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A Priori Estimates and Continuous Dependence of Solutions of Mixed Problems for Parabolic Equations As Nonlocal Boundary Conditions Pass into Local Ones

N. I. Yurchuk and Charie Koku
Belarus State University, Minsk, Belarus
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Abstract—In the rectangle $G = (0,1) \times (0,T)$, we consider the family of problems

$$\frac{1}{a(x,t)}\frac{\partial u_\alpha}{\partial t} - \frac{\partial^2 u_\alpha}{\partial x^2} = f(x,t), \quad u_\alpha(x,0) = \varphi_\alpha(x), \quad u_\alpha(0,t) = 0, \quad 0 \le \alpha \le 1,$$

$$u_0(1,t) = h(t), \quad \frac{\partial u_1(1,t)}{\partial x} = h(t), \quad \frac{u_\alpha(1,t) - u_\alpha(\alpha,t)}{1-\alpha} = h(t), \quad 0 < \alpha < 1,$$

$$a_1 \ge a(x,t) \ge a_0 > 0, \quad h \in W_2^1(0,T), \quad \varphi_\alpha \in W_2^1(0,T), \quad \varphi_\alpha(0) = 0, \quad 0 \le \alpha \le 1,$$

$$\varphi_0(1) = h(0), \quad \varphi_1'(1) = h(0), \quad \frac{\varphi_\alpha(1) - \varphi_\alpha(0)}{1-\alpha} = h(0), \quad 0 < \alpha < 1, \quad f \in L_2(G).$$

It is well known that, for $\alpha = 0$ and $\alpha = 1$, the corresponding problems with local conditions are solvable, and the solutions are unique and belong to $W_2^{2,1}(G)$.

We prove the existence and uniqueness of solutions of the family of problems with nonlocal conditions for each $\alpha \in (0,1)$. For the differences $u_\alpha - u_0$ and $u_\alpha - u_1$ $(0 < \alpha < 1)$, we establish a priori estimates and use them to prove that if $\varphi_\alpha \to \varphi_0$ as $\alpha \to 0$, then $u_\alpha \to u_0$ and if $\varphi_\alpha \to \varphi_1$ as $\alpha \to 1$, then $u_\alpha \to u_1$.

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1. STATEMENT OF THE PROBLEM

In the rectangle $G = (0,1) \times (0,T)$, we consider the family of mixed problems with nonlocal conditions

$$\frac{1}{a(x,t)}\frac{\partial u_\alpha}{\partial t} - \frac{\partial^2 u_\alpha}{\partial x^2} = f(x,t), \qquad 0 < \alpha < 1, \tag{1}$$

$$u_\alpha(x,0) = \varphi_\alpha(x), \qquad u_\alpha(0,t) = 0, \qquad \frac{u_\alpha(1,t) - u_\alpha(\alpha,t)}{1-\alpha} = h(t) \tag{2}$$

and two mixed problems with local conditions:

$$\frac{1}{a(x,t)}\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} = f(x,t), \tag{3}$$

$$u_0(x,0) = \varphi_0(x), \qquad u_0(0,t) = 0, \qquad u_0(1,t) = h(t), \tag{4}$$

$$\frac{1}{a(x,t)}\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = f(x,t), \tag{5}$$

$$u_1(x,0) = \varphi_1(x), \qquad u_1(0,t) = 0, \qquad \frac{\partial u_1(1,t)}{\partial x} = h(t). \tag{6}$$

We assume that the coefficient $a(x,t)$ in Eqs. (1), (3), and (5) is a continuously differentiable function and

$$a_1 \ge a(x,t) \ge a_0 > 0, \qquad f \in L_2(G), \qquad h \in W_2^1(0,T),$$
$$\varphi_\alpha, \varphi_0, \varphi_1 \in W_2^1(0,1), \qquad \varphi_\alpha(0) = \varphi_0(0) = \varphi_1(0) = 0,$$
$$\varphi_0(1) = h(0), \qquad \varphi_1'(1) = h(0), \qquad \frac{\varphi_\alpha(1) - \varphi_\alpha(\alpha)}{1-\alpha} = h(0).$$

It is known that if these assumptions hold, then there exist unique solutions of problem (3), (4) and (5), (6), and these solutions have first derivatives with respect to t and derivatives of order ≤ 2 with respect to x almost everywhere in G. If the problem data are smoother, then so are the solutions. In the present paper, we prove the existence and uniqueness of solutions of the family of problems (1), (2) for each $0 < \alpha < 1$.

Problem (3), (4) is the limit of the family of problems (1), (2) as $\alpha \to 0$, and problem (5), (6) is the limit of the family of problems (1), (2) as $\alpha \to 1$.

In the present paper, we prove a priori estimates for the differences $u_\alpha - u_0$ and $u_\alpha - u_1$ and use them to show that if $\varphi_\alpha \to \varphi_0$ as $\alpha \to 0$, then $u_\alpha \to u_0$ and if $\varphi_\alpha \to \varphi_1$ as $\alpha \to 1$, then $u_\alpha \to u_1$. Thus, we establish the following new important property: the solutions of mixed problems for parabolic equations with nonlocal conditions in (2) change continuously as these conditions pass into the local ones.

Note that a similar property was obtained in [1] for the mixed problems (1), (2) for the case in which the nonlocal condition in (2) is replaced by the nonlocal integral condition

$$\frac{1}{\alpha} \int\limits_{1-\alpha}^{1} u_\alpha(x,t)\, dx = h(t) \tag{7}$$

and $u_\alpha \to u_0$ as $\alpha \to 0$, where u_0 is the solution of problem (3), (4). Problems with a condition of the form (7) were earlier studied in [2–7].

2. A PRIORI ESTIMATES AND THE EXISTENCE OF GENERALIZED SOLUTIONS OF PROBLEMS (1), (2)

If we replace the unknown functions in problem (1), (2) by the formula

$$u_\alpha(x,t) = v_\alpha(x,t) + xh(t), \tag{8}$$

then, for the new functions $v_\alpha(x,t)$, we obtain the problem

$$\frac{1}{a(x,t)} \frac{\partial v_\alpha}{\partial t} - \frac{\partial^2 v_\alpha}{\partial x^2} = \tilde{f}(x,t) \equiv f(x,t) - \frac{x}{a} h'(t), \tag{9}$$

$$v_\alpha(x,0) = \tilde{\varphi}_\alpha(x) \equiv \varphi_\alpha(x) - xh(0), \qquad v_\alpha(0,t) = 0, \qquad \frac{v_\alpha(1,t) - v_\alpha(\alpha,t)}{1-\alpha} = 0 \tag{10}$$

for each $\alpha \in (0,1)$.

Theorem 1. *The solutions* v_α *of problem* (9), (10) *satisfy the inequality*

$$\sup_{0 \le t \le T} \int\limits_0^1 \psi_\alpha(x) v_\alpha^2(x,t)\, dx + \int\limits_0^T \int\limits_0^1 \psi_\alpha(x) \left(\frac{\partial v_\alpha(x,t)}{\partial x} \right)^2 dx\, dt$$
$$\le C \left(\int\limits_0^1 \psi_\alpha(x) \tilde{\varphi}_\alpha^2(x)\, dx + \int\limits_0^T \int\limits_0^1 \psi_\alpha(x) \tilde{f}^2(x,t)\, dx\, dt \right), \tag{11}$$

where the constant C is independent of v_α and

$$\psi_\alpha = \begin{cases} 1 & \text{if } 0 \le x \le \alpha \\ (1-x)/(1-\alpha) & \text{if } \alpha \le x \le 1. \end{cases}$$

Proof. By integrating by parts, we obtain the identities

$$\int_0^1 \frac{\psi_\alpha}{a} \frac{\partial v_\alpha}{\partial t} v_\alpha \, dx = \frac{1}{2} \frac{\partial}{\partial t} \int_0^1 \frac{\psi_\alpha}{a} v_\alpha^2 \, dx - \frac{1}{2} \int_0^1 \frac{\partial}{\partial t} \left(\frac{\psi_\alpha}{a} \right) v_\alpha^2 \, dx, \tag{12}$$

$$-\int_0^1 \psi_\alpha \frac{\partial^2 v_\alpha}{\partial x^2} v_\alpha \, dx = -\int_0^\alpha \frac{\partial^2 v_\alpha}{\partial x^2} v_\alpha \, dx - \int_\alpha^1 \frac{1-x}{1-\alpha} \frac{\partial^2 v_\alpha}{\partial x^2} v_\alpha \, dx, \tag{13}$$

$$-\int_0^\alpha \frac{\partial^2 v_\alpha}{\partial x^2} v_\alpha \, dx = \int_0^\alpha \left(\frac{\partial v_\alpha}{\partial x} \right)^2 dx - \frac{\partial v_\alpha(\alpha - 0, t)}{\partial x} v_\alpha(\alpha - 0, t), \tag{14}$$

$$-\int_\alpha^1 \frac{1-x}{1-\alpha} \frac{\partial^2 v_\alpha}{\partial x^2} v_\alpha \, dx = \int_\alpha^1 \frac{1-x}{1-\alpha} \left(\frac{\partial v_\alpha}{\partial x} \right)^2 dx + \frac{\partial v_\alpha(\alpha + 0, t)}{\partial x} v_\alpha(\alpha + 0, t) - \frac{1}{1-\alpha} \int_\alpha^1 \frac{\partial v_\alpha}{\partial x} v_\alpha \, dx, \tag{15}$$

$$2\int_\alpha^1 \frac{\partial v_\alpha}{\partial x} v_\alpha \, dx = v_\alpha^2(1, t) - v_\alpha^2(\alpha, t) = 0. \tag{16}$$

By multiplying both sides of Eq. (9) by $e^{c(\tau - t)} \psi_\alpha v_\alpha$, by integrating the resulting relation with respect to x from 0 to 1 and with respect to t from 0 to $0 < \tau \le T$, and by using identities (12)–(16), we obtain

$$\frac{1}{2} \int_0^1 \frac{\psi_\alpha(x)}{a(x,\tau)} v_\alpha^2(x,\tau) \, dx + \int_0^\tau \int_0^1 e^{c(\tau - t)} \psi_\alpha(x) \left(\frac{\partial v_\alpha(x,\tau)}{\partial x} \right)^2 dx \, dt$$
$$= \frac{e^{c\tau}}{2} \int_0^1 \frac{\psi_\alpha(x)}{a(x,0)} \tilde{\varphi}_\alpha^2(x) \, dx + \int_0^\tau \int_0^1 e^{c(\tau - t)} \psi_\alpha(x) \tilde{f}(x,t) v_\alpha(x,t) \, dx \, dt$$
$$- \frac{1}{2} \int_0^\tau \int_0^1 e^{c(\tau - t)} \frac{\psi_\alpha(x)}{a^2(x,t)} \left(\frac{\partial a(x,t)}{\partial t} + c a(x,t) \right) v_\alpha^2(x,t) \, dx \, dt. \tag{17}$$

We choose the constant c in (17) so as to ensure that $-\dfrac{\partial a(x,t)}{\partial t} \le c a(x,t)$. Then the third term on the right-hand side in (17) is nonpositive, and we omit it. The second term on the right-hand side in (17) can be estimated by the quantity

$$\frac{\varepsilon}{4} T \sup_{0 \le t \le T} \int_0^1 \psi_\alpha(x) v_\alpha^2(x,t) \, dx + \frac{e^{2cT}}{\varepsilon} \int_0^T \int_0^1 \psi_\alpha(x) \tilde{f}^2(x,t) \, dx \, dt,$$

where $\varepsilon > 0$ is an arbitrary number. After that, from (17), we obtain the inequality

$$\frac{1}{2a_1}\int_0^1 \psi_\alpha(x) v_\alpha^2(x,t)\,dx + \int_0^\tau\int_0^1 \psi_\alpha(x)\left(\frac{\partial v_\alpha(x,t)}{\partial x}\right)^2 dx\,dt$$
$$\le \frac{e^{cT}}{2a_0}\int_0^1 \psi_\alpha(x)\tilde{\varphi}_\alpha^2(x)\,dx + \frac{e^{2cT}}{\varepsilon}\int_0^T\int_0^1 \psi_\alpha(x)\tilde{f}^2(x,t)\,dx\,dt$$
$$+\frac{\varepsilon}{4}T\sup_{0\le t\le T}\int_0^1 \psi_\alpha(x)v_\alpha^2(x,t)\,dx, \tag{18}$$

whose right-hand side is independent of τ. Then we take the least upper bound (sup) with respect to $0 \le \tau \le T$ on the left-hand side of this inequality and set $\varepsilon = 1/(a_1 T)$. As a result, we obtain inequality (11), where

$$C = e^{cT}\max\left(2a_1/a_0, 4Te^{cT}a_1^2\right).$$

The proof of Theorem 1 is complete.

Let us proceed to the proof of the existence of a strong generalized solution of problem (9), (10). For the solution of problem (9), (10), we introduce the space E_α with the norm

$$\|v_\alpha\|_{E_\alpha}^2 = \sup_{0\le t\le T}\int_0^1 \psi_\alpha(x)v_\alpha^2(x,t)\,dx + \int_0^T\int_0^1 \psi_\alpha(x)\left(\frac{\partial v_\alpha(x,t)}{\partial x}\right)^2 dx\,dt$$

given by the left-hand side of (11). The right-hand side $\tilde{f}$ of Eq. (9) and the initial data $\tilde{\varphi}_\alpha$ are considered in the space F_α of vector functions $\tilde{F}_\alpha = \left(\tilde{\varphi}_\alpha, \tilde{f}\right)$ with the norm

$$\left\|\tilde{F}_\alpha\right\|_{F_\alpha}^2 = \int_0^1 \psi_\alpha(x)\tilde{\varphi}^2(x)\,dx + \int_0^T\int_0^1 \psi_\alpha(x)\tilde{f}^2(x,t)\,dx\,dt$$

given by the right-hand side of inequality (11). Problem (9), (10) corresponds to the operator $L_\alpha = (\mathcal{L}_\alpha, l)$, $\mathcal{L}_\alpha v_\alpha \equiv \dfrac{1}{a}\dfrac{\partial v_\alpha}{\partial t} - \dfrac{\partial^2 v_\alpha}{\partial x^2}$, $lv_\alpha = v_\alpha(x,0)$, mapping E_α into F_α with domain

$$D\left(L_\alpha\right) = \left\{v_\alpha \in W_2^{2,1}(G):\ v_\alpha(0,t) = 0,\ \frac{v_\alpha(1,t) - v_\alpha(\alpha,t)}{1-\alpha} = 0\right\}.$$

In a standard way, one can show [8, p. 92 of the Russian translation] that the operator $L_\alpha : E_\alpha \to F_\alpha$ is closable. Its closure will be denoted by $\bar{L}_\alpha$, and the domain of $\bar{L}_\alpha$ will be denoted by $D\left(\bar{L}_\alpha\right)$.

Definition. A solution of the equation $\bar{L}_\alpha v_\alpha = \tilde{F}_\alpha \in F_\alpha$ is called a *strong generalized solution* of problem (9), (10).

In other words, a function v_α is called a strong generalized solution of problem (9), (10) if there exists a sequence of functions $v_{\alpha,n} \in D\left(L_\alpha\right)$ such that $\|v_{\alpha,n} - v_\alpha\|_{E_\alpha} \to 0$ and $\|L_\alpha v_{\alpha,n} - \tilde{F}_\alpha\|_{F_\alpha} \to 0$ as $n \to \infty$. For the sequences $v_{\alpha,n} \in D\left(L_\alpha\right)$, we have the inequalities

$$\|v_{\alpha,n}\|_{E_\alpha}^2 \le C\,\|L_\alpha v_{\alpha,n}\|_{F_\alpha}^2, \tag{19}$$

which follow from Theorem 1. By passing to the limit in (19), we obtain the inequality

$$\|v_\alpha\|_{E_\alpha}^2 \le C\left\|\bar{L}_\alpha v_\alpha\right\|_{F_\alpha}^2, \tag{20}$$

which implies that the strong generalized solution of problem (9), (10) is unique and the range of $\bar{L}_\alpha$ satisfies $R\left(\bar{L}_\alpha\right) = \overline{R(L_\alpha)}$. Therefore, to prove the existence of a strong generalized solution, one should show that $R\left(L_\alpha\right)$ is dense in F_α. In turn, since the range of the trace operator l is dense in $L_2(0,1)$ with weight ψ_α, it suffices to show that the relation

$$\int_0^T\int_0^1 \left(\frac{1}{a}\,\frac{\partial v_\alpha}{\partial t} - \frac{\partial^2 v_\alpha}{\partial x^2}\right)\psi_\alpha(x) g(x,t)\,dx\,dt = 0, \tag{21}$$

where v_α ranges over $D_0\left(L_\alpha\right) = \{v_\alpha \in D(L_\alpha):\ v_\alpha(x,0) = 0\}$ and $g \in L_2(G)$, implies that $g \equiv 0$.

In (21), we set

$$v_\alpha(x,t) = \int_0^t e^{c(T-\tau)}\left[\int_0^x (x-\xi)\frac{g(\xi,\tau)}{a(\xi,\tau)}\,d\xi - x\int_0^1 \psi_\alpha(\xi)\frac{g(\xi,\tau)}{a(\xi,\tau)}\,d\xi\right]d\tau. \tag{22}$$

Then $g(x,t) = a(x,t)\dfrac{\partial^3 v_\alpha(x,t)}{\partial x^2 \partial t}$, and from (21), we obtain

$$\int_0^T\int_0^1 e^{c(T-t)}\psi_\alpha(x)\left(\frac{\partial v_\alpha}{\partial t}\,\frac{\partial^3 v_\alpha}{\partial x^2\partial t} - a\frac{\partial^2 v_\alpha}{\partial x^2}\,\frac{\partial^3 v_\alpha}{\partial t\partial x^2}\right)dx\,dt = 0. \tag{23}$$

Just as in (12)–(16), we prove the identities

$$\int_0^T\int_0^1 e^{c(T-t)}\psi_\alpha(x)\frac{\partial v_\alpha}{\partial t}\,\frac{\partial^3 v_\alpha}{\partial x^2\partial t}\,dx\,dt = -\int_0^T\int_0^1 e^{c(T-t)}\psi_\alpha(x)\left(\frac{\partial^2 v_\alpha}{\partial x\partial t}\right)^2 dx\,dt, \tag{24}$$

$$\int_0^T\int_0^1 e^{c(T-t)}\psi_\alpha(x)a\frac{\partial^2 v_\alpha}{\partial x^2}\,\frac{\partial^3 v_\alpha}{\partial t\partial x^2}\,dx\,dt = \int_0^1 \psi_\alpha(x)a\left(\frac{\partial^2 v_\alpha}{\partial x^2}\right)^2 dx\Bigg|_{t=T} + \int_0^T\int_0^1\left(ac - \frac{\partial a}{\partial t}\right)e^{c(T-t)}\psi_\alpha(x)\left(\frac{\partial^2 v_\alpha}{\partial x^2}\right)^2 dx\,dt. \tag{25}$$

Take $ac \ge \partial a/\partial t$. Then, by using identities (24) and (25), from (23), we obtain the inequality

$$\int_0^T e^{c(T-t)}\psi_\alpha(x)\left(\frac{\partial^2 v_\alpha}{\partial x\partial t}\right)^2 dx\,dt \le 0,$$

which implies that $v_\alpha \equiv 0$. By virtue of (22), we conclude that $g \equiv 0$. Thus, there exists a strong generalized solution of problem (9), (10), and inequality (18) holds.

3. A PRIORI ESTIMATES FOR THE DIFFERENCES $u_\alpha - u_0$ AND $u_\alpha - u_1$

By $u(x,t)$ we denote either the solution u_0 of problem (3), (4) or the solution u_1 of problem (5), (6). Let us introduce the new function

$$v(x,t) = u(x,t) - x\frac{u(1,t) - u(\alpha,t)}{1-\alpha}. \tag{26}$$

The function v is a strong generalized solution of the problem

$$\frac{1}{a}\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = f(x,t) - \frac{x}{a(1-\alpha)}\left[\frac{\partial u(1,t)}{\partial t} - \frac{\partial u(\alpha,t)}{\partial t}\right],$$

$$v(x,0) = \varphi(x) - x\frac{u(1,0)-u(\alpha,0)}{1-\alpha},$$

$$v(0,t) = 0, \qquad \frac{v(1,t)-v(\alpha,t)}{1-\alpha} = 0,$$

where $\varphi(x)$ is one of the functions $\varphi_0(x)$ and $\varphi_1(x)$, depending on the problem to be considered. The difference $w_\alpha = v - v_\alpha$ is a strong generalized solution of the problem

$$\frac{1}{a}\frac{\partial w_\alpha}{\partial t} - \frac{\partial^2 w_\alpha}{\partial x^2} = \frac{x}{a}\left[h'(t) - \frac{\dfrac{\partial u(1,t)}{\partial t} - \dfrac{\partial u(\alpha,t)}{\partial t}}{1-\alpha}\right],$$

$$w_\alpha(x,0) = \varphi - \varphi_\alpha - x\left[\frac{u(1,0)-u(\alpha,0)}{1-\alpha} - h(0)\right],$$

$$w_\alpha(0,t) = 0, \qquad \frac{w_\alpha(1,t)-w_\alpha(\alpha,t)}{1-\alpha} = 0,$$

and w_α satisfies the inequality [see (11)]

$$\sup_{0\le t\le T}\int_0^1 \psi_\alpha(x) w_\alpha^2(x,t)\,dx + \int_0^T\!\!\int_0^1 \psi_\alpha(x)\left(\frac{\partial w_\alpha(x,t)}{\partial x}\right)^2 dx\,dt$$
$$\le 2C\left(\int_0^1 \psi_\alpha(x)\left|\varphi(x)-\varphi_\alpha(x)\right|^2 dx + \int_0^1 \psi_\alpha(x)x^2\left|\frac{u(1,0)-u(\alpha,0)}{1-\alpha} - h(0)\right|^2 dx\right.$$
$$\left. + \int_0^T\!\!\int_0^1 \psi_\alpha\frac{x^2}{a^2(x,t)}\left|h'(t) - \frac{\dfrac{\partial u(1,t)}{\partial t} - \dfrac{\partial u(\alpha,t)}{\partial t}}{1-\alpha}\right|^2 dx\,dt\right). \qquad (27)$$

Inequality (27) remains valid if we use the inequality $\psi_\alpha(x) \ge 1-x$ on its left-hand side and the inequality $\psi_\alpha(x) \le 1$ on the right-hand side. We apply the resulting inequality to the difference

$$u - u_\alpha = w_\alpha + x\left[\frac{u(1,t)-u(\alpha,t)}{1-\alpha} - h(t)\right],$$

where $u(x,t)$ is either the solution u_0 of problem (3), (4) or the solution u_1 of problem (5), (6). As a result, we obtain the a priori estimate

$$\sup_{0\le t\le T}\int_0^1 (1-x)|u_0-u_\alpha|^2\,dx + \int_0^T\!\!\int_0^1 (1-x)\left|\frac{\partial u_0}{\partial x} - \frac{\partial u_\alpha}{\partial x}\right|^2 dx\,dt$$
$$\le C\left\{\int_0^1 \left|\varphi_0(x)-\varphi_\alpha(x)\right|^2 dx + \int_0^T\left[\left|\frac{\alpha h(t)-u_0(\alpha,t)}{1-\alpha}\right|^2 + \left|\frac{\alpha h'(t) - \dfrac{\partial u_0(\alpha,t)}{\partial t}}{1-\alpha}\right|^2\right] dt\right.$$
$$\left. + \left|\frac{\alpha h(0)-\varphi_0(\alpha)}{1-\alpha}\right|^2\right\}, \qquad (28)$$

where u_0 is the solution of problem (3), (4), and the a priori estimate

$$\sup_{0\le t\le T}\int_0^1 (1-x)\left|u_1-u_\alpha\right|^2 dx + \int_0^T\int_0^1 (1-x)\left|\frac{\partial u_1}{\partial x}-\frac{\partial u_\alpha}{\partial x}\right|^2 dx\,dt$$
$$\le C\left\{\int_0^1 \left|\varphi_1(x)-\varphi_\alpha(x)\right|^2 dx + \int_0^T\left[\left|\frac{u_1(1,t)-u_1(\alpha,t)}{1-\alpha}-h(t)\right|^2\right.\right.$$
$$\left.\left.+\left|\frac{\dfrac{\partial u_1(1,t)}{\partial t}-\dfrac{\partial u_1(\alpha,t)}{\partial t}}{1-\alpha}-h'(t)\right|^2\right]dt+\left|\frac{\varphi_1(1)-\varphi_1(\alpha)}{1-\alpha}-h(0)\right|^2\right\}, \qquad (29)$$

where u_1 is the solution of problem (5), (6). Note that the constants C in inequalities (11) and (27)–(29) are in general different but independent of the parameter α, the solutions u_0, u_α, and u_1, the initial data of the problems, the coefficient a, the right-hand side f of the equations, and the function h.

4. CONTINUOUS DEPENDENCE OF SOLUTIONS OF PROBLEM (1), (2) ON THE PARAMETER α

Theorem 2. *If*

$$\lim_{\alpha\to 0}\int_0^T \left|\varphi_\alpha(x)-\varphi_0(x)\right|^2 dx = 0, \qquad (30)$$

then

$$\lim_{\alpha\to 0}\left[\sup_{0\le t\le T}\int_0^1 (1-x)\left|u_\alpha-u_0\right|^2 dx + \int_0^T\int_0^1 (1-x)\left|\frac{\partial u_\alpha}{\partial x}-\frac{\partial u_0}{\partial x}\right|^2 dx\,dt\right] = 0. \qquad (31)$$

Proof. Since the solutions of problem (3), (4) satisfy the relations

$$\lim_{\alpha\to 0}\int_0^T u_0(\alpha,t)\,dt = 0, \qquad \lim_{\alpha\to 0}\int_0^T \left|\frac{\partial u_0(\alpha,t)}{\partial t}\right|^2 dx = 0$$

and $\lim_{\alpha\to 0}\varphi_0(\alpha)=0$, it follows from (30) and inequality (28) that relation (31) holds.

To complete the considerations, it remains to show that, for each function $\varphi_0 \in W_2^1(0,T)$ satisfying the conditions $\varphi_0(0)=0$ and $\varphi_0(1)=h(0)$, there exist functions $\varphi_\alpha \in W_2^1(0,T)$ such that $\varphi_\alpha(0)=0$, $(\varphi_\alpha(1)-\varphi_\alpha(\alpha))/(1-\alpha)=h(0)$, and relation (30) is valid. Obviously, one can take $\varphi_\alpha(x)=\varphi_0(x)-x\left((\alpha h(0)-\varphi_0(\alpha))/(1-\alpha)\right)$. The proof of Theorem 2 is complete.

Theorem 3. *If*

$$\lim_{\alpha\to 1}\int_0^1 \left|\varphi_\alpha(x)-\varphi_1(x)\right|^2 dx = 0, \qquad (32)$$

then

$$\lim_{\alpha\to 0}\left[\sup_{0\le t\le T}\int_0^1 (1-x)\left|u_\alpha-u_1\right|^2 dx + \int_0^T\int_0^1 (1-x)\left|\frac{\partial u_\alpha}{\partial x}-\frac{\partial u_1}{\partial x}\right|^2 dx\,dt\right] = 0. \qquad (33)$$

Proof. Since the solutions of problem (5), (6) satisfy the relation

$$\lim_{\alpha\to 1}\int_0^T\left[\left|\frac{u_1(1,t)-u_1(\alpha,t)}{1-\alpha}-h(t)\right|^2+\left|\frac{\dfrac{\partial u_1(1,t)}{\partial t}-\dfrac{\partial u_1(\alpha,t)}{\partial t}}{1-\alpha}-h'(t)\right|^2\right]dt=0$$

and $\lim\limits_{\alpha\to 1}\dfrac{\varphi_1(x)-\varphi_1(\alpha)}{1-\alpha}=h(0)$, it follows from condition (32) and inequality (29) that relation (31) is valid. To complete the considerations, it remains to show that, for each function $\varphi_1\in W_2^1(0,T)$ satisfying the conditions $\varphi_1(0)=0$ and $\varphi_1'(1)=h(0)$, there exist functions $\varphi_\alpha\in W_2^1(0,T)$ such that $\varphi_\alpha(0)=0$, $(\varphi_\alpha(1)-\varphi_\alpha(\alpha))/(1-\alpha)=h(0)$, and relation (32) holds. Indeed, one can take

$$\varphi_\alpha(x)=\varphi_1(x)-x\left(\frac{\varphi_1(1)-\varphi_1(\alpha)}{1-\alpha}-h(0)\right).$$

The proof of Theorem 3 is complete.

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