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CONTINUOUS DEPENDENCE OF SOLUTIONS TO MIXED BOUNDARY VALUE PROBLEMS FOR A PARABOLIC EQUATION

MOUSSA ZAKARI DJIBIBE, KOKOU TCHARIE, NIKOLAY IOSSIFOVICH YURCHUK

Abstract. We prove the continuous dependence, upon the data, of solutions to second-order parabolic equations. We study two boundary-value problems: One has a nonlocal (integral) condition and the another has a local boundary condition. The proofs are based on a priori estimate for the difference of solutions.

1. Introduction

This paper is devoted to the proof of the continuous dependence, upon the data, of generalized solutions of a second order parabolic equation. The boundary conditions are of mixed type. This article contributes to the development of the a priori estimates method for solving such problems. The questions related to these problems are so miscellaneous that the elaboration of a general theory is still premature. Therefore, the investigation of these problems requires at every time a separate study.

The importance of problems with integral condition has been pointed out by Samarskii [23]. Mathematical modelling by evolution problems with a nonlocal constraint of the form $\frac{1}{1-\alpha}\int_\alpha^1 u(x,t)\,dx = E(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic. See for instance Benouar-Yurchuk [1], Benouar-Bouziani [2]-[3], Bouziani [4]-[7], Cannon et al [13]-[15], Ionkin [17]-[18], Kamynin [19] and Yurchuk [25]-[27]. Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Bouziani [7]-[9], Byszewski et al [10]-[12], Gasymov [16], Ionkin [17]-[18], Lazhar [21], Mouravey-Philipovski [22] and Said-Nadia [24]. The results and the method used here are a further elaboration of those in [1]. We should mention here that the presence of integral term in the boundary condition can greatly complicate the application of standard functional and numerical techniques. This work can be considered as a continuation of the results in [7], [25] and [27].

The remainder of the paper is divided into four section. In Section 2, we give the statement of the problem. Then in Section 3, we establish a priori estimate.

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Finally, in section 4, we show the continuous dependence of a solution upon the data.

2. Statement of the problem

In the rectangle $G = \{(x,t) : 0 < x < 1,\ 0 < t < T\}$, we consider following two mixed boundary value problems for parabolic equations:

The mixed problem with boundary integral condition

$$\frac{\partial u_\alpha}{\partial t} - \frac{\partial}{\partial x}\big(a(x,t)\frac{\partial u_\alpha}{\partial x}\big) = f(x,t), \tag{2.1}$$

$$u_\alpha(x,0) = \varphi_\alpha(x), \quad \frac{\partial u_\alpha(0,t)}{\partial x} = 0 \tag{2.2}$$

$$\frac{1}{1-\alpha}\int_\alpha^1 u_\alpha(x,t)\,dx = h(t), \tag{2.3}$$

where $0 \le \alpha < 1$.

The mixed problem with local boundary conditions

$$\frac{\partial u_1}{\partial t} - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u_1}{\partial x}) = f(x,t) \tag{2.4}$$

$$u_1(x,0) = \varphi_1(x), \quad \frac{\partial u_1(0,t)}{\partial x} = 0 \tag{2.5}$$

$$u_1(1,t) = h(t). \tag{2.6}$$

We assume the following conditions:

(A1) The coefficient $a(x,t)$ in (2.1) and (2.4) is a continuous differentiable function, and $0 < a_0 \le a(x,t) \le a_1$.

(A2) $f \in L_2(G)$, $h \in W_2^1(0,T)$, φ_α, $\varphi_1 \in W_2^1(0,T)$, $\varphi'_\alpha(0) = \varphi'_1(0) = 0$, $\varphi_1(1) = h(0)$, $\frac{1}{1-\alpha}\int_\alpha^1 \varphi_\alpha(x)\,dx = h(0)$.

In [25], it is shown that if the conditions (A1)–(A2) are satisfied, then there exists a unique solution to (2.1)–(2.3) and (2.4)–(2.6), and that the solution is differentiable almost everywhere on G. It is clear that for every function $g \in C([0,1]) : g(1) = \lim_{\alpha\to 1}\frac{1}{1-\alpha}\int_\alpha^1 g(x)\,dx$ Therefore, the problem (2.4)–(2.6) is the limit when $\alpha \to 1$ of the problem (2.1)–(2.3). In this paper, we establish a priori estimate for the difference $u_\alpha - u_1$ and use it to prove that if $\alpha \to 1$ and $\varphi_\alpha \to \varphi_1$ then $u_\alpha \to u_1$. In this way we prove a new important property; a continuous dependence of solutions of mixed problems for parabolic equations of the form of boundary conditions.

3. A priori estimate

Theorem 3.1. *Under assumptions* (A1)–(A2)*, there exists a positive constant* c *independent of* u_α, u_1 *and* α *such that*

$$\begin{aligned}
&\int_0^1\int_0^T (1-x)[|\frac{\partial u_\alpha}{\partial t}-\frac{\partial u_1}{\partial t}|^2+|\frac{\partial^2 u_\alpha}{\partial x^2}-\frac{\partial^2 u_1}{\partial x^2}|^2]\,dx\,dt\\
&+\sup_{0\le t\le T}\int_0^1 [|u_\alpha-u_1|^2+(1-x)|\frac{\partial u_\alpha}{\partial x}-\frac{\partial u_1}{\partial x}|^2]\,dx\\
&\le c\Big\{\int_0^1 |\varphi'_\alpha(x)-\varphi'_1(x)|^2\,dx+\big|h(0)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx\big|^2\\
&\quad+\int_0^T \big[\big|h'(t)-\frac{1}{1-\alpha}\int_\alpha^1\frac{\partial u_1(x,t)}{\partial t}\,dx\big|^2+\big|h(t)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx\big|^2\big]\,dt\Big\}
\end{aligned} \tag{3.1}$$

Proof. Consider the problem (2.1)–(2.3), with

$$u_\alpha = v_\alpha + \frac{3x^2}{1+\alpha+\alpha^2}h(t) \tag{3.2}$$

where v_α is a solution of the problem

$$\begin{gathered}
\frac{\partial v_\alpha}{\partial t}-\frac{\partial}{\partial x}(a(x,t)\frac{\partial v_\alpha}{\partial x}) = g_\alpha(x,t)\\
v_\alpha(x,0)=\psi_\alpha(x),\quad \frac{\partial v_\alpha(0,t)}{\partial x}=0,\quad \frac{1}{1-\alpha}\int_\alpha^1 v_\alpha(x,t)\,dx=0\,,
\end{gathered}$$

where

$$\begin{gathered}
g_\alpha(x,t)=f(x,t)-\frac{3x^2}{1+\alpha+\alpha^2}h'(t)+\frac{6h(t)}{1+\alpha+\alpha^2}(a(x,t)+x\frac{\partial a(x,t)}{\partial x})\\
\psi_\alpha(x)=\varphi_\alpha(x)-\frac{3x^2}{1+\alpha+\alpha^2}h(0)
\end{gathered}$$

In (2.4)-(2.6) we also pose

$$u_1 = v_1+\frac{3x^2}{1-\alpha^3}\int_\alpha^1 u_1(x,t)\,dx \tag{3.3}$$

where v_1 is a solution of the problem

$$\begin{gathered}
\frac{\partial v_1}{\partial t}-\frac{\partial}{\partial x}\big(a(x,t)\frac{\partial v_1}{\partial x}\big)=g_1(x,t)\\
v_1(x,0)=\psi_1(x),\quad \frac{\partial v(0,t)}{\partial x}=0,\\
v_1(1,t)=h(t)-\frac{3}{1-\alpha^3}\int_\alpha^1 u_1(x,t)\,dx
\end{gathered}$$

where

$$\begin{aligned}
&g_1(x)\\
&=f(x,t)-\frac{3x^2}{1-\alpha^3}\int_\alpha^1\frac{\partial u_1(x,t)}{\partial t}\,dx+\frac{6}{1-\alpha^3}(a(x,t)+x\frac{\partial a(x,t)}{\partial x})\int_\alpha^1 u_1(x,t)\,dx,
\end{aligned}$$

$$\psi_(x)=\varphi_1(x)-\frac{3x^2}{1-\alpha^3}\int_\alpha^1 u_1(x,0)\,dx\,.$$

Then the function $w_\alpha = v_1 - v_\alpha$ is a solution of the problem

$$\frac{\partial w_\alpha}{\partial t} - \frac{\partial}{\partial x}(a(x,t)\frac{\partial w_\alpha}{\partial x}) = F_\alpha(x,t) \tag{3.4}$$

$$w_\alpha(x,0) = \phi_\alpha(x), \quad \frac{\partial w_\alpha(0,t)}{\partial x} = 0, \quad \frac{1}{1-\alpha}\int_\alpha^1 w_\alpha(x,t)\,dx = 0\,, \tag{3.5}$$

where

$$\begin{aligned} F_\alpha(x,t) = &-\frac{6}{1+\alpha+\alpha^2}\big(a(x,t) + x\frac{\partial a(x,t)}{\partial x}\big)\big[h(t) - \frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx\big] \\ &+ \frac{3x^2}{1+\alpha+\alpha^2}\big[h'(t) - \frac{1}{1-\alpha}\int_\alpha^1 \frac{\partial u_1(x,t)}{\partial t}\,dx\big] \end{aligned} \tag{3.6}$$

$$\phi_\alpha(x) = \varphi_1(x) - \varphi_\alpha(x) + \frac{3x^2}{1+\alpha+\alpha^2}\big(h(0) - \frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx\big)\,. \tag{3.7}$$

For this purpose we need the following result.

Lemma 3.2. *For a function w_α satisfying (3.4))–(3.5), there exists a positive constant c independent of w_α, ϕ_α and F_α such that*

$$\begin{aligned} &\int_G \psi_\alpha(x)\big[|\frac{\partial w_\alpha}{\partial t}|^2 + |\frac{\partial^2 w_\alpha}{\partial x^2}|^2\big]\,dx\,dt + \sup_{0\le t\le T}\int_0^1 \big[\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2 + |w_\alpha|^2\big]\,dx \\ &\le c_1\big\{\int_0^1 \psi_\alpha(x)|\phi'_\alpha(x)|^2\,dx + \int_G |F_\alpha(x,t)|^2\,dx\,dt\big\} \end{aligned} \tag{3.8}$$

where $\psi_\alpha(x) = \begin{cases} 1 & \text{if } 0 \le x \le \alpha \\ \frac{1-x}{1-\alpha} & \text{if } \alpha \le x \le 1\,. \end{cases}$

Suppose, for the moment, that Lemma 3.2 has been proved, and turn to the proof of Theorem 3.1. In accordance with the equality

$$u_1 - u_\alpha = w_\alpha - \frac{3x^2}{1+\alpha+\alpha^2}\big[h(t) - \frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx\big],$$

and the estimates

$$\begin{aligned} &\int_G (1-x)|\frac{\partial u_1}{\partial t} - \frac{\partial u_\alpha}{\partial t}|^2\,dx\,dt \\ &\le \frac{3}{5}\int_0^T |h'(t) - \frac{1}{1-\alpha}\int_\alpha^1 \frac{\partial u_1(x,t)}{\partial t}\,dx|^2\,dt + 2\int_G (1-x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt\,, \end{aligned}$$

$$\begin{aligned} &\int_G (1-x)|\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_\alpha}{\partial x^2}|^2\,dx\,dt \\ &\le 36\int_0^T |h(t) - \frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx|^2\,dt + 2\int_G (1-x)|\frac{\partial^2 w_\alpha}{\partial x^2}|^2\,dx\,dt\,, \end{aligned}$$

$$\begin{aligned} &\sup_{0\le t\le T}\int_0^1 (1-x)|\frac{\partial u_1}{\partial x} - \frac{\partial u_\alpha}{\partial x}|^2\,dx \\ &\le 6\sup_{0\le t\le T}|h(t) - \frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx|^2 + 2\sup_{0\le t\le T}\int_0^1 (1-x)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\,, \end{aligned}$$

$$\begin{aligned}
&\sup_{0\le t\le T}\int_0^1 |u_1-u_\alpha|^2\,dx\\
&\le \frac{18}{5}\sup_{0\le t\le T}|h(t)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx|^2+2\sup_{0\le t\le T}\int_0^1|w_\alpha|^2\,dx\,,
\end{aligned}$$

$$\int_0^1|\phi'_\alpha|^2\,dx\le 2\int_0^1|\varphi'_1(x)-\varphi'_\alpha|^2\,dx+24|h(0)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx|^2,$$

$$\begin{aligned}
\int_G|F_\alpha(x,t)|^2\,dx\,dt\le 72\int_G|a(x,t)+x\frac{\partial a(x,t)}{\partial x}|^2\,dx\,dt&\\
\times\int_0^T|h(t)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx|^2\,dt&\\
+\frac{18}{5}\int_0^T|h'(t)-\frac{1}{1-\alpha}\int_\alpha^1\frac{\partial u_1(x,t)}{\partial t}\,dx|^2\,dt\,,&
\end{aligned}$$

$$\begin{aligned}
&\sup_{0\le t\le T}|h(t)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx|^2\\
&\le 2|h(0)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx|^2+2T\int_0^T|h'(t)-\frac{1}{1-\alpha}\int_\alpha^1\frac{\partial u_1(x,t)}{\partial t}\,dx|^2\,dt\,.
\end{aligned}$$

We have the estimate

$$\int_0^1|\varphi_1(x)-\varphi_\alpha(x)|^2\,dx\le 2\int_0^1|\varphi'_1(x)-\varphi'_\alpha(x)|^2\,dx+2|\varphi_1(0)-\varphi_\alpha(0)|^2\,. \tag{3.9}$$

Then we obtain

$$\begin{aligned}
&\int_G(1-x)\big[|\frac{\partial u_1}{\partial t}-\frac{\partial u_\alpha}{\partial t}|^2+|\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial^2 u_\alpha}{\partial x^2}|^2\big]\,dx\,dt\\
&+\sup_{0\le t\le T}\int_0^1\big[(1-x)|\frac{\partial u_1}{\partial x}-\frac{\partial u_\alpha}{\partial x}|^2+|u_1-u_\alpha|^2\big]\,dx\,dt\\
&\le 4c_1\int_0^1|\varphi'_1(x)-\varphi'_\alpha(x)|^2\,dx+48(\frac{5c_1+2}{5})\big|h(0)-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx\big|^2\\
&\quad+3(\frac{12c_1+1+32T}{5})\int_0^T\big|h'(t)-\frac{1}{1-\alpha}\int_\alpha^1\frac{\partial u_1(x,t)}{\partial t}\,dx\big|^2\,dt\\
&\quad+\big(36+144c_1\max_{0\le t\le T}\int_0^1|a(x,t)+x\frac{\partial a(x,t)}{\partial x}|^2\,dx\big)\int_0^T\big|h(t)\\
&\quad-\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx\big|^2\,dt\,.
\end{aligned}$$

From the above inequality, it follows that (3.1) holds with

$$c=\max\Big(4c_1,\ 48(\frac{5c_1+2}{5}),\ \max\big(3(\frac{12c_1+1+32T}{5}),\ N\big)\Big)\,, \tag{3.10}$$

where

$$N=36+144c_1\max_{0\le t\le T}\int_0^1|a(x,t)+x\frac{\partial a(x,t)}{\partial x}|^2\,dx\,.$$

To complete the proof of Theorem 3.1, it remains to prove the Lemma.

Proof of Lemma 3.2. Let

$$M_\alpha w_\alpha = \begin{cases} \frac{\partial w_\alpha}{\partial t} & \text{if } 0 \le x \le \alpha \\ \frac{1-x}{1-\alpha}\frac{\partial w_\alpha}{\partial t} + \frac{1}{1-\alpha} I_\alpha \frac{\partial w_\alpha}{\partial t} & \text{if } \alpha \le x \le 1\,, \end{cases}$$

where $I_\alpha w_\alpha = \int_\alpha^x w_\alpha(\xi,t)\,d\xi$. Multiplying both sides of (3.4) by $M_\alpha w_\alpha$ and integrating the result with respect to $x \in [0;1]$, we obtain

$$\int_0^1 \frac{\partial w_\alpha}{\partial t} M_\alpha w_\alpha\,dx - \int_0^1 \frac{\partial}{\partial x}(a(x,t)\frac{\partial w_\alpha}{\partial x}) M_\alpha w_\alpha\,dx = \int_0^1 F_\alpha(x;t) M_\alpha w_\alpha\,dx\,dt\,. \tag{3.11}$$

Integrating by parts the first two integrals on the left-hand of (3.11), and using the boundary conditions (3.5), we obtain

$$\int_0^1 \frac{\partial w_\alpha}{\partial t} M_\alpha w_\alpha\,dx = \int_0^1 \psi_\alpha(x)(\frac{\partial w_\alpha}{\partial t})^2\,dx; \tag{3.12}$$

and

$$\begin{aligned} &-\int_0^1 \frac{\partial}{\partial x}(a(x,t)\frac{\partial w_\alpha}{\partial x}) M_\alpha w_\alpha\,dx \\ &= \frac{1}{2}\frac{\partial}{\partial t}\Big(\int_0^1 \psi_\alpha(x) a(x;t)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\Big) - \frac{1}{2}\int_0^1 \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial w_\alpha}{\partial x}|^2\,dx\,. \end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11), we obtain

$$\begin{aligned} &\int_0^1 \psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx + \frac{1}{2}\frac{\partial}{\partial t}\Big(\int_0^1 \psi_\alpha(x) a(x;t)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\Big) \\ &= \frac{1}{2}\int_0^1 \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial w_\alpha}{\partial x}|^2\,dx + \int_0^1 F_\alpha(x;t) M_\alpha w_\alpha\,dx\,dt\,. \end{aligned}$$

Multiplying both sides of the above equality by $e^{c(\tau-t)}$, and integrating the result with respect $t \in [0;\tau]$, we obtain

$$\begin{aligned} &\int_0^\tau \int_0^1 e^{c(\tau-t)}\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt + \frac{1}{2}\int_0^1 a(x,t)\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\;|_{t=\tau} \\ &= \int_0^\tau \int_0^1 e^{c(\tau-t)} F_\alpha(x,t) M_\alpha w_\alpha\,dx\,dt + \frac{1}{2}\int_0^\tau \int_0^1 e^{c(\tau-t)}\frac{\partial a(x,t)}{\partial t}\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\,dt \\ &\quad + \frac{1}{2}\int_0^1 e^{c\tau} a(x,0)\psi_\alpha(x)|\phi'(x)|^2\,dx - \frac{1}{2}\int_0^\tau \int_0^1 c a(x,t) e^{c(\tau-t)}\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\,dt\,. \end{aligned} \tag{3.14}$$

Using the following estimates

$$\begin{aligned} &\int_0^1 F_\alpha(x,t) M_\alpha w_\alpha\,dx \\ &\le \int_0^1 |F_\alpha(x,t)|^2\,dx + \frac{1}{4}\int_0^1 \psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx + \frac{1}{1-\alpha}\int_\alpha^1 F_\alpha(x,t) I_\alpha \frac{\partial w_\alpha}{\partial t}\,dx\,, \end{aligned}$$

$$\frac{1}{1-\alpha}\int_\alpha^1 F_\alpha(x,t) I_\alpha \frac{\partial w_\alpha}{\partial t}\,dx \le 2\int_\alpha^1 |F_\alpha(x,t)|^2\,dx + \frac{1}{8(1-\alpha)^2}\int_\alpha^1 |I_\alpha \frac{\partial w_\alpha}{\partial t}|^2\,dx\,,$$

$$\frac{1}{8(1-\alpha)^2}\int_\alpha^1 |I_\alpha \frac{\partial w_\alpha}{\partial t}|^2\,dx \le \frac{1}{2}\int_\alpha^1 \frac{(1-x)^2}{(1-\alpha)^2}|\frac{\partial w_\alpha}{\partial t}|^2\,dx \le \frac{1}{2}\int_0^1 \psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx;$$

and choosing a constant c in (3.14) such that $ca_0 \geq |\frac{\partial a(x,t)}{\partial t}|$, we deduce the inequality

$$\begin{aligned}&\frac{1}{4}\int_0^\tau\int_0^1 e^{c(\tau-t)}\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt+\frac{1}{2}a_0\int_0^1\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2\,dx\,|_{t=\tau}\\&\leq 3\int_0^\tau\int_0^1 e^{c(\tau-t)}|F_\alpha(x,t)|^2\,dx\,dt+\int_0^1 e^{c\tau}a(x,t)\psi_\alpha(x)|\phi'(x)|^2\,dx\end{aligned}$$

which can be written as

$$\begin{aligned}&\frac{1}{4}\int_0^\tau\int_0^1 \psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt+a_0\int_0^1\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,|_{t=\tau}\;dx\\&\leq 3\int_G e^{cT}|F_\alpha(x,t)|^2\,dx\,dt+\int_0^1 e^{cT}\psi_\alpha(x)a(x,t)|\phi'_\alpha(x)|^2\,dx\,.\end{aligned}\tag{3.15}$$

For the function $w_\alpha(x,t)$, there hold the relation

$$\int_0^1|w_\alpha(x,t)|^2\,dx\leq 4\int_0^1(1-x)^2|\frac{\partial w_\alpha(x,t)}{\partial x}|^2\,dx\leq 4\int_0^1\psi_\alpha(x)|\frac{\partial w_\alpha(x,t)}{\partial x}|^2\,dx\,.\tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned}&\frac{1}{4}\int_0^\tau\int_0^1 \psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt+\frac{a_0}{2}\int_0^1\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2\,|_{t=\tau}\;dx+\frac{a_0}{8}\int_0^1|w_\alpha|^2\,|_{t=\tau}\;dx\\&\leq e^{cT}\int_0^1 a(x,t)\psi_\alpha(x)|\phi'_\alpha(x)|^2\,dx+3e^{cT}\int_G|F_\alpha(x,t)|^2\,dx\,dt\,.\end{aligned}\tag{3.17}$$

Since the right-hand side of (3.17) does not depend on τ, we take the supremum of the left-hand side of (3.17) with respect to $0\leq t\leq T$; we obtain

$$\begin{aligned}&\frac{1}{4}\int_G\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt+\frac{1}{2}a_0\sup_{0\leq t\leq T}\int_0^1[\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2+\frac{1}{4}|w_\alpha|^2]\,dx\\&\leq e^{cT}\int_0^1 a(x,t)\psi_\alpha(x)|\phi'_\alpha(x)|^2\,dx+3e^{cT}\int_G|F_\alpha(x,t)|^2\,dx\,dt\,.\end{aligned}\tag{3.18}$$

It follows from (3.4) that

$$\begin{aligned}&a_0\int_G\psi_\alpha(x)|\frac{\partial w_\alpha^2}{\partial x^2}|^2\,dx\,dt\\&\leq 2\int_G\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial t}|^2\,dx\,dt+2T\max_{0\leq t\leq t}|\frac{\partial a(x,t)}{\partial t}|^2\\&\times\sup_{0\leq t\leq T}\int_0^1\psi_\alpha(x)|\frac{\partial w_\alpha}{\partial x}|^2\,dx+2\int_G\psi_\alpha(x)|F_\alpha(x,t)|^2\,dx\,dt\,.\end{aligned}$$

Combining this inequality and (3.18), we obtain (3.8), with

$$c_1=\frac{\frac{48e^{cT}\max(a_0,3)\max(1,T|\frac{\partial a}{\partial t}|^2)}{\min(a_0,2)}+3}{\min(a_0,3)\max(1,T|\frac{\partial a}{\partial t}|^2)}\,.\tag{3.19}$$

Thus the proof is complete, which also proves Theorem 3.1. $\square$

Corollary 3.3. *Since $1-x \le 1-\alpha < 1$ for $\alpha \le x \le 1$ it follows that $1-x \le \psi_\alpha(x) \le 1$ and the inequality* (3.8) *yields*

$$\begin{aligned}
&\int_G (1-x)\big[|\frac{\partial w_\alpha}{\partial t}|^2 + |\frac{\partial^2 w_\alpha}{\partial x^2}|^2\big]\,dx\,dt + \sup_{0\le t\le T}\int_0^1 [(1-x)|\frac{\partial w_\alpha}{\partial x}|^2 + |w_\alpha|^2]\,dx \\
&\le c_1\{\int_0^1 |\phi_\alpha'(x)|^2\,dx + \int_G |F_\alpha(x,t)|^2\,dx\,dt\}\,.
\end{aligned}$$

4. Continuous dependence of solutions

Using a priori estimate (3.1) for the difference $(u_\alpha - u_1)$ of solution u_α of the mixed problem (2.1)–(2.3) with integral boundary condition, and the solution u_1 of the the mixed problem (2.4)–(2.6) with the local condition, we come to the following result. We remark that this important property has not been established prior to this work.

Theorem 4.1. *Assume that* (A1)–(A2) *hold. If*

$$\lim_{\alpha\to 1}\int_0^1 |\varphi_1'(x) - \varphi_\alpha'(x)|\,dx = 0, \tag{4.1}$$

then

$$\begin{aligned}
&\lim_{\alpha\to 1}\Big\{\int_0^1\int_0^T (1-x)\big[|\frac{\partial u_\alpha}{\partial t} - \frac{\partial u_1}{\partial t}|^2 + |\frac{\partial^2 u_\alpha}{\partial x^2} - \frac{\partial u_1}{\partial t}|^2\big]\,dx\,dt \\
&+ \sup_{0\le t\le T}\int_0^1 \big[(1-x)|\frac{\partial u_\alpha}{\partial x} - \frac{\partial u_1}{\partial t}|^2 + |u_\alpha - u_1|^2\big]\,dx\Big\} = 0\,.
\end{aligned} \tag{4.2}$$

Proof. Since (A1) and (A2) are satisfied, the a priori estimate (3.1) holds for the difference $u_\alpha - u_1$. Moreover, the equality $u_1(1,t) = h(t)$ yields

$$\begin{gathered}
\lim_{\alpha\to 1}\big|\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx - h(t)\big| = \lim_{\alpha\to 1}\big|\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,t)\,dx - u_1(1,t)\big|^2 = 0 \\
\lim_{\alpha\to 1}\big|\frac{1}{1-\alpha}\int_\alpha^1 \frac{\partial u_1(x,t)}{\partial t} - h'(t)\big|^2 = \lim_{\alpha\to 1}\big|\frac{1}{1-\alpha}\int_\alpha^1 \frac{\partial u_1(x,t)}{\partial t}\,dx - \frac{\partial u_1(1,t)}{\partial t}\big|^2 = 0 \\
\lim_{\alpha\to 1}\big|\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx - h(0)|^2 = \lim_{\alpha\to 1}\big|\frac{1}{1-\alpha}\int_\alpha^1 u_1(x,0)\,dx - u_1(1,0)|^2 = 0\,.
\end{gathered}$$

From these three equalities, (4.1), and (3.1), we obtain (4.3). Thus theorem is proved. □

To complete our investigation we show that for any function $\varphi_1 \in W_2^1(0,T)$ satisfying the conditions $\varphi_1'(0) = 0$ and $\varphi_1(1) = h(0)$, there exist functions $\varphi_\alpha \in W_2^1(0,T)$ such that

$$\varphi_\alpha'(0) = 0, \quad \frac{1}{1-\alpha}\int_\alpha^1 \varphi_\alpha(x)\,dx = h(0) \tag{4.3}$$

and the equality (4.1) is valid. Set

$$\varphi_\alpha(x) = \varphi_1(x) + \frac{3x^2}{1+\alpha+\alpha^2}\big(h(0) - \frac{1}{1-\alpha}\int_\alpha^1 \varphi_1(x)\,dx\big)\,.$$

Then $\varphi_\alpha \in W_2^1(0,T)$, $\varphi'_\alpha(0) = \varphi'_1(0) = 0$,

$$\begin{aligned}
\frac{1}{1-\alpha}\int_\alpha^1 \varphi_\alpha(x)\,dx &= \frac{1}{1-\alpha}\int_\alpha^1 \varphi_1(x)\,dx \\
&\quad + \frac{3}{1-\alpha^3}\int_\alpha^1 x^2\,dx\Big(h(0) - \frac{1}{1-\alpha}\int_\alpha^1 \varphi_1(x)\,dx\Big) \\
&= \frac{1}{1-\alpha}\int_\alpha^1 \varphi_1(x)\,dx + \Big(h(0) - \frac{1}{1-\alpha}\int_\alpha^1 \varphi_1(x)\,dx\Big) \\
&= h(0)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\alpha\to 1}\int_0^1 |\varphi'_\alpha(x) - \varphi'_1(x)|\,dx &= \lim_{\alpha\to 1}\frac{3}{1+\alpha+\alpha^2}\big|h(0) - \frac{1}{1-\alpha}\int_\alpha^1 \varphi_1(x)\big| \\
&= |\varphi_1(1) - h(0)| = 0\,,
\end{aligned}$$

$$\lim_{\alpha\to 1}\frac{1}{1-\alpha}\int_\alpha^1 |\varphi_1(x)\,dx = \varphi_1(1) = h(0)\,.$$

This completes the proof. □

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Moussa Zakari Djibibe
University of Lomé - Togo, Department of Mathematics, PO Box 1515 Lomé, Togo
E-mail address: `mdjibibe@tg.refer.org zakari.djibibe@gmail.com tel +228 924 45 21`

Kokou Tcharie
University of Lomé - Togo, Department of Mathematics, PO Box 1515 Lomé, Togo
E-mail address: `tkokou@yahoo.fr`

Nikolay Iossifovich Yurchuk
Dept. of Mechanics and Mathematics, Belarussian State University, 220050, Minsk, Belarus
E-mail address: `yurchuk@bsu.by`