at formula (7).

Thus, formula (6) provides a classical solution of Eq. (1) with the initial conditions for $x - at \ge 0$, and formula (7) provides a classical solution of Eq. (1) with the boundary condition (3) for $x - at \le 0$. It remains to verify that the functions u_+ and u_- and their derivatives of order ≤ 2 coincide on the characteristic $x - at = 0$.

By setting $x = at$ in the expressions (6) and (7) and their derivatives, we obtain

$$u_+|_{x=at} = \frac{\varphi(2x)+\varphi(0)}{2} + \frac{1}{2a}\int\limits_0^{2x} \psi(\xi)\,d\xi, \tag{9}$$

$$u_-|_{x=at} = \frac{\varphi(2x)-\varphi(0)}{2} + \frac{1}{2a}\int\limits_0^{2x} \psi(\xi)\,d\xi + \varphi(0), \tag{10}$$

$$\left.\frac{\partial u_-}{\partial t}\right|_{x=at} = \left.\frac{\partial u_+}{\partial t}\right|_{x=at} - \frac{a}{a\alpha(0)-\beta(0)}(J_1 - g(0)), \tag{11}$$

$$\left.\frac{\partial u_-}{\partial x}\right|_{x=at} = \left.\frac{\partial u_+}{\partial x}\right|_{x=at} + \frac{1}{a\alpha(0)-\beta(0)}(J_1 - g(0)), \tag{12}$$

$$\left.\frac{\partial^2 u_-}{\partial t^2}\right|_{x=at} = \left.\frac{\partial^2 u_+}{\partial t^2}\right|_{x=at} - \frac{a}{a\alpha(0)-\beta(0)}\,(J_2 - g'(0)) + \frac{a(a\alpha'(0)-\beta'(0)+a\gamma(0))}{(a\alpha(0)-\beta(0))^2}\,(J_1 - g(0)), \tag{13}$$

$$\left.\frac{\partial^2 u_-}{\partial x^2}\right|_{x=at} = \left.\frac{\partial^2 u_+}{\partial x^2}\right|_{x=at} - \frac{1}{a(a\alpha(0)-\beta(0))}\,(J_2 - g'(0)) + \frac{a\alpha'(0)-\beta'(0)+a\gamma(0)}{a(a\alpha(0)-\beta(0))^2}\,(J_1 - g(0)), \tag{14}$$

where J_1 and J_2 are defined in the coordination conditions (4) and (5). By virtue of these conditions, $J_1 = g(0)$ and $J_2 = g'(0)$. Therefore, relations (9)–(14) imply that the functions u_+ and u_- and their derivatives of order ≤ 2 coincide on the characteristic $x - at = 0$. The uniqueness of the solution follows from the unique derivation of formulas (6) and (7). The proof of Theorem 1 is complete.

3. Consider problem (1)–(3) for $a\alpha(t) = \beta(t)$.

Theorem 2. *Suppose that* $\varphi \in C^3[0,\infty)$, $\psi \in C^2[0,\infty)$, $\alpha, \beta, \gamma, g \in C^2[0,\infty)$, *the coordination conditions* (4) *and* (5) *with* $a\alpha(0) = \beta(0)$ *hold for the initial and boundary data, the coordination conditions*

$$\begin{aligned} J_3 \equiv a\beta(0)\psi''(0) + a^2\beta(0)\varphi'''(0) + a^2\gamma(0)\varphi''(0) + 2a\beta'(0)\varphi''(0) + 2\beta'(0)\psi'(0) \\ + 2\gamma'(0)\psi(0) + \frac{\beta''(0)}{a}\psi(0) + \beta''(0)\varphi'(0) + \gamma''(0)\varphi(0) = g''(0) \end{aligned} \tag{15}$$

hold, and $a\alpha(t) = \beta(t)$ *and* $\gamma(t) \neq 0$ *in the boundary condition* (3). *Then there exists a unique classical solution of problem* (1)–(3), *which can be represented by formulas* (6) *and*

$$\begin{aligned} \overline{u}_-(x,t) = \frac{\varphi(at+x) - \varphi(at-x)}{2} + \frac{1}{2a}\int\limits_{at-x}^{at+x} \psi(\xi)\,d\xi \\ - \frac{\beta(t-x/a)}{\gamma(t-x/a)}\left[\varphi'(at-x) + \frac{\psi(at-x)}{a}\right] + \frac{g(t-x/a)}{\gamma(t-x/a)}, \qquad x - at \le 0. \end{aligned} \tag{16}$$

Proof. Formula (15) can be obtained by analogy with (7) for $a\alpha(t) \neq \beta(t)$. Note that a higher smoothness than in Theorem 1 is required of the functions φ, ψ, α, β, γ, and g for the function $\overline{u}_-(x,t)$ to be twice continuously differentiable.

We need to verify that the functions u_+ and $\overline{u}_-$ and their derivatives of order ≤ 2 coincide on the characteristic $x - at = 0$.

By setting $x = at$ in the expressions (6) and (16) and their derivatives, we obtain

$$\overline{u}_-|_{x=at} = u_+|_{x=at} - \frac{1}{\gamma(0)}(J_1 - g(0)), \tag{17}$$

$$\left.\frac{\partial \overline{u}_-}{\partial t}\right|_{x=at} = \left.\frac{\partial u_+}{\partial t}\right|_{x=at} - \frac{1}{\gamma(0)}[J_2 - g'(0)] + \frac{\gamma'(0)}{\gamma^2(0)}[J_1 - g(0)], \tag{18}$$

$$\left.\frac{\partial \overline{u}_-}{\partial x}\right|_{x=at} = \left.\frac{\partial u_+}{\partial x}\right|_{x=at} + \frac{1}{a\gamma(0)}[J_2 - g'(0)] - \frac{\gamma'(0)}{a\gamma^2(0)}[J_1 - g(0)], \tag{19}$$

$$\begin{aligned} \left.\frac{\partial^2 \overline{u}_-}{\partial t^2}\right|_{x=at} = \left.\frac{\partial^2 u_+}{\partial t^2}\right|_{x=at} - \frac{1}{\gamma(0)}[J_3 - g''(0)] + 2\frac{\gamma'(0)}{\gamma^2(0)}[J_2 - g'(0)] \\ + \frac{\gamma''(0)\gamma(0) - 2(\gamma'(0))^2}{\gamma^3(0)}[J_1 - g(0)], \end{aligned} \tag{20}$$

$$\begin{aligned} \left.\frac{\partial^2 \overline{u}_-}{\partial x^2}\right|_{x=at} = \left.\frac{\partial^2 u_+}{\partial x^2}\right|_{x=at} - \frac{1}{a^2\gamma(0)}[J_3 - g''(0)] + \frac{2}{a^2}\frac{\gamma'(0)}{\gamma^2(0)}[J_2 - g'(0)] \\ + \frac{\gamma''(0)\gamma(0) - 2(\gamma'(0))^2}{a^2\gamma^3(0)}[J_1 - g(0)], \end{aligned} \tag{21}$$

where J_1 and J_2 are defined in the coordination conditions (4) and (5), respectively, for $a\alpha(0) = \beta(0)$ and J_3 is given in condition (15). Since, by virtue of these conditions, $J_1 = g(0)$, $J_2 = g'(0)$, and $J_3 = g''(0)$, it follows from (17)–(21) that the functions u_+ and $\overline{u}_-$ and their derivatives of order ≤ 2 coincide on the characteristic $x - at = 0$. The uniqueness of the solution follows from the unique derivation of formulas (6) and (16). The proof of Theorem 2 is complete.

4. Let us establish a relationship between u_- and $\overline{u}_-$. By integrating by parts in the third and fourth terms in formula (7), we obtain

$$u_-(x,t) - \overline{u}_-(x,t) = \int\limits_0^{t-x/a} \frac{d}{d\tau}\left[\frac{\beta(\tau)}{\gamma(\tau)}\left(\varphi'(\tau) + \frac{\psi(\tau)}{a}\right)\right] \exp\left\{\int\limits_{t-x/a}^{\tau} \frac{a\gamma(\xi)\,d\xi}{a\alpha(\xi) - \beta(\xi)}\right\} d\tau$$
$$- a\int\limits_0^{t-x/a} \frac{d}{d\tau}\left(\frac{g(\tau)}{\gamma(\tau)}\right) \exp\left\{\int\limits_{t-x/a}^{\tau} \frac{a\gamma(\xi)\,d\xi}{a\alpha(\xi) - \beta(\xi)}\right\} d\tau. \qquad (22)$$

If $\gamma(\xi)(a\alpha(\xi) - \beta(\xi)) > 0$ and $a\alpha(\xi) \to \beta(\xi)$, so that

$$\int\limits_{t-x/a}^{\tau} \frac{d\xi}{|a\alpha(\xi) - \beta(\xi)|} \to \infty,$$

then it follows from (22) that $u_-(x,t) \to \overline{u}_-(x,t)$. But if $\gamma(\xi)(a\alpha(\xi) - \beta(\xi)) < 0$ and $a\alpha(\xi) \to \beta(\xi)$, then the absolute value $|u_-(x,t)|$ increases infinitely.

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